NETHERLANDS GEODETIC COMMISSION

PUBLICATIONS ON GEODESY NEW SERIES

VOLUME 7 NUMBER 1

A FURTHER INQUIRY INTO THE THEORY OF S-TRANSFORMATIONS AND CRITERION MATRICES

by

M. MOLENAAR

INTERNATIONAL INSTITUTE FOR AERIAL SURVEY AND EARTH SCIENCES (ITC) – Enschede

1981

RIJKSCOMMISSIE VOOR GEODESIE, THIJSSEWEG 11, DELFT, THE NETHERLANDS

PRINTED BY W. D. MEINEMA B.V., DELFT, THE NETHERLANDS

ISBN 906132226X

To my dearest Heidi and David,
who always, when I was wandering about in the realm of abstractions too long, managed to call me back to the real world of family life, and who were always able to prove that one finds the greatest happiness and satisfaction in the role of husband and father.

PREFACE

The present publication is the result of a study which lasted from the end of 1974 until the spring of 1980. The earliest ideas, however, occurred in the last part of my period as a student at the Technical University of Delft. At that time I was first confronted with the theory of S–transformations and criterion matrices as it was presented, during the courses, for use in planimetric networks. Then the question arose of how to apply these ideas in three-dimensional pointfields. Professor Baarda of T.U. - Delft studied this problem in the beginning of the seventies, later on I was forced in that direction by difficulties I met when studying the precision and reliability of aerotriangulation blocks. When my research was progressing, Professor Baarda showed interest and was willing to act as my promotor on this subject. I am very much indebted to him for the many valuable discussions we had during the last six years. His criticisms and suggestions were very stimulating for my research. His assistant, Dr. v. Daalen, gave valuable suggestions for the part of the thesis concerning the choice of covariance functions for the criterion matrices. He was also very helpful in reading the thesis and checking some of the derivations.

Furthermore, thanks are due to Miss Hunter who turned my English into real English, and to Mr Rogge for editing the text. Last but certainly not least I would like to thank Mrs Lefers who never lost her good humour during this difficult typing job.

CONTENTS

CHAPTER I INTRODUCTION

1.1 Why this study?

Trained at the geodetic institute of the technical university of Delft, the author joined the International Institute for Aerial Survey and Earth Sciences - ITC Enschede, with a vivid interest in the application of mathematical statistics to geodesy. With this attitude a study was initiated into which way the 'Delft techniques' [4a, 5, 6] could be applied to photogrammetry. In the initial stage of this research emphasis was put on methods for error-detection in aero-triangulation blocks. The problem was twofold:

- A direct question from photogrammetric practice asked for an error-detection technique, which could be applied at an early stage of aero-triangulation projects e.g.: preadjustment error-detection.
- More theoretical, but nevertheless very important for practice as well, was the question about the reliability of photogrammetric data.

The second question was, and still is, tackled by W. Förstner [16, 17], Grün [26, 27] and the photogrammetry group of the T.U. Delft [39].

A group at ITC took up the challenge of the first question, as experience showed that 'gross data errors' prevented the convergence of the solution for block-adjustments. Because the mathematical relationships for independent model blocks are relatively simple, a pilot study has been done for this method of adjustment.

A preadjustment error-detection method should be based on a test of the misclosures of condition equations for observations. The difficulty is the formulation of such condition equations, because independent model blocks require the connection of models, which is in fact the connection of coordinate systems [1, 30]. The transformation elements for these connections are found as a result of the adjustment.

An alternative is the connection of models before adjustment by means of S-transformations [6]. Therefore basepoints should be chosen in the overlap of two models to be connected. When both models have been transformed to that base, condition equations can be formulated for other points in the overlap, similar to ([6] Ch. 17).

This approach is complicated and could be avoided for planimetric independent model (i.m.) blocks [13, 34, 36]. For this case, use of complex number algebra according to [3] made a simple elimination of transformation elements possible. For three-dimensional i.m. blocks such a simple elimination doesn't seem possible. There the formulation of condition equations should be based on the use of S-transformations. Unpublished studies by Baarda and Molenaar show that a similar conclusion is valid for a block-adjustment using photobundles. Thus the necessity is felt for the use of three-dimensional S-transformations in photogrammetry.

Besides error-detection techniques, attention was paid to the connection of photogrammetric blocks to ground control. When rigorous block-adjustment procedures became available, photogrammetrists wanted to investigate how well their blocks fit to given terrestrial coordinates. Many research projects and experiments began in this field. A summary and description of techniques applied in this research is given in [35] and its references.

One of the main problems indicated in that paper was the lack of knowledge about the variance-covariance matrices for the photo-block coordinates and for the terrestrial coordinates. The former matrix can be found by some extra numerical effort as a byproduct of the block-adjustment. The latter one is more difficult to obtain, when use is made of old pointfields, from which not all data used for the computation of coordinates are available anymore. Baarda proposed in [6] to replace the real matrix for such pointfields by an artificial one. This suggestion is not only of importance for use in testfields, but also for ordinary block-adjustments. As photogrammetric practice shows a tendency to the use of larger blocks, the method should be developed for extensive pointfields.

So it should be studied how the criterion matrix of Baarda can be generalized to cover large areas. The use of such matrices will require the use of S-bases again. The solution of these problems will facilitate the set up of block-adjustments with a proper use of stochastic groundcontrol.

1.2 A sketch of the problem

Of course, the question arises about the meaning of S-bases. Therefore we shall start with a short explanation and a sketch of some related problems which will be treated to a greater extent in the following chapters.

The geodesist considers it as one of his main tasks to find the relative positioning of points on and near the earth's surface. Therefore he measures angles and length ratios. These observations are stochastic and in general they are assumed to be normally distributed about an unknown expected value. From these observations coordinates will be computed, which are also stochastic and have an unknown expected value. To initiate the computations in geodetic practice, use is made of approximate values for these variates, which will differ very little from the unknown expected values. Using these approximate values, the original relationships between observations and coordinates can be expanded in a Taylor series, where second and higher order terms are neglegible with respect to first order terms. Under this assumption corrections to the approximate values of the coordinates can be considered as linear functions of corrections to the approximate values for the observations. This means that the normal distribution will be conserved. Then, the precision of the coordinates is completely described by their variance-covariance matrix, which is a function of the variance-covariance matrix of the observations. Tests on the precision of the coordinates should be based on that matrix.

The problem is that starting from a set of observations, different sets of coordinates can be computed, depending on the choice of coordinate system. This choice is made by means of some parameters in the relationships, by which the coordinates are computed. Values for these parameters can not be obtained by measurements, thus they should be introduced as non-stochastic quantities. Of course this is of importance for the precision of the coordinates, as different choices will lead to different values for their variance-covariance matrix. The theory of S-transformations describes the relation between such different coordinate systems.

In general geodetic coordinates should be computed in a three-dimensional Euclidian space R_3, but in many cases the objective of the geodetic survey is formulated so that it allows the processing of data in two subspaces i.e.: the one-dimensional space R_1 related to the direction of the local gravity vector for height measurements and, orthogonal to that, a two-dimensional space R_2 for horizontal control. This approach makes it possible to use a simplified mathematical model, especially when the measurements cover a limited area. Then rectalinear coordinate systems can be used in R_1 and R_2. Baarda dealt with this situation in [6]. He studied the stochastic consequences of the introduction of coordinate systems for planimetric and for levelling networks, and gave criteria for their precision. In his study Baarda formulated the concept of 'S-transformations' which proved to be essential for developing such criteria.

In this paper we shall investigate the application of Baarda's ideas to more extensive pointfields.

These pointfields should be considered as three-dimensional, which implies a more complicated algebraic structure. Often geodesists prefer an (approximate) description in a curved two-dimensional space for 'horizontal' point positioning as an alternative and additional to that they use a one-dimensional space for 'height'. The spherical or ellipsoidal shape of the earth makes such an approach very attractive.

None of these solutions allows a direct application of the theory developed in [6]. This situation led to the present study dealing with the following problems:
The concept of S-transformations needs a more general formulation, from which the specific forms for distinct mathematical models describing different types of pointfields can be found. The S-bases for three-dimensional networks and spherical triangulation will be derived in this way. Baarda demonstrated in [6] that the use of S-bases is necessary for tests on the precision of networks. Based on the generalization in this paper an alternative proof will be given.

After these preparations it is possible to define 'pointfields with a homogeneous and isotropic inner precision', this definition giving a sufficient base for the design of a criterion matrix for large pointfields over a sphere.

We should finally prove that this matrix is consistent with the matrix for the complex plane as given in [6].

1.3 A guide for the reader

The problems mentioned are treated in the following chapters.
Chapter II defines S-systems and S-transformations in a general way. In relation to these, definitions for K-systems and K-transformations follow. The former give stochastic coordinate transformations, whereas the latter give non-stochastic transformations.
Chapter III starts with a definition of 'measurable', 'estimable' and 'non-estimable' quantities. Based on these definitions rules are given for the choice of S-bases. This chapter concludes with two examples: the choice of an S-base in the complex plane and one for spherical triangulation.
Chapter IV makes use of the theory developed in chapter II and III for finding the S-base in R_3 and for finding the related S- and K-transformations. The derivations are using quaternion algebra to show the structure of the coordinate computation in R_3 clearly.
Chapter V first demonstrates why S-transformations are necessary for tests on the precision of pointfields. Then a proposal follows for the structure of criterion matrices for more extended geodetic networks. This proposal is followed by a definition of homogeneous and isotropic inner precision for such networks. The relation between S-bases on the sphere and S-bases in R_3 and R_2 will be given. The chapter concludes with the design of a criterion matrix for networks over a sphere and with considerations about the semi-positive definiteness of such a matrix.

At the end of this introduction it is stressed once more that the theory is based on the assumption formulated in section 1.2:
Observations in geodesy are normally distributed, and functional relationships can be linearized so that the normal distribution is conserved for functions of these observations.

CHAPTER II A GENERALIZED FORMULATION OF S- AND K-TRANSFORMATIONS

2.1 The basic relationships

From the discussion in section 1.2, it may be clear that the theory to be developed deals with the relationships between three sets of variates: observations or measurable quantities, parameters which define a coordinate system, and coordinates. These variates will be denoted as follows:

1. measurable quantities; values for these are directly or indirectly obtainable by measurements. In terrestrial geodesy, these are angles and ratios of length. A definition of the term 'measurable' will be given in chapter III, although we shall already use it in this chapter. The notation is:

   $\{\ldots\ldots \tilde{x}^i \ldots\ldots\}$ for expected values

   $\{\ldots\ldots \underline{x}^i \ldots\ldots\}$ for stochastic quantities or variates

   $\{\ldots\ldots x^i_o \ldots\ldots\}$ for approximate values $\qquad x^i_o \approx \tilde{x}^i;\ i = 1, \ldots\ldots, n$

2. parameters, which define a coordinate system, further on called definition parameters:

   $\{\ldots\ldots \tilde{x}^u \ldots\ldots\}$ for expected values

   $\{\ldots\ldots \underline{x}^u \ldots\ldots\}$ for stochastic quantities or variates

   $\{\ldots\ldots x^u_o \ldots\ldots\}$ for approximate values $\qquad x^u_o \approx \tilde{x}^u;\ u = n+1, \ldots\ldots, n+b$

3. coordinates, which will be computed from the measurable quantities by means of a set of functional relationships:

$$
\begin{aligned}
&\text{a)} \quad (\tilde{y}^r) = (y^r\{\ldots \tilde{x}^i \ldots\ldots \tilde{x}^u \ldots\}) && i = 1, \ldots\ldots\ldots, n \\
&\text{b)} \quad (\underline{y}^r) = (y^r\{\ldots \underline{x}^i \ldots\ldots \underline{x}^u \ldots\}) && u = n+1, \ldots, n+b \qquad (2.1) \\
&\text{c)} \quad (y^r_o) = (y^r\{\ldots x^i_o \ldots\ldots x^u_o \ldots\}) && r = 1, \ldots\ldots\ldots, n+b
\end{aligned}
$$

This set of relationships will be chosen so that inversion is possible:

$$
\text{a)} \quad \begin{pmatrix} \tilde{x}^i \\ \tilde{x}^u \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\ldots \tilde{y}^r \ldots\ldots\} \\ x^u\{\ldots\ldots \tilde{y}^r \ldots\ldots\} \end{pmatrix}
$$

$$
\text{b)} \quad \begin{pmatrix} \underline{x}^i \\ \underline{x}^u \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\ldots \underline{y}^r \ldots\ldots\} \\ x^u\{\ldots\ldots \underline{y}^r \ldots\ldots\} \end{pmatrix} \qquad (2.1')
$$

$$
\text{c)} \quad \begin{pmatrix} x^i_o \\ x^u_o \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\ldots y^r_o \ldots\ldots\} \\ x^u\{\ldots\ldots y^r_o \ldots\ldots\} \end{pmatrix}
$$

For the given variates the following difference quantities will be used:

$$\begin{array}{lll} \Delta \tilde{x}^i = \tilde{x}^i - x^i_o & \Delta \tilde{x}^u = \tilde{x}^u - x^u_o & \Delta \tilde{y}^r = \tilde{y}^r - y^r_o \\ \Delta \underline{x}^i = \underline{x}^i - x^i_o & \Delta \underline{x}^u = \underline{x}^u - x^u_o & \Delta \underline{y}^r = \underline{y}^r - y^r_o \end{array} \qquad (2.2)$$

If we neglect the second and higher order terms, a Taylor expansion using approximate values x_o gives for (2.1):

$$(\tilde{y}^r) = (y^r\{\ldots x^i_o \ldots\ldots x^u_o \ldots\}) + (U^r_i \;\; U^r_u)\begin{pmatrix}\Delta \tilde{x}^i \\ \Delta \tilde{x}^u\end{pmatrix}, \text{ with } (U^r_i) = \left(\frac{\partial y^r}{\partial x^i}\right)_{x^i_o,\, x^u_o}$$

$$(\underline{y}^r) = (y^r\{\ldots x^i_o \ldots\ldots x^u_o \ldots\}) + (U^r_i \;\; U^r_u)\begin{pmatrix}\Delta \underline{x}^i \\ \Delta \underline{x}^u\end{pmatrix}, \text{ and } (U^r_u) = \left(\frac{\partial y^r}{\partial x^u}\right)_{x^i_o,\, x^u_o}$$

with (2.1) and (2.2) this gives:

$$\begin{array}{ll} \text{a)} & (\tilde{y}^r - y^r_o) = (\Delta \tilde{y}^r) = (U^r_i \;\; U^r_u)\begin{pmatrix}\Delta \tilde{x}^i \\ \Delta \tilde{x}^u\end{pmatrix} \\ \text{b)} & (\underline{y}^r - y^r_o) = (\Delta \underline{y}^r) = (U^r_i \;\; U^r_u)\begin{pmatrix}\Delta \underline{x}^i \\ \Delta \underline{x}^u\end{pmatrix} \end{array} \qquad (2.3)$$

In a similar way (2.1′) leads to:

$$\begin{array}{llll} \text{a)} & \begin{pmatrix}\Delta \tilde{x}^i \\ \Delta \tilde{x}^u\end{pmatrix} = \begin{pmatrix}A^i_r \\ A^u_r\end{pmatrix}(\Delta \tilde{y}^r) & \text{with} & (A^i_r) = \left(\frac{\partial x^i}{\partial y^r}\right)_{y^r_o} \\ & & \text{and} & (A^u_r) = \left(\frac{\partial x^u}{\partial y^r}\right)_{y^r_o} \\ \text{b)} & \begin{pmatrix}\Delta \underline{x}^i \\ \Delta \underline{x}^u\end{pmatrix} = \begin{pmatrix}A^i_r \\ A^u_r\end{pmatrix}(\Delta \underline{y}^r) & & \end{array} \qquad (2.3')$$

The unknown variates $\tilde{y}$ and $\tilde{x}$ in the original relationships are now replaced by the difference quantities $\Delta\tilde{y}$ and $\Delta\tilde{x}$ in the linearized forms. The following study will consider the latter. The coefficient matrices obtained by the linearization are related by:

$$\begin{array}{l} (U^r_i \;\; U^r_u)\begin{pmatrix}A^i_{r'} \\ A^u_{r'}\end{pmatrix} = (\delta^r_{r'}) \\ \begin{pmatrix}A^i_r \\ A^u_r\end{pmatrix}(U^r_{i'} \;\; U^r_u) = \begin{pmatrix}\delta^i_{i'} & 0 \\ 0 & \delta^u_{u'}\end{pmatrix} \end{array} \qquad (2.3'')$$

2.2 S-systems

In the relationships (2.1) the variates y^r have been expressed as a function of x^i and x^u. From now on we shall discuss the situation in which there are no observations avaliable for x^u. Thus in real computations only approximate values can be used and (2.1) will be then:

$$
\begin{aligned}
&\text{a)} \quad (\tilde{y}^{(u)^r}) = (y^r\{\ldots\tilde{x}^i\ldots\ldots x_o^u\ldots\}) \\
&\text{b)} \quad (\underline{y}^{(u)^r}) = (y^r\{\ldots\underline{x}^i\ldots\ldots x_o^u\ldots\}) \\
&\text{c)} \quad (y_o^{(u)^r}) = (y_o^r) = (y^r\{\ldots x_o^i\ldots\ldots x_o^u\ldots\})
\end{aligned}
\tag{2.4}
$$

Because (2.4) made use of x_o^u, the y-variates on the left side are given the upper index (u). We define:

Def. 2.I

> A set of relationships as given in (2.4) will be called an 'S-system'. In (2.4) the parameters (x_o^u) form the 'S-base' of the S-system which can be specified here as the (u)-system.

The names 'S-system', 'S-transformation' and 'S-base' are used to follow Baarda, though due to the generalizations made here, these names may lose the strict meaning they had in [6]. In the applications given later in this publication, they will return, however, to that strict meaning.

The assumption that no observations are available for x^u comes from the practical situation where a geodesist has to define a coordinate system e.g.: the origin, direction of coordinate axes and length scale should be defined. These parameters cannot be derived from observations. Some authors (Grafarend [24], Meissl [32], Bjerhammar [11]) deal with this problem in the so-called free net adjustments. They allow rank deficiencies in the coefficient matrices of the unknown coordinates in the least squares adjustment, according to the Gauss-Markov model. These rank deficiencies are due to the lack of definition of coordinate systems. The adjustment of such networks is solved by means of generalized matrix inverses. This complicates the interpretation of the final results as the definition of coordinate systems is then not so transparent. Baarda found that by the use of S-systems these problems are avoided, because their coordinate systems are defined in a simple way.

Introduction of $(y^{(u)^r})$ in the inverse relationships (2.1′) gives:

$$
\begin{aligned}
&\text{a)} \quad \begin{pmatrix} \tilde{x}^i \\ x_o^u \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\tilde{y}^{(u)^r}\ldots\} \\ x^u\{\ldots\tilde{y}^{(u)^r}\ldots\} \end{pmatrix} \\
&\text{b)} \quad \begin{pmatrix} \underline{x}^i \\ x_o^u \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\underline{y}^{(u)^r}\ldots\} \\ x^u\{\ldots\underline{y}^{(u)^r}\ldots\} \end{pmatrix} \\
&\text{c)} \quad \begin{pmatrix} x_o^i \\ x_o^u \end{pmatrix} = \begin{pmatrix} x^i\{\ldots y_o^{(u)^r}\ldots\} \\ x^u\{\ldots y_o^{(u)^r}\ldots\} \end{pmatrix} = \begin{pmatrix} x^i\{\ldots y_o^r\ldots\} \\ x^u\{\ldots y_o^r\ldots\} \end{pmatrix}
\end{aligned}
\tag{2.4'}
$$

For computation in the (u)-system the difference quantities for y in (2.2) should be replaced by:

$$
\begin{aligned}
(\Delta\tilde{y}^{(u)^r}) &= (\tilde{y}^{(u)^r} - y_o^{(u)^r}) = (\tilde{y}^{(u)^r} - y_o^r) \\
(\Delta\underline{y}^{(u)^r}) &= (\underline{y}^{(u)^r} - y_o^{(u)^r}) = (\underline{y}^{(u)^r} - y_o^r)
\end{aligned}
\tag{2.5}
$$

The Taylor expansion of (2.4.a) neglecting second and higher order terms leads to:

$$(\widetilde{y}^{(u)^r}) = (y^r\{\ldots x^i_o\ldots\ldots x^u_o\ldots\}) + (U^r_i)\ (\Delta\widetilde{x}^i) \tag{2.6.a}$$

and for (2.4.b):

$$(\underline{y}^{(u)^r}) = (y^r\{\ldots x^i_o\ldots\ldots x^u_o\ldots\}) + (U^r_i)\ (\Delta\underline{x}^i) \tag{2.6.b}$$

This with (2.1.c) and (2.5) gives:

$$\begin{aligned}&\text{a)}\quad (\widetilde{y}^{(u)^r} - y^r_o) = (\Delta\widetilde{y}^{(u)^r}) = (U^r_i)\ (\Delta\widetilde{x}^i)\\ &\text{b)}\quad (\underline{y}^{(u)^r} - y^r_o) = (\Delta\underline{y}^{(u)^r}) = (U^r_i)\ (\Delta\underline{x}^i)\end{aligned} \tag{2.7}$$

and similar for (2.4'):

$$\begin{aligned}&\text{a)}\quad (\Delta\widetilde{x}^i) = (A^i_r)\ (\Delta\widetilde{y}^{(u)^r})\\ &\text{b)}\quad (\Delta\underline{x}^i) = (A^i_r)\ (\Delta\underline{y}^{(u)^r})\end{aligned} \tag{2.7'}$$

In the S–system formulated in (2.4), the derived quantities (y-variates) can be considered as a function of the measurable quantities x^i only. Whether it is possible to linearize these functional relationships according to (2.6) depends on the choice of the approximate values (x^i_o) and not on (x^u_o). This is the importance of using S–systems. In geodesy this means that coordinates computed in an S–system are (linearized) functions of only the measured angles and ratios of length. So statements about the coordinates concerning precision and reliability are in fact statements about the internal geometry of the pointfield under consideration.

2.3 S–transformations

In geodetic practice it may occur that for a pointfield, coordinates have been computed in more than one S–system e.g.: besides the (u)–system we could have another system defined by the parameters x^p. These systems are related by transformations which leave the internal geometry of the pointfield invariant. In terms of section 2.1 this means that the relationship (2.1) can be replaced by:

$$\begin{aligned}&\text{a)}\quad (\widetilde{y}'^r) = (y'^r\{\ldots\widetilde{x}^i\ldots\ldots\widetilde{x}^p\ldots\})\\ &\text{b)}\quad (\underline{y}'^r) = (y'^r\{\ldots\underline{x}^i\ldots\ldots\underline{x}^p\ldots\}) \qquad ;\ p = n+1,\ldots,n+b\\ &\text{c)}\quad (y'^r_o) = (y'^r\{\ldots x^i_o\ldots\ldots x^p_o\ldots\})\end{aligned} \tag{2.8}$$

where the parameters x^u have been replaced by x^p. (2.8) makes use of the same variates, x^i, as used in (2.1), so these quantities must be invariant under a transformation from (2.1) to (2.8).

The inverted relationships of (2.8) are:

$$\text{a)}\quad \begin{pmatrix} \tilde{x}^i \\ \tilde{x}^p \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\tilde{y}'^r\ldots\} \\ x^p\{\ldots\tilde{y}'^r\ldots\} \end{pmatrix}$$

$$\text{b)}\quad \begin{pmatrix} \underline{x}^i \\ \underline{x}^p \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\underline{y}'^r\ldots\} \\ x^p\{\ldots\underline{y}'^r\ldots\} \end{pmatrix} \qquad (2.8')$$

$$\text{c)}\quad \begin{pmatrix} x^i_o \\ x^p_o \end{pmatrix} = \begin{pmatrix} x^i\{\ldots y'^r_o\ldots\} \\ x^p\{\ldots y'^r_o\ldots\} \end{pmatrix}$$

Linearization of (2.8) gives:

$$\text{a)}\quad (\Delta\tilde{y}'^r) = (V^r_i \;\; V^r_p)\begin{pmatrix} \Delta\tilde{x}^i \\ \Delta\tilde{x}^p \end{pmatrix} \qquad (V^r_i) = \left(\frac{\partial y'^r}{\partial x^i}\right)_{x^i_o,\, x^p_o}$$

$$\text{b)}\quad (\Delta\underline{y}'^r) = (V^r_i \;\; V^r_p)\begin{pmatrix} \Delta\underline{x}^i \\ \Delta\underline{x}^p \end{pmatrix} \qquad (V^r_p) = \left(\frac{\partial y'^r}{\partial x^p}\right)_{x^i_o,\, x^p_o} \qquad (2.9)$$

and (2.8') gives:

$$\text{a)}\quad \begin{pmatrix} \Delta\tilde{x}^i \\ \Delta\tilde{x}^p \end{pmatrix} = \begin{pmatrix} B^i_r \\ B^p_r \end{pmatrix}(\Delta\tilde{y}'^r) \qquad (B^i_r) = \left(\frac{\partial x^i}{\partial y'^r}\right)_{y'^r_o}$$

$$\text{b)}\quad \begin{pmatrix} \Delta\underline{x}^i \\ \Delta\underline{x}^p \end{pmatrix} = \begin{pmatrix} B^i_r \\ B^p_r \end{pmatrix}(\Delta\underline{y}'^r) \qquad (B^p_r) = \left(\frac{\partial x^p}{\partial y'^r}\right)_{y'^r_o} \qquad (2.9')$$

In the sequel we assume that the relationships (2.1) and (2.8) have been formulated so that:

$$(\tilde{y}^r) \approx (\tilde{y}'^r) \qquad \text{a)}$$

$$(\underline{y}^r) \approx (\underline{y}'^r) \qquad \text{b)} \qquad (2.10)$$

$$\text{and for approximate values} \quad (y^r_o) = (y'^r_o) \qquad \text{c)}$$

The relationships b) just mean that the values of $\underline{y}^r$ and $\underline{y}'^r$ obtained from the same set of actual values for $\underline{x}^i$ do not differ very much, however the stochastic properties may differ considerably. If no observations are available for x^p, only x^p_o can be used in (2.8), we will assume that these approximate values have been computed in the same system as x^u_o. Then the p-system is defined by:

$$\text{a)}\quad (\tilde{y}'^{(p)^r}) = (y'^r\{\ldots\tilde{x}^i\ldots\ldots x^p_o\ldots\})$$

$$\text{b)}\quad (\underline{y}'^{(p)^r}) = (y'^r\{\ldots\underline{x}^i\ldots\ldots x^p_o\ldots\}) \qquad (2.11)$$

$$\text{c)}\quad (y'^{(p)^r}_o) = (y'^r\{\ldots x^i_o\ldots\ldots x^p_o\ldots\}) = (y'^r_o)$$

Because of (2.4.c), (2.10.c) and (2.11.c) the symbol ′ can be omitted on the left-hand side of expression (2.11), this will cause no loss of clarity.

$$
\begin{aligned}
&\text{a)} \quad (\tilde{y}^{(p)'^r}) = (y'^r\{\ldots \tilde{x}^i \ldots\ldots x_o^p \ldots\})\\
&\text{b)} \quad (\underline{y}^{(p)'^r}) = (y'^r\{\ldots \underline{x}^i \ldots\ldots x_o^p \ldots\})\\
&\text{c)} \quad (y_o^{(p)'^r}) = (y'^r\{\ldots x_o^i \ldots\ldots x_o^p \ldots\}) = (y_o^r)
\end{aligned}
\qquad (2.11')
$$

Here is:

$$y_o'^{(p)'^r} = y_o^{(p)'^r} = y_o^r \quad \text{and} \quad y_o^{(p)'^r} = y_o^{(u)'^r}$$

Inversion of (2.11′) gives:

$$
\begin{aligned}
&\text{a)} \quad \begin{pmatrix} \tilde{x}^i \\ x_o^p \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\ldots \tilde{y}^{(p)'^r} \ldots\ldots\} \\ x^p\{\ldots\ldots \tilde{y}^{(p)'^r} \ldots\ldots\} \end{pmatrix}\\
&\text{b)} \quad \begin{pmatrix} \underline{x}^i \\ x_o^p \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\ldots \underline{y}^{(p)'^r} \ldots\ldots\} \\ x^p\{\ldots\ldots \underline{y}^{(p)'^r} \ldots\ldots\} \end{pmatrix}\\
&\text{c)} \quad \begin{pmatrix} x_o^i \\ x_o^p \end{pmatrix} = \begin{pmatrix} x^i\{\ldots\ldots y_o^{(p)'^r} \ldots\ldots\} \\ x^p\{\ldots\ldots y_o^{(p)'^r} \ldots\ldots\} \end{pmatrix}
\end{aligned}
\qquad (2.11'')
$$

Difference quantities for the y-variates in the (p)-system are:

$$
\begin{aligned}
(\Delta\tilde{y}^{(p)'^r}) &= (\tilde{y}^{(p)'^r} - y_o^{(p)'^r}) = (\tilde{y}^{(p)'^r} - y_o^r)\\
(\Delta\underline{y}^{(p)'^r}) &= (\underline{y}^{(p)'^r} - y_o^{(p)'^r}) = (\underline{y}^{(p)'^r} - y_o^r)
\end{aligned}
\qquad (2.12)
$$

If second and higher order terms are neglected, a Taylor expansion of (2.11′.a) leads to:

$$(\tilde{y}^{(p)'^r}) = (y'^r\{\ldots x_o^i \ldots\ldots x_o^p \ldots\}) + (V_i^r)\,(\Delta\tilde{x}^i)$$

and for (2.11′.b):

$$(\underline{y}^{(p)'^r}) = (y'^r\{\ldots x_o^i \ldots\ldots x_o^p \ldots\}) + (V_i^r)\,(\Delta\underline{x}^i)$$

With (2.8.c), (2.10.c) and (2.12) we get:

$$
\begin{aligned}
&\text{a)} \quad (\Delta\tilde{y}^{(p)'^r}) = (V_i^r)\,(\Delta\tilde{x}^i)\\
&\text{b)} \quad (\Delta\underline{y}^{(p)'^r}) = (V_i^r)\,(\Delta\underline{x}^i)
\end{aligned}
\qquad (2.13)
$$

In a similar way we obtain from (2.11″):

$$\text{a)}\quad (\Delta\tilde{x}^{i}) = (B^{i}_{r})\ (\Delta\tilde{y}^{(p)^{r}})$$

$$\text{b)}\quad (\Delta\underline{x}^{i}) = (B^{i}_{r})\ (\Delta\underline{y}^{(p)^{r}}) \qquad (2.13')$$

(2.13) and (2.7′) show that a transformation from (u) – to (p) –system is possible indeed, if condition (2.10) is satisfied. This transformation is:

$$\text{a)}\quad (\Delta\tilde{y}^{(p)^{r}}) = (V^{r}_{i})\ (A^{i}_{r'})\ (\Delta\tilde{y}^{(u)^{r'}})$$

$$\text{b)}\quad (\Delta\underline{y}^{(p)^{r}}) = (V^{r}_{i})\ (A^{i}_{r'})\ (\Delta\underline{y}^{(u)^{r'}}) \qquad (2.14)$$

The inverse transformation is:

$$\text{a)}\quad (\Delta\tilde{y}^{(u)^{r}}) = (U^{r}_{i})\ (B^{i}_{r'})\ (\Delta\tilde{y}^{(p)^{r'}})$$

$$\text{b)}\quad (\Delta\underline{y}^{(u)^{r}}) = (U^{r}_{i})\ (B^{i}_{r'})\ (\Delta\underline{y}^{(p)^{r'}}) \qquad (2.14')$$

Def. 2.II

Transformations given by (2.14) and (2.14′) are ‘S–transformations’.

S–transformations perform a transition from S–base (u) to S–base (p). This means that linearized relationships (2.7) will be replaced by (2.13). That is why S–transformations can be considered as transformations of coefficient matrices used in the propagation laws for stochastic properties e.g. : deviations and variances and covariances.

In (2.14) and (2.14′) $i = 1,\dots,n$ and $r,r' = 1,\dots\dots, n+b$, so the products $(V^{r}_{i})\,(A^{i}_{r'})$ and $(U^{r}_{i})\,(B^{i}_{r'})$ give singular matrices. The derivation leading to (2.14) and (2.14′) made use of the fact that x^{i} should be invariant under S–transformations as required in the first part of this section.

2.4 Another derivation of S–transformations

The S–transformations (2.14) and (2.14′) have been found under the assumption that the S–base in the (u) – and the (p)-system both are known. This is not a necessary condition. To demonstrate that, we first state that because of (2.4), (2.11′) and $x^{i}_{o} \approx \tilde{x}^{i}$ and $x^{i}_{o} \approx \underline{x}^{i}$, (2.10) can be replaced by:

$$\text{a)}\quad (\tilde{y}^{(u)^{r}}) \approx (\tilde{y}^{(p)^{r}})$$

$$\text{b)}\quad (\underline{y}^{(u)^{r}}) \approx (\underline{y}^{(p)^{r}}) \qquad (2.15)$$

$$\text{c)}\quad (y^{(u)^{r}}_{o}) = (y^{(p)^{r}}_{o}) = (y^{r}_{o})$$

The relationships b) have a similar meaning as (2.10.b).

Introduction of $y^{(u)^r}$ in (2.11″) will give (see (2.4′)):

$$\text{a)}\quad \begin{pmatrix} \tilde{x}^i \\ \tilde{x}^{(u)^p} \end{pmatrix} = \begin{pmatrix} x^i\{\ldots.\tilde{y}^{(u)^r}\ \ldots.\} \\ x^p\{\ldots.\tilde{y}^{(u)^r}\ \ldots.\} \end{pmatrix}$$

$$\text{b)}\quad \begin{pmatrix} \underline{x}^i \\ \underline{x}^{(u)^p} \end{pmatrix} = \begin{pmatrix} x^i\{\ldots.\underline{y}^{(u)^r}\ \ldots.\} \\ x^p\{\ldots.\underline{y}^{(u)^r}\ \ldots.\} \end{pmatrix} \qquad (2.16)$$

$$\text{c)}\quad \begin{pmatrix} x^i_o \\ x^{(u)^p}_o \end{pmatrix} = \begin{pmatrix} x^i\{\ldots.y^{(u)^r}_o\ \ldots.\} \\ x^p\{\ldots.y^{(u)^r}_o\ \ldots.\} \end{pmatrix}$$

The variates x^p get the superscript (u), because they have been computed from $y^{(u)^r}$, that is they have been computed in the (u)-system. As the relationships (2.11″) are the inverse of (2.11′), the latter will give:

$$\text{a)}\quad (\tilde{y}^{(u)^r}) = (y'^r\{\ldots\tilde{x}^i\ \ldots..\tilde{x}^{(u)^p}\ \ldots\})$$

$$\text{b)}\quad (\underline{y}^{(u)^r}) = (y'^r\{\ldots\underline{x}^i\ \ldots..\underline{x}^{(u)^p}\ \ldots\}) \qquad (2.16')$$

$$\text{c)}\quad (y^{(u)^r}_o) = (y'^r\{\ldots x^i_o\ \ldots..x^{(u)^p}_o\ \ldots\}) = (y^r_o)$$

To find $y^{(u)^r}$, (2.4) can be replaced by (2.16′). According to (2.9) the linearized equations are:

$$\text{a)}\quad (\Delta\tilde{y}^{(u)^r}) = (V^r_i\ \ V^r_p)\begin{pmatrix} \Delta\tilde{x}^i \\ \Delta\tilde{x}^{(u)^p} \end{pmatrix} = (V^r_i)\,(\Delta\tilde{x}^i) + (V^r_p)\,(\Delta\tilde{x}^{(u)^p})$$

$$(2.17)$$

$$\text{b)}\quad (\Delta\underline{y}^{(u)^r}) = (V^r_i\ \ V^r_p)\begin{pmatrix} \Delta\underline{x}^i \\ \Delta\underline{x}^{(u)^p} \end{pmatrix} = (V^r_i)\,(\Delta\underline{x}^i) + (V^r_p)\,(\Delta\underline{x}^{(u)^p})$$

whereas (2.9′) and (2.16) lead to:

$$\text{a)}\quad \begin{pmatrix} \Delta\tilde{x}^i \\ \Delta\tilde{x}^{(u)^p} \end{pmatrix} = \begin{pmatrix} B^i_r \\ B^p_r \end{pmatrix}(\Delta y^{(u)^r})$$

$$(2.17')$$

$$\text{b)}\quad \begin{pmatrix} \Delta\underline{x}^i \\ \Delta\underline{x}^{(u)^p} \end{pmatrix} = \begin{pmatrix} B^i_r \\ B^p_r \end{pmatrix}(\Delta\underline{y}^{(u)^r})$$

With (2.13) and (2.17) the S-transformation from (u) – to (p)-system is:

$$\text{a)}\quad (\Delta\tilde{y}^{(p)^r}) = (\Delta\tilde{y}^{(u)^r}) - (V^r_p)\,(\Delta\tilde{x}^{(u)^p})$$

$$(2.18)$$

$$\text{b)}\quad (\Delta\underline{y}^{(p)^r}) = (\Delta\underline{y}^{(u)^r}) - (V^r_p)\,(\Delta\underline{x}^{(u)^p})$$

This derivation shows indeed that a transition from the (u) – to the (p)-system requires no knowledge of the S-base in the former system. The transformation in this form only makes use of $x^{(u)^p}$ or $\Delta x^{(u)^p}$, i.e.: the values of the new base parameters x^p expressed in the (u)-system. This makes it possible to transform a system where no S-base has been specified, to a new system with a specified S-base:

Def. 2.III $\boxed{\text{An S-system where no S-base has been specified is an ‘(a)-system’}}$

Equations (2.18) express $y^{(p)^r}$ as a function of $y^{(u)^r}$ and $x^{(u)^p}$, whereas (2.14) gives a relationship between the y-variates in the old and the new system, without reference to the x-variates and not using the base parameters explicity. Starting from (2.18) a similar expression can be found. Therefore (2.17′) should be used to eliminate $x^{(u)^p}$

$$\begin{aligned} &\text{a)} \quad (\Delta\tilde{y}^{(p)^r}) = (\Delta\tilde{y}^{(u)^r}) - (V^r_p)(B^p_{r'})(\Delta\tilde{y}^{(u)^{r'}}) = [(\delta^r_{r'}) - (V^r_p)(B^p_{r'})](\Delta\tilde{y}^{(u)^{r'}}) \\ &\text{b)} \quad (\Delta\underline{y}^{(p)^r}) = (\Delta\underline{y}^{(u)^r}) - (V^r_p)(B^p_{r'})(\Delta\underline{y}^{(u)^{r'}}) = [(\delta^r_{r'}) - (V^r_p)(B^p_{r'})](\Delta\underline{y}^{(u)^{r'}}) \end{aligned} \tag{2.19}$$

Introduction of an (a)-system with:

$$\begin{aligned} &\text{a)} \quad (\tilde{x}^i) = (x^i\{\ldots\ldots\tilde{y}^{(p)^r}\ldots\ldots\}) = (x^i\{\ldots\ldots\tilde{y}^{(a)^r}\ldots\ldots\}) \\ &\text{b)} \quad (\underline{x}^i) = (x^i\{\ldots\ldots\underline{y}^{(p)^r}\ldots\ldots\}) = (x^i\{\ldots\ldots\underline{y}^{(a)^r}\ldots\ldots\}) \\ &\text{c)} \quad (x^i_o) = (x^i\{\ldots\ldots y_o^{(p)^r}\ldots\ldots\}) = (x^i\{\ldots\ldots y_o^{(a)^r}\ldots\ldots\}) \\ &\text{d)} \quad (y_o^{(p)^r}) = (y_o^{(a)^r}) \end{aligned} \tag{2.20}$$

in (2.16) will give $x^{(a)^p}$ instead of $x^{(u)^p}$. With this, we find in (2.19):

$$\begin{aligned} &\text{a)} \quad (\Delta\tilde{y}^{(p)^r}) = [(\delta^r_{r'}) - (V^r_p)(B^p_{r'})](\Delta\tilde{y}^{(a)^{r'}}) \\ &\text{b)} \quad (\Delta\underline{y}^{(p)^r}) = [(\delta^r_{r'}) - (V^r_p)(B^p_{r'})](\Delta\underline{y}^{(a)^{r'}}) \end{aligned} \tag{2.19′}$$

In (2.19) and (2.19′) the left sides of the equations, as well as the coefficient matrices within [ ] on the right sides, are identical. Only the variates y^r on the right sides have been expressed in different S-systems. This leads to the conclusion that every S-system satisfying (2.20) will be transformed to the (p)-system by this transformation. The coefficient matrix on the right side is called an ‘S-matrix’ and is denoted by:

$$(\delta^r_{r'}) - (V^r_p)(B^p_{r'}) = (S^{(p)^r}_{r'}) \tag{2.21}$$

The kernal letter S gets a superscript (p) to indicate the transformation to the (p)-system. In this notation (2.19′) is:

$$\begin{aligned} &\text{a)} \quad (\Delta\tilde{y}^{(p)^r}) = (S^{(p)^r}_{r'})(\Delta\tilde{y}^{(a)^{r'}}) \\ &\text{b)} \quad (\Delta\underline{y}^{(p)^r}) = (S^{(p)^r}_{r'})(\Delta\underline{y}^{(a)^{r'}}) \end{aligned} \tag{2.22}$$

In a similar way the transformation to the (u) - system is:

$$\text{a)} \quad (\Delta\tilde{y}^{(u)^r}) = (S^{(u)^r}_{r'})(\Delta\tilde{y}^{(a)^{r'}})$$

$$\text{b)} \quad (\Delta\underline{y}^{(u)^r}) = (S^{(u)^r})(\Delta\underline{y}^{(a)^{r'}}) \qquad (2.22')$$

From (2.19.a), (2.21) and (2.22′.a) it follows:

$$(\Delta\tilde{y}^{(p)^r}) = (S^{(p)^r}_{r'})(\Delta\tilde{y}^{(u)^{r'}}) = (S^{(p)^r}_{r'})(S^{(u)^{r'}}_{r''})(\Delta\tilde{y}^{(a)^{r''}})$$

and with (2.22.a):

$$(S^{(p)^r}_{r''}) = (S^{(p)^r}_{r'})(S^{(u)^{r'}}_{r''})$$

and similarly $\qquad (2.23)$

$$(S^{(u)^r}_{r''}) = (S^{(u)^r}_{r'})(S^{(p)^{r'}}_{r''})$$

According to (2.23) the product of two S-matrices always gives the lefthand matrix. Therefore the final result of a transformation from the (a)– to the (p)-system via the (u)-system is equal to the result of a direct transformation. From (2.14), (2.19) and (2.21) follows:

$$(S^{(u)^r}_{r'}) = (V^r_i)(A^i_{r'})$$

the S-matrix is singular (see discussions of (2.14) and (2.14′)).

2.5 K-transformations

2.5.1 K-systems and K-transformations

In this section we will give the relationship between two different coordinate systems which both have been based on the variates $\underline{x}^i$. Let the first system be defined by:

$$(\tilde{x}^u) = (\tilde{x}^u_{(1)})$$

$$\text{and} \quad (x^u_o) = (x^u_{(1)_o})$$

then for (2.1.a) and (2.1.c) we get:

$$\text{a)} \quad (\tilde{y}^r_{(1)}) = (y^r\{\ldots\tilde{x}^i\ldots\ldots\tilde{x}^u_{(1)}\ldots\})$$

$$\text{c)} \quad (y^r_{(1)_o}) = (y^r\{\ldots x^i_o\ldots\ldots x^u_{(1)_o}\ldots\}) \qquad (2.24)$$

Def. 2.IV $\boxed{\text{The set of relationships (2.24) forms a K-system (coordinate system)}}$

In (2.24) the K-system will be called the (1)-system because of the parameter values $x^u_{(1)}$. Here (2.3.a) is:

$$(\Delta\tilde{y}^r_{(1)}) = (U^r_{(1)_i} \;\; U^r_{(1)_u}) \begin{pmatrix} \Delta\tilde{x}^i \\ \Delta\tilde{x}^u_{(1)} \end{pmatrix} \qquad (2.25)$$

whereas (2.1'.a) and (2.1'.c) become:

$$\text{a)}\quad \begin{pmatrix} \tilde{x}^i \\ \\ \tilde{x}^u_{(1)} \end{pmatrix} = \begin{pmatrix} x^i\{\ldots \tilde{y}^r_{(1)} \ldots\} \\ \\ x^u\{\ldots \tilde{y}^r_{(1)} \ldots\} \end{pmatrix}$$

$$\text{c)}\quad \begin{pmatrix} x^i_o \\ \\ x^u_{(1)_o} \end{pmatrix} = \begin{pmatrix} x^i\{\ldots y^r_{(1)_o} \ldots\} \\ \\ x^u\{\ldots y^r_{(1)_o} \ldots\} \end{pmatrix} \tag{2.24'}$$

and (2.3'.a):

$$\begin{pmatrix} \Delta\tilde{x}^i \\ \\ \Delta\tilde{x}^u_{(1)} \end{pmatrix} = \begin{pmatrix} A^i_{(1)_r} \\ \\ A^u_{(1)_r} \end{pmatrix} (\Delta\tilde{y}^r_{(1)}) \tag{2.25'}$$

Let the second system be defined by:

$$(\tilde{x}^u) = (\tilde{x}^u_{(2)})$$

$$(x^u_o) = (x^u_{(2)_o})$$

Now we define:

$$(\tilde{x}^u_{(2)}) - (\tilde{x}^u_{(1)}) = (\tilde{\theta}^u_{(12)})$$

$$(x^u_{(2)_o}) - (x^u_{(1)_o}) = (\theta^u_{(12)_o}) \tag{2.26}$$

$$(\Delta\tilde{x}^u_{(2)}) - (\Delta\tilde{x}^u_{(1)}) = (\Delta\tilde{\theta}^u_{(12)})$$

Then we get instead of (2.24):

$$\text{a)}\quad (\tilde{y}^r_{(2)}) = (y^r\{\ldots \tilde{x}^i \ldots \tilde{x}^u_{(2)} \ldots\})$$

$$\text{c)}\quad (y^r_{(2)_o}) = (y^r\{\ldots x^i_o \ldots x^u_{(2)_o} \ldots\}) \tag{2.27}$$

This is the (2)-system, where instead of (2.25)

$$(\Delta\tilde{y}^r_{(2)}) = (U^r_{(2)_i} \;\; U^r_{(2)_u}) \begin{pmatrix} \Delta\tilde{x}^i \\ \\ \Delta\tilde{x}^u_{(2)} \end{pmatrix} = (U^r_{(2)_i} \;\; U^r_{(2)_u}) \begin{pmatrix} \Delta\tilde{x}^i \\ \\ \Delta\tilde{x}^u_{(1)} \end{pmatrix} + (U^r_{(2)_u})\,(\Delta\tilde{\theta}^u_{(12)}) \tag{2.28}$$

The inverse of (2.27) is:

$$\text{a)}\quad \begin{pmatrix} \tilde{x}^i \\ \\ \tilde{x}^u_{(2)} \end{pmatrix} = \begin{pmatrix} x^i\{\ldots \tilde{y}^r_{(2)} \ldots\} \\ \\ x^u\{\ldots \tilde{y}^r_{(2)} \ldots\} \end{pmatrix}$$

$$\text{c)}\quad \begin{pmatrix} x^i_o \\ \\ x^u_{(2)_o} \end{pmatrix} = \begin{pmatrix} x^i\{\ldots y^r_{(2)_o} \ldots\} \\ \\ x^u\{\ldots y^r_{(2)_o} \ldots\} \end{pmatrix} \tag{2.27'}$$

with:

$$\begin{pmatrix} \Delta\tilde{x}^i \\ \\ \Delta\tilde{x}^u_{(2)} \end{pmatrix} = \begin{pmatrix} A^i_{(2)_r} \\ \\ A^u_{(2)_r} \end{pmatrix} (\Delta\tilde{y}^r_{(2)}) \tag{2.28'}$$

Here the transition $\left\{ \begin{array}{l} \tilde{x}^u_{(1)} \longrightarrow \tilde{x}^u_{(2)} \\ x^u_{(1)_o} \longrightarrow x^u_{(2)_o} \end{array} \right\}$ implies $\left\{ \begin{array}{l} \tilde{y}^r_{(1)} \longrightarrow \tilde{y}^r_{(2)} \\ y^r_{(1)_o} \longrightarrow y^r_{(2)_o} \end{array} \right\}$

We write the latter as relationships which express $y^r_{(2)}$ as functions of $y^r_{(1)}$ and a set of transformation parameters τ^γ :

$$\begin{array}{ll} \text{(a)} & (\tilde{y}^r_{(2)}) = (\kappa^r\{ \;\ldots\; \tilde{y}^{r'}_{(1)} \ldots \tilde{\tau}^\gamma_{(12)} \ldots \}) \\ \text{(c)} & (y^r_{(2)_o}) = (\kappa^r\{ \;\ldots\; y^{r'}_{(1)_o} \ldots \tau^\gamma_{(12)_o} \ldots \}) \end{array} \tag{2.29}$$

Def. 2.V $\boxed{\text{The set of relationships (2.29) perform a K-transformation (coordinate transformation)}}$

(2.24) and (2.27) show that $y^r_{(1)}$ and $y^r_{(2)}$ are functions of the same variates x^i, therefore the latter are invariant under the transformation (2.29). In a practical situation this means that (2.24) and (2.27) give two different sets of coordinates for one pointfield. These two sets are related by a similarity transformation which leaves angles and length ratios invariant. The difference equation of (2.29) is:

$$(\Delta\tilde{y}^r_{(2)}) = (K^r_{(12)_{r'}}, K^r_{(12)_\gamma}) \begin{pmatrix} \Delta\tilde{y}^{r'}_{(1)} \\ \\ \Delta\tilde{\tau}^\gamma_{(12)} \end{pmatrix} = (K^r_{(12)_{r'}})(\Delta\tilde{y}^{r'}_{(1)}) + (K^r_{(12)_\gamma})(\Delta\tilde{\tau}^\gamma_{(12)}) \tag{2.30}$$

where:

$$(K^r_{(12)_{r'}}) = \left(\frac{\partial\kappa^r}{\partial y^{r'}_{(1)}}\right)_{y^{r'}_{(1)_o},\ \tau^\gamma_{(12)_o}} = \left(\frac{\partial y^r_{(2)}}{\partial y^{r'}_{(1)}}\right)_{y^{r'}_{(1)_o},\ \tau^\gamma_{(12)_o}}$$

and:

$$(K^r_{(12)_\gamma}) = \left(\frac{\partial\kappa^r}{\partial\tau^\gamma_{(12)}}\right)_{y^r_{(1)_o},\ \tau^\gamma_{(12)_o}}$$

From (2.28) and (2.25′) we obtain:

$$(\Delta y^r_{(2)}) = (U^r_{(2)_i} \quad U^r_{(2)_u}) \begin{pmatrix} \Delta \tilde{x}^i \\ \Delta \tilde{x}^u_{(1)} \end{pmatrix} + (U^r_{(2)_u})(\Delta \tilde{\theta}^u_{(12)}) = \tag{2.31}$$

$$= (U^r_{(2)_i} \quad U^r_{(2)_u}) \begin{pmatrix} A^i_{(1)_{r'}} \\ A^u_{(1)_{r'}} \end{pmatrix} (\Delta y^{r'}_{(1)}) + (U^r_{(2)_u})(\Delta \tilde{\theta}^u_{(12)})$$

From (2.30) and (2.31) it is obvious that:

$$(U^r_{(2)_i} \quad U^r_{(2)_u}) \begin{pmatrix} A^i_{(1)_{r'}} \\ A^u_{(1)_{r'}} \end{pmatrix} = (K^r_{(12)_{r'}}) \tag{2.32.1}$$

and also:

$$(U^r_{(2)_u})(\Delta \tilde{\theta}^u_{(12)}) = (K^r_{(12)_\gamma})(\Delta \tilde{\tau}^\gamma_{(12)}) \tag{2.32.2}$$

Hence the K–transformation (2.29) is defined by b parameters, which are only dependent on $(\theta^u_{(12)})$ from (2.26) and thus on the values of the parameters x^u in the (1)– and the (2)–system.

2.5.2 S–systems and K–transformations

If the difference between two coordinate systems is only due to the fact that different approximate values have been used for the parameters of the S-base x^u, then we find for the (1)–system:

$$\begin{aligned} &\text{b)} \quad (\underline{y}^{(u)^r}_{(1)}) = (y^r\{\ldots\underline{x}^i\ldots x^u_{(1)_o}\ldots\}) \\ &\text{c)} \quad (y^{u^r}_{(1)_o}) = (y^{u^r}\{\ldots x^i_o\ldots x^u_{(1)_o}\ldots\}) \end{aligned} \tag{2.33}$$

and for the (2)–system:

$$\begin{aligned} &\text{b)} \quad (\underline{y}^{(u)^r}_{(2)}) = (y^r\{\ldots\underline{x}^i\ldots x^u_{(2)_o}\ldots\}) \\ &\text{c)} \quad (y^{u^r}_{(2)_o}) = (y^r\{\ldots x^i_o\ldots x^u_{(2)_o}\ldots\}) \end{aligned} \tag{2.33′}$$

In this case (2.29) becomes:

$$\begin{aligned} &\text{b)} \quad (\underline{y}^{(u)^r}_{(2)}) = (\kappa^r\{\ldots\underline{y}^{(u)^{r'}}\ldots\tilde{\tau}^\gamma_{(12)}\ldots\}) \\ &\text{c)} \quad (y^{(u)^r}_{(2)_o}) = (\kappa^r\{\ldots y^{(u)^r}_{(1)_o}\ldots\tau^\gamma_{(12)_o}\ldots\}) \end{aligned} \tag{2.34}$$

with the difference equations according to (2.30):

$$(\Delta\underline{y}^{(u)^r}_{(2)}) = (K^r_{(12)_{r'}})(\Delta\underline{y}^{(u)^{r'}}_{(1)}) + (K^r_{(12)_\gamma})(\Delta\underline{\tau}^\gamma_{(12)}) \tag{2.35}$$

It follows however from (2.7) and (2.7′) that:

$$(\Delta \underline{y}^{(u)^r}_{(2)}) = (U^r_{(2)_i} \quad U^r_{(2)_u}) \begin{pmatrix} \Delta \underline{x}^i \\ 0 \end{pmatrix} = (U^r_{(2)_i} \quad U^r_{(2)_u}) \begin{pmatrix} A^i_{(1)_{r'}} \\ A^u_{(1)_{r'}} \end{pmatrix} (\Delta \underline{y}^{(u)^{r'}}_{(1)})$$

which with (2.32.1) and (2.35) results in:

$$(K^r_{(12)_\gamma})(\Delta \underline{\tau}^\gamma_{(12)}) = (0) \longrightarrow (\Delta \underline{\tau}^\gamma_{(12)}) = (0)$$

Thus the transformation parameters τ^γ are not stochastic in this case. This conclusion agrees with the statement under (2.32.2) that $\tau^\gamma_{(12)}$ is only dependent on $x^u_{(1)}$ and $x^u_{(2)}$. As only approximate values are available for the parameters of the S-base, the K-transformation can only be performed using $\tau^\gamma_{(12)_o}$. This can also be formulated as follows:

The transition $\qquad x^u_{(1)_o} \longrightarrow x^u_{(2)_o}$

leads to a K-transformation

$$(\underline{y}^{(u)^r}_{(2)}) = (\kappa^r \{ \ldots . \; \underline{y}^{(u)^{r'}}_{(1)} \; \ldots . \; \tau^\gamma_{(12)_o} \; \ldots . \})$$

with $\qquad$ (2.34′)

$$(\Delta \underline{y}^{(u)^r}_{(2)}) = (K^r_{(12)r'})(\Delta \underline{y}^{(u)^{r'}}_{(1)})$$

In relation to the statement in section 2.2, i.e. that values for the parameters x^u cannot be obtained by ovservations, we can state now that the definition of K-systems should refer to the choice of approximate values for x^u, and thus for the coordinates. The choice of S-system defines the structure of the coefficient matrices for the application of the laws of propagation for stochastic properties, and the choice of K-system determines the actual values of the elements of these matrices and the values of the coordinates to be computed.

2.5.3 S-transformations interpreted as differential K-transformations

According to (2.18) two different S-systems for which the approximate values of all variates are equal, are related by:

$$(\Delta \tilde{y}^{(u)^r}) = (\Delta \tilde{y}^{(p)^r}) - (U^r_u)(\Delta \tilde{x}^{(p)^u}) \qquad (2.36)$$

In terms of (2.31) and (2.32.1) this can be read as a differential K-transformation where:

$$(K^r_{(12)_{r'}}) = (K^{(pu)^r}_{r'}) = (\delta^r_{r'})$$

$$(\Delta \tilde{\theta}^u_{(12)}) = (\Delta \tilde{\theta}^{(pu)^u}) = (-\Delta \tilde{x}^{(p)^u}) \qquad (2.37)$$

Matrix $(K^r_{r'})$ and vector $(\Delta\widetilde{\theta}^u)$ have the superscript (pu) here instead of the index (12). to indicate an S–transformation. This transition seems to be a differential K–transformation for $\widetilde{y}^r$ whereas y^r_o is not altered. Therefore (2.29) becomes:

$$\begin{aligned} &\text{a)} \quad (\widetilde{y}^{(u)^r}) = (\kappa^r\{\ldots \widetilde{y}^{(p)^r} \ldots\ldots \widetilde{\tau}^{(pu)^\gamma} \ldots\}) \\ &\text{c)} \quad (y_o^{(u)^r}) = (y_o^{(p)^r}) \end{aligned} \tag{2.29'}$$

from which follows:

$$(\Delta\widetilde{y}^{(u)^r}) = (\Delta\widetilde{y}^{(p)^r}) + (K^{(pu)^r}_\gamma)\,(\Delta\widetilde{\tau}^{(pu)^\gamma}) \tag{2.30'}$$

As (2.30') is equivalent to (2.36) the derivation of the S–transformation (2.22) can start from (2.30') instead of (2.18) as was done in section 2.4. The transition from the (p) - to the (u)-system then requires the solution of b parameters $\Delta\widetilde{\tau}^{(pu)^\gamma}$, using b quantities $\Delta y_o^{(u)^{r'}}$ for which the corresponding submatrix $(K^{(pu)^{r'}}_\gamma)$ is nonsingular. Chapter IV gives a situation where such quantities $\Delta\widetilde{y}^{(u)^r}$ follow directly from the choice of the S–base. The S–base in the new system only effects the elimination process of $\Delta\tau$. The approach given in this section has been used by Baarda in [6] and it will be used in chapter IV of this paper as well.

2.6 Epilogue to chapter II

The previous sections deal with the description of some physical object (earth's surface) by means of a mathematical model. Therefore measurable quantities which have a clear interpretation in the model, should be defined on the object. For a good link, between the measuring procedure and the mathematical model, they should be designed in close relation with each other. Under this provision, the measurements will give values for the variates x^i which, in the model, give a complete description of the object. In many cases preference is given to a description by means of variates y^r which are functions of the variates x^i. These functions are given in (2.1) and they require the use of parameters x^u. The S–theory is based on the assumption that no observations are available for the latter. Therefore (2.1) must be replaced by (2.4) where the introduction of x^u_o defines an S–system. If one replaces (2.4) by (2.11'), using the same set of variates x^i, but different parameters, and if one takes care of (2.15), then a transition of the (u)-system to the (p)-system is possible by means of an S–transformation. The variates x^i are invariant to such transformations.

In section 2.5 the necessity of K–transformations appeared to be a consequence of the fact that various sets of values could be introduced for the parameters x^u. The quantities x^i are invariant to these transformations as well. Section 2.5.3 proved the S–transformation to be a differential K–transformation.

A transformation of $y^{(u)^r}_{(1)}$ to $y^{(p)^r}_{(2)}$ has to be made in two steps: First an S–transformation is made according to (2.22); this transforms $y^{(u)^r}_{(1)}$ to $y^{(p)^r}_{(1)}$. Then a K–transformation according to (2.34') transforms $y^{(p)^r}_{(1)}$ to $y^{(p)^r}_{(2)}$.

This chapter made clear that if a mathematical model is based on a set of measurable quantities, only the existence of S–bases gives rise to the possibility of S– and K–transformations. The measurable quantities should be invariant, by definition, to these transformations. If this is not so, the mathematical model is wrong.

CHAPTER III ON THE CHOICE OF S–BASES

3.1 Estimable quantities

3.1.1 Definitions

The introduction of an S-base in section 2.2 was necessary because of the fact that no observations were available for the variates x^u in (2.4). These variables represented parameters which cannot be measured, such as the parameters which define a coordinate system in geodesy. So, the values x^u_o had to be used in (2.4). These values can be chosen such that they define a coordinate system which best fits the aims of the user. Once this choice has been made,the observations for $\underline{x}^i$ are introduced to compute coordinates. In practice these observations follow from one single measurement, but the definition of variates x^i should enable a repetition of measurements to be possible:

Def. 3.I.1

> Let p be any positive integer, then a quantity $\tilde{x}^i$ is said to be 'measurable', if p repetitions of a measuring process are described by the stochastic variates:
>
> $$\underline{x}^i_1\ ,\ \ \underline{x}^i_2\ , \ldots\ldots ,\ \ \underline{x}^i_p$$
>
> which are identical and can therefore be denoted by $\underline{x}^i$ which has the expectation:
>
> $$E\{\,\underline{x}^i\,\} = \tilde{x}^i$$

Note: The word 'measurable' used here should not be mixed up with the similar word used in functional analysis for 'measurable sets', etc. In this paper the word is strictly related to the experimental behaviour of a variate.

Now a distinction can be made between functions of only x^i and functions which need the introduction of extra parameters, x^u.

The former can be written as:

$$\begin{aligned}
&\text{a)} \quad (\tilde{y}^\alpha) = (y^\alpha\{\ldots\ldots\tilde{x}^i\ldots\ldots\}) \\
&\text{b)} \quad (\underline{y}^\alpha) = (y^\alpha\{\ldots\ldots\underline{x}^i\ldots\ldots\}) \\
&\text{c)} \quad (y^\alpha_o) = (y^\alpha\{\ldots\ldots x^i_o\ldots\ldots\})
\end{aligned} \tag{3.1}$$

Then with

$$\tilde{y}^\alpha - y^\alpha_o = \Delta\,\tilde{y}^\alpha \qquad \underline{y}^\alpha - y^\alpha_o = \Delta\,\underline{y}^\alpha \tag{3.2}$$

and

$$U^\alpha_i = \left.\frac{\partial y^\alpha}{\partial x^i}\right|_{x^i_o}$$

the linearization of (3.1) is:

$$\begin{aligned}
&\text{a)} \quad (\Delta\tilde{y}^\alpha) = (U^\alpha_i)\,(\Delta\tilde{x}^i) \\
&\text{b)} \quad (\Delta\underline{y}^\alpha) = (U^\alpha_i)\,(\Delta\underline{x}^i)
\end{aligned} \tag{3.3}$$

With these relationships, estimable functions can be defined as follows:

Def. 3.I.2

Functions y^{α} are 'unbiasedly estimable' if they can be expressed as: $\tilde{y}^{\alpha} = y^{\alpha}\{\ldots\tilde{x}^{i}\ldots\}$ or in linearized form: $y^{\alpha}_{o} + \Delta\tilde{y}^{\alpha} = y^{\alpha}\{\ldots x^{i}_{o}\ldots\} + U^{\alpha}\Delta\tilde{x}^{i}$

In this publication, the expression 'estimable' will be used as a synonym for 'unbiasedly estimable'.

Rao limits his definition of unbiasedly estimable functions ([41] p. 137) to linear functions of the observations. Def. 3.I.2 has been formulated for non-linear functions as well, but under the restriction that good approximate values are available so that second and higher order terms in a Taylor expansion of y^{α} are negligable compared to the first order terms. Under this assumption

$$\tilde{y}^{\alpha} = y^{\alpha}_{o} + \Delta\tilde{y}^{\alpha}$$

The definition can be read as follows:

All y^{α} which are functions of only the measurable quantities x^{i} are unbiasedly estimable. This implies of course that the measurable quantities themselves are unbiasedly estimable.

Examples will be given in sections 3.3.2 and 3.3.3.

Besides the measurable and estimable quantities,there is a third group of variates:

Def. 3.I.3

All quantities which do not satisfy Def. 3.I.1 or Def. 3.I.2 are 'not unbiasedly estimable' or, in short, 'not estimable'.

From chapter II it is clear that the variates x^{u} and y^{r} fall under this latter definition, so these are not estimable.

3.1.2 S- and K-transformations and their invariants

Baardà's choice for complex number algebra and quaternion algebra to describe two and three dimensional survey systems, has been based on the fact that these (division) algebras facilitate network analyses in terms of quantities which are invariant with respect to similarity transformations. Hence an inquiry into the reliability and precision of networks is made independent of the choice of coordinate systems. Grafarend et al. already demonstrated the similarity of estimable quantities and invariants with regard to K-transformations. In this section this similarity will be discussed from a different point of view.

With (3.1) and (2.1′) y^{α} can be expressed as a function of y^{r}:

$$(y^{\alpha}) = (y^{\alpha}\{\ldots x^{i}\ldots\}) = (y^{\alpha}\{\ldots x^{i}\{\ldots y^{r}\ldots\}\ldots\})$$

or, in short:

$$\begin{array}{ll} \text{a)} & (\tilde{y}^{\alpha}) = (\nu^{\alpha}\{\ldots\ \tilde{y}^{r}\ \ldots\}) \\ \text{b)} & (\underline{y}^{\alpha}) = (\nu^{\alpha}\{\ldots\ \underline{y}^{r}\ \ldots\}) \\ \text{c)} & (y^{\alpha}_{o}) = (\nu^{\alpha}\{\ldots\ y^{r}_{o}\ \ldots\}) \end{array} \tag{3.4}$$

The variates y^r can only be computed in an S-system, so they are not invariant to S-transformations and K-transformations. The definition of y^{α}, however, implies that they are invariant with respect to such transformations, because no S-base is needed for their computation. This means with (2.4') and (2.11''):

$$\begin{array}{lll} \text{a)} & (\tilde{y}^{\alpha}) & = (\nu^{\alpha}\{\ldots\ \tilde{y}^{(u)^r}\ \ldots\}) = (\nu^{\alpha}\{\ldots\ \tilde{y}^{(p)^r}\ \ldots\}) \\ \text{b)} & (\underline{y}^{\alpha}) & = (\nu^{\alpha}\{\ldots\ \underline{y}^{(u)^r}\ \ldots\}) = (\nu^{\alpha}\{\ldots\ \underline{y}^{(p)^r}\ \ldots\}) \\ \text{c)} & (y^{\alpha}_{o}) & = (\nu^{\alpha}\{\ldots\ y_o^{(u)^r}\ \ldots\}) = (\nu^{\alpha}\{\ldots\ y_o^{(p)^r}\ \ldots\}) \end{array} \tag{3.5}$$

and with (2.33), (2.33'), (2.24') and (2.27') given in (p)-system:

$$\begin{array}{lll} \text{a)} & (\tilde{y}^{\alpha}) & = (\nu^{\alpha}\{\ldots\ \tilde{y}^{(p)^r}_{(1)}\ \ldots\}) = (\nu^{\alpha}\{\ldots\ \tilde{y}^{(p)^r}_{(2)}\ \ldots\}) \\ \text{b)} & (\underline{y}^{\alpha}) & = (\nu^{\alpha}\{\ldots\ \underline{y}^{(p)^r}_{(1)}\ \ldots\}) = (\nu^{\alpha}\{\ldots\ \underline{y}^{(p)^r}_{(2)}\ \ldots\}) \\ \text{c)} & (y^{\alpha}_{o}) & = (\nu^{\alpha}\{\ldots\ y^{(p)^r}_{(1)\,o}\ \ldots\}) = (\nu^{\alpha}\{\ldots\ y^{(p)^r}_{(2)\,o}\ \ldots\}) \end{array} \tag{3.6}$$

Let:

$$\left.\frac{\partial y^{\alpha}}{\partial y^{r}}\right|_{y^{r}_{o}} = \Lambda^{\alpha}_{r}$$

then the linearization of (3.5a) is:

$$(\Delta\tilde{y}^{\alpha}) = (\Lambda^{\alpha}_{r})(\Delta\tilde{y}^{(u)^r}) = (\Lambda^{\alpha}_{r})(\Delta\tilde{y}^{(p)^r})$$

This gives with (2.13.a) and (2.17.a):

$$(\Delta\tilde{y}^{\alpha}) = (\Lambda^{\alpha}_{r})(V^{r}_{i}\ V^{r}_{p})\begin{pmatrix}\Delta\tilde{x}^{i} \\ \Delta\tilde{x}^{(u)^p}\end{pmatrix} = (\Lambda^{\alpha}_{r})(V^{r}_{i})(\Delta\tilde{x}^{i}) \tag{3.7}$$

hence

$$(\Lambda^{\alpha}_{r})(V^{r}_{i})(\Delta\tilde{x}^{i}) + (\Lambda^{\alpha}_{r})(V^{r}_{p})(\Delta\tilde{x}^{(u)^p}) = (\Lambda^{\alpha}_{r})(V^{r}_{i})(\Delta\tilde{x}^{i})$$

from which follows:

$$(\Lambda^{\alpha}_{r})(V^{r}_{p})(\Delta\tilde{x}^{(u)^p}) = (0)$$

so

$$\left(\frac{\partial y^{\alpha}}{\partial x^{p}}\right) = (\Lambda^{\alpha}_{r})(V^{r}_{p}) = (0) \tag{3.8}$$

The interpretation of (3.8) is: If y^{α} is invariant to S-transformations, then y^{α} can be expressed as a function of only x^{i}. Then (3.5) and (3.8) together lead to the conclusion that:

Theorem (3.9)	Estimable quantities are invariant to S-transformations and quantities which are invariant to S-transformations are estimable	(3.9)

Starting from (3.6) with:

$$\left(\frac{\partial y^{\alpha}}{\partial y^{r}_{(1)}}\right)_{y^{r}_{(1)o}} = (\Lambda^{\alpha}_{(1)\,r}) \quad \text{and} \quad \left(\frac{\partial y^{\alpha}}{\partial y^{r}_{(2)}}\right)_{y^{r}_{(2)o}} = (\Lambda^{\alpha}_{(2)\,r})$$

the linearisation of (3.6.a) is:

$$(\Delta\tilde{y}^{\alpha}) = (\Lambda^{\alpha}_{(1)\,r})(\Delta\tilde{y}^{(p)r}_{(1)}) = (\Lambda^{\alpha}_{(2)r})(\Delta\tilde{y}^{(p)r}_{(2)})$$

With (2.13.a) for (1)– and for (2)–system we get:

$$(\Delta\tilde{y}^{\alpha}) = (\Lambda^{\alpha}_{(1)r})(V^{r}_{(1)i})(\Delta\tilde{x}^{i}) = (\Lambda^{\alpha}_{(2)r})(V^{r}_{(2)i})(\Delta\tilde{x}^{i}) \qquad (3.10)$$

From (3.10) it is clear that y^{α} can be expressed as a function of x^{i} only, thus Def. 3.I.2 applies to y^{α}. This means that besides (3.9) we have:

Theorem (3.9′)	Estimable quantities are invariant to K–transformations and quantities which are invariant to K–transformations are estimable.	(3.9′)

Theorem (3.9) and (3.9′) are closely related, which may also be clear from section 2.5.3, where S–transformations appeared to be differential K–transformations.

Compare (3.7) with (3.3.a):

$$(\Lambda^{\alpha}_{r})(V^{r}_{i}) = (U^{\alpha}_{i}) \text{ and also (see (2.7)) } (\Lambda^{\alpha}_{r})(U^{r}_{i}) = (U^{\alpha}_{i}) \qquad (3.11)$$
$$\text{where} \quad (\Lambda^{\alpha}_{r})(U^{r}_{u}) = (0)$$

This relationship has been derived in a general form, so it will be true for any K–system. Therefore we find:

$$(\Lambda^{\alpha}_{(1)\,r})(V^{r}_{(1)i}) = (\Lambda^{\alpha}_{(2)r})(V^{r}_{(2)i}) = (U^{\alpha}_{i})$$

whereas (3.8) leads to:

$$(\Lambda^{\alpha}_{(1)\,r})(V^{r}_{(1)p}) = (\Lambda^{\alpha}_{(2)\,r})(V^{r}_{(2)p}) = (0)$$

The preceding proof of equivalence of estimable and invariant quantities agrees with the results found by Grafarend et al. in [25] theorem (3.1). Our proof seems to be more direct because it shows that (3.9) and (3.9′) follow directly from Def. 3.I.2. In our approach the mathematical model should be developed in relation to the possible measuring procedure. The measuring procedure defines the measurable quantities, then estimable functions can be defined within the context of the mathematical model. Now transformations are allowed for the functions y^r, but only under the restriction that they leave the estimable quantities invariant. In fact these transformations are only made possible by the introduction of non-estimable quantities as stated in section 2.6. Grafarend and Schaffrin started their reasoning from a different point of view, because in their paper they define first the allowable transformations, then they prove that invariant quantities are estimable. This is a difference in attitude which may have consequences in further research on related items, like the design of criterion matrices for geodetic pointfields (see sections 5.2 and 5.6).

3.2 The procedure for the choice of an S-base

In the preceding sections the concept 'mathematical model' has been used, without definition. In this publication we shall use the term mathematical model for a complete set of mutually consistent relationships, based on the use of a set of measurable quantities. Within the context of such a mathematical model it can be decided which functions are estimable, and also within the context of such a model, S-systems can be defined. The geodesist, who has angles and ratios of length as measurable quantities, will have a geometry as the mathematical model, e.g.: the Euclid space R_3 which in some cases, under extra assumptions, may be reduced to the planimetry of R_2 or perhaps the geometry of the spherical surface or ellipsoid. In such a geometry, functions of the angles and length ratios can be computed. A special set of functions are the coordinates of points which must be computed in an S-system.

From the preceding section it is clear that the base of an S-system consists of non estimable parameters; here we can state more exactly:

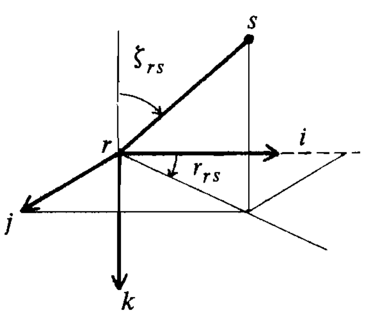
The base parameters of an S-system are the non-estimable variates x^u, which have to be introduced during the computing process of the variates y^r and which cannot be expressed as a function of the variates introduced in an earlier stage of the process.

3.3 Examples

3.3.1 Measurable quantities in terrestrial geodesy

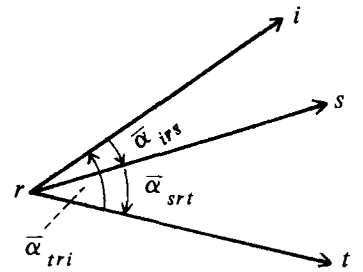
As preparation for the following examples we define the measurable quantities in terrestrial geodesy. For the relative positioning of points on or near the earth's surface the surveyor can make use of theodolite and distance measurements. For these, the following considerations are valid:

– Theodolite measurements:



A right-handed orthonormal i.j.k.-system is connected to the theodolite. The origin is the intersection of the horizontal and vertical axes of the theodolite. The k-axis is pointing downwards parallel to the vertical axis, the i-axis points orthogonal to the vertical axis in the zero direction of the horizontal circle. The j-axis completes the system ([2], [40]).

Measurement from point r to point s gives r_{rs} and ζ_{rs} as indicated in the figure. When taking up position in r for the first time, the surveyor observes points s, t and i :



$$
\begin{array}{ll}
r_{rs}^{(1)} & \zeta_{rs}^{(1)} \\
r_{rt}^{(1)} & \zeta_{rt}^{(1)} \\
r_{ri}^{(1)} & \zeta_{ri}^{(1)}
\end{array}
$$

Index (1) indicates this first measurement.

These observations give:

$$
\begin{aligned}
\cos \bar{\alpha}_{srt}^{(1)} &= \cos (r_{rt}^{(1)} - r_{rs}^{(1)}) \sin \zeta_{rt}^{(1)} \sin \zeta_{rs}^{(1)} + \cos \zeta_{rt}^{(1)} \cos \zeta_{rs}^{(1)} \\
\cos \bar{\alpha}_{tri}^{(1)} &= \cos (r_{ri}^{(1)} - r_{rt}^{(1)}) \sin \zeta_{ri}^{(1)} \sin \zeta_{rt}^{(1)} + \cos \zeta_{ri}^{(1)} \cos \zeta_{rt}^{(1)} \\
\cos \bar{\alpha}_{irs}^{(1)} &= \cos (r_{rs}^{(1)} - r_{ri}^{(1)}) \sin \zeta_{rs}^{(1)} \sin \zeta_{ri}^{(1)} + \cos \zeta_{rs}^{(1)} \cos \zeta_{ri}^{(1)}
\end{aligned}
\qquad (3.12)
$$

If the theodolite has been set up in r for a second measurement, the new i, j, k–axes in general will be rotated with respect to those of the first measurement. The new observations are:

$$
\begin{array}{llcl}
r_{rs}^{(2)} & \zeta_{rs}^{(2)} & & \bar{\alpha}_{srt}^{(2)} \\
r_{rt}^{(2)} & \zeta_{rt}^{(2)} & \text{computation according to (3.12) gives} & \bar{\alpha}_{tri}^{(2)} \\
r_{ri}^{(2)} & \zeta_{ri}^{(2)} & & \bar{\alpha}_{irs}^{(2)}
\end{array}
$$

Because of the difference in i, j, k–system for first and second measurement we find for the observed bearings:

$$
E\{\underline{r}_{ri}^{(1)}\} \neq E\{\underline{r}_{ri}^{(2)}\} \quad \text{and} \quad E\{\underline{\zeta}_{ri}^{(1)}\} \neq E\{\underline{\zeta}_{ri}^{(2)}\} \ \text{etc.} \qquad (3.13.1)
$$

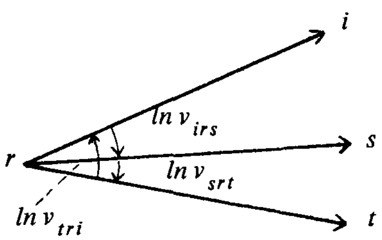
but for the angles:

$$
E\{\underline{\bar{\alpha}}_{sri}^{(1)}\} = E\{\underline{\bar{\alpha}}_{sri}^{(2)}\} \ \text{etc.} \qquad (3.13.2)
$$

(3.13.2) shows that the angles $\bar{\alpha}$ are measurable quantities in the sense of Def. 3.I.1, whereas according to (3.13.1) this is not true for the bearings.

– *Distance measurements:*

An instrument (1) at point r gives for the distance measurements to points s, t and i :



$$
\begin{array}{lcl}
l_{rs}^{(1)} & & \ln l_{rs}^{(1)} \\
l_{rt}^{(1)} & \text{or} & \ln l_{rt}^{(1)} \\
l_{ri}^{(1)} & & \ln l_{ri}^{(1)}
\end{array}
$$

index (1) indicates instrument (1)

From these we obtain:

$$
\begin{aligned}
ln\, v^{(1)}_{srt} &= ln\, l^{(1)}_{rt} - ln\, l^{(1)}_{rs} \\
ln\, v^{(1)}_{tri} &= ln\, l^{(1)}_{ri} - ln\, l^{(1)}_{rt} \\
ln\, v^{(1)}_{irs} &= ln\, l^{(1)}_{rs} - ln\, l^{(1)}_{ri} = ln\, v^{(1)}_{tri} - ln\, v^{(1)}_{srt}
\end{aligned}
\tag{3.14}
$$

An instrument (2) will give:

$ln\, l^{(2)}_{rs}$		$ln\, v^{(2)}_{srt}$
$ln\, l^{(2)}_{rt}$	and according to (3.14)	$ln\, v^{(2)}_{tri}$
$ln\, l^{(2)}_{ri}$		$ln\, v^{(2)}_{irs}$

Different instruments may have different length scale-factors. That means for the observed distances:

$$E\{\, ln\, l^{(1)}_{ri} \,\} \neq E\{\, ln\, l^{(2)}_{ri} \,\} \quad \text{etc.} \tag{3.15.1}$$

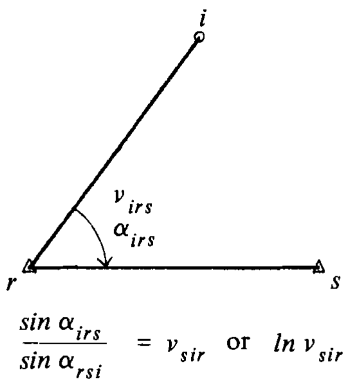
but for the ratios of length:

$$E\{\, ln\, v^{(1)}_{irs} \,\} = E\{\, ln\, v^{(2)}_{irs} \,\} \quad \text{etc.} \tag{3.15.2}$$

From (3.15.2) it follows that length ratios are measurable quantities in the sense of Def. 3.I.1, whereas observed distances are not. So the geodesist should use the angles $\bar{\alpha}$ and the ratios of length v as x^i-variates, when introducing S-systems. Transformation of these S-systems should leave these variates invariant, hence only similarity transformations are allowed.

3.3.2 The S-base in planimetry

In the first example we suppose that all points are in one plane, so the mathematical model involves plane geometry. Then, if the plane is horizontal, the theodolite measurements can be simplified if the k-axis of the theodolite is taken perpendicular to the plane. Then



$$\alpha_{jik} = r_{ik} - r_{ij}\,.$$

The triangle is a basic planimetric configuration. The measurable elements of a triangle are: three angles and three ratios of length. Applying the sine and cosine rules to these elements gives:

$$\frac{sin\, \alpha_{irs}}{sin\, \alpha_{rsi}} = v_{sir} \quad \text{or} \quad ln\, v_{sir} = ln\, (sin\, \alpha_{irs}) - ln\, (sin\, \alpha_{rsi})$$

$$cos\, \alpha_{irs} = \tfrac{1}{2}\, [\, v_{rsi}\, v_{ris} - v_{irs} - v_{sri}\,]$$

Other angles and length ratios can be computed from the measurements without introduction of extra parameters, so these are estimable quantities in planimetry.

For the computation of the coordinates of the points in a network, the algebra of complex numbers will be used to demonstrate the structure of these computations clearly. In this algebra a point i is denoted by: $z_i = x_i + i \cdot y_i$ (see [3, 4b, 6]). Let point r be the first point:

$$z_r = x_r + i \cdot y_r$$

No observations are available for x_r and y_r, they are the first non-estimable quantities in this computation, so they have to be part of the S-base. Hence we must use :

$$z_r^o = x_r^o + i \cdot y_r^o$$

The second point follows from (see [4.b] (3.7.b)) :

$$z_s = z_r^o + e^{\ln s_{rs} + iA_{rs}}$$

s_{rs} is the length of side r, s and A_{rs} is the bearing. These quantities are not estimable, nor can they be computed from parameters or other variates introduced sofar. So according to section 3.2 they have to be part of the S-base. The computation will make use of:

$$z_s^o = z_r^o + e^{\ln s_{rs}^o + iA_{rs}^o}$$

A third point i follows from (see [4.b] (4.1)) :

$$z_i = z_r^o + e^{\ln v_{sri} + i\,\alpha_{sri}} \cdot (z_s^o - z_r^o)$$

v_{sri} and α_{sri} are measurable quantities, so no further extension of the S-base is needed. The complete S-base is:

$$z_r^o \ , \ z_s^o$$

Computation of other points gives: $z_i^{(rs)}$, the indices (r, s) indicate the S-base. The meaning in terms of chapter II is:

$$\text{(see (2.4))} \qquad \{\ldots . x_o^u \ldots ..\} \longrightarrow \{ z_r^o \ , \ z_s^o \} \tag{3.16}$$

$$\text{(2.4.a)} \qquad \begin{pmatrix} \tilde{z}_r^{(rs)} \\ \tilde{z}_s^{(rs)} \\ \tilde{z}_i^{(rs)} \end{pmatrix} = \begin{pmatrix} z_r^o \\ z_s^o \\ z_r^o + e^{\ln \tilde{v}_{sri} + i\tilde{\alpha}_{sri}} \cdot (z_s^o - z_r^o) \ \underline{\underline{\text{say}}} \ z_r^o + e^{\tilde{\pi}_{sri}} \cdot z_{rs}^o \end{pmatrix} \tag{3.16'}$$

and

$$\text{(2.6.a)} \qquad \begin{pmatrix} \Delta\tilde{z}_r^{(rs)} \\ \Delta\tilde{z}_s^{(rs)} \\ \Delta\tilde{z}_i^{(rs)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z_{ri} \cdot (\frac{\tilde{\Delta} v_{sri}}{v_{sri}} + i\,\Delta\,\tilde{\alpha}_{sri}) = z_{ri} \cdot \Delta\tilde{\pi}_{sri} \end{pmatrix}$$

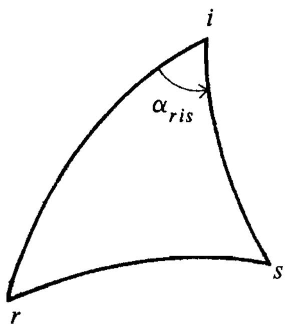
and

$$
(2.4'.a) \qquad \begin{pmatrix} z_r^o \\ z_s^o \\ \tilde{\pi}_{sri} \end{pmatrix} = \begin{pmatrix} \tilde{z}_r^{(rs)} \\ \tilde{z}_s^{(rs)} \\ \ln\left(\dfrac{\tilde{z}_{ri}^{(rs)}}{\tilde{z}_{rs}^{(rs)}}\right) \end{pmatrix}
$$

An extensive treatment of the S–transformation in planimetry has been given by Baarda in [6] .

3.3.3 The S–base for spherical triangulation

For the second example the points involved are supposed to be on a spherical surface. The length of the radius of the sphere is not known. Theodolite measurements will get a simplified interpretation by the assumption that the k-axis always points towards the centre of the sphere. Measurement in point i gives the angle between large circles connecting i respectively with r and s.



This is:

$$\alpha_{ris} = r_{is} - r_{ir}$$

Here again the triangle is the basic configuration. Suppose the three angles of triangle, r, s, i, have been observed:

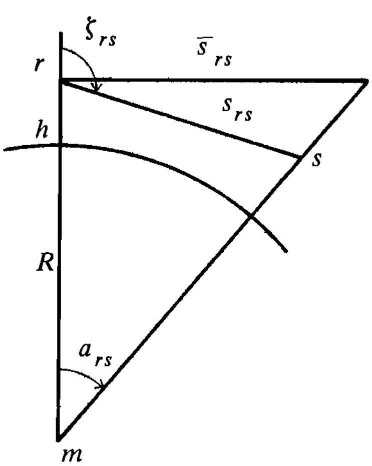
$$\alpha_{irs} , \ \alpha_{rsi} , \ \alpha_{sir}$$

then by means of the cosine rule for spherical triangles we find:

$$\cos a_{rs} = \frac{\cos \alpha_{sir} + \cos \alpha_{rsi} \cdot \cos \alpha_{irs}}{\sin \alpha_{rsi} \cdot \sin \alpha_{irs}}$$

a_{rs} is the angle between the radius to r and the radius to s . The cosine rule shows that a_{rs}, a_{si} and a_{ir} are estimable.

Distance measurements over the sphere can be interpreted as follows :



Suppose an instrument has been set up at station r and the distance is observed to s, with zenith angle ζ_{rs} which is supposed to be corrected for refraction. Now distance measurements must be linked up with angular measurements. Angles are measured in a plane orthogonal to the vertical at r , therefore distances will be projected to such a plane as well:

$$\bar{s}_{rs} = s_{rs} (\sin \zeta_{rs} - \cos \zeta_{rs} \ tg \ a_{rs})$$

but we find also:

$$\bar{s}_{rs} = \lambda (R + h) \ tg \ a_{rs}$$

where R is the radius of the sphere, h the height of r with respect to the sphere and λ is a scale factor relating the length unit of the instrument to R.

These relationships lead to:

$$\bar{s}_{rs} = s_{rs}\,(\sin \zeta_{rs} - \cos \zeta_{rs}\, tg\, a_{rs}) = \lambda\,(R + h)\; tg\, a_{rs}$$

and for the side $r.i$ we get:

$$\bar{s}_{ri} = s_{ri}\,(\sin \zeta_{ri} - \cos \zeta_{ri}\; tg\, a_{ri}) = \lambda\;(R + h)\; tg\, a_{rs}$$

elimination of the factor $\lambda\,(R + h)$ gives:

$$\begin{aligned} \ln \bar{v}_{sri} &= \ln \bar{s}_{ri} - \ln \bar{s}_{rs} \\ &= \ln (s_{ri}\,(\sin \zeta_{ri} - \cos \zeta_{ri}\; tg\, a_{ri})\,) - \ln (s_{rs}(\sin \zeta_{rs} - \cos \zeta_{rs}\; tg\, a_{rs})\,) \\ &= \ln (tg\, a_{ri}) - \ln (tg\, a_{rs}) \end{aligned}$$

Rewriting this expression we get:

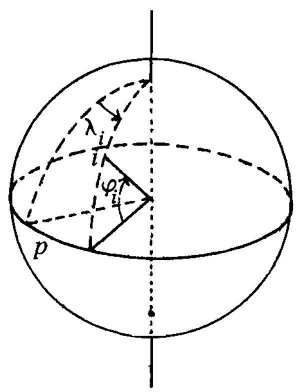
$$\begin{aligned} \ln v_{sri} &= (\ln (tg\, a_{ri}) - \ln (\sin \zeta_{ri} - \cos \zeta_{ri} tg\, a_{ri})\,) \\ &\quad - (\ln (tg\, a_{rs}) - \ln (\sin \zeta_{rs} - \cos \zeta_{rs}\; tg\, a_{rs})\,) \end{aligned}$$

where $\qquad \ln v_{sri} = \ln s_{ri} - \ln s_{rs}$

If the vertical axis of the theodolite is always pointing towards the centre of the sphere, then the zenith angles, ζ, are measurable. In that case, the given relationship links up the distance measurements with the angular measurements.

In these formulae the length-scale factor λ cannot be separated from the factor $(R + h)$. If we write $R + h = R\,.\,(1 + \frac{h}{R})$, then we see that the effect of the assumption that λ is unknown is similar to the effect of the assumption that R is unknown.

The preceding considerations lead to the conclusion that for spherical triangulation the estimable quantities are the central angles a_{rs} etc. and the measurable quantities α_{irs} and v_{irs} etc. Computation of spherical coordinates requires the choice of an equatorial plane. The angle between radius R_i and this plane will be denoted by φ_i. On the large circle (equator) which is the intersection of the sphere and the equatorial plane, a point p must be chosen which gets the coordinates $(\varphi_p\,,\,\lambda_p) = (0,\,0)$. The angle between R_p and the orthogonal projection of R_i on the equatorial plane will be denoted by λ_i, the second coordinate of i. φ and λ are counted as indicated in the figure. The coordinate computation will start from point r. The coordinates of this point do not belong to the set of estimable quantities nor can they be computed from earlier introduced variates, therefore they belong to the S–base. Hence further computations make use of: $\varphi_r^o\,,\;\lambda_r^o$.



Point s can now be obtained from:

$$\lambda_s = \lambda_r + \lambda_{rs} = \lambda_r + \operatorname{arc\,cot} \left\{ \frac{\cot a_{rs}\,.\,\cos \varphi_r - \sin \varphi_r\,.\,\cos A_{rs}}{\sin A_{rs}} \right\}$$

$$\varphi_s = \operatorname{arc\,cos} \left\{ \frac{\sin A_{rs}\,.\,\sin a_{rs}}{\sin \lambda_{rs}} \right\}$$

A_{rs} is the azimuth of side r, s, the angle between the meridian of r and the side r, s. This angle is not estimable, nor can it be computed from variates introduced before. Therefore A_{rs} is an element of the S-base. So point s is given by:

$$\tilde{\lambda}_s^{(\;)} = \lambda_r^o + arc\ cot \left\{ \frac{cot\ \tilde{a}_{rs} \,.\, cos\ \varphi_r^o - sin\ \varphi_r^o \,.\, cos\ A_{rs}^o}{sin\ A_{rs}^o} \right\}$$

$$\tilde{\varphi}_s^{(\;)} = arc\ cos \left\{ \frac{sin\ A_{rs}^o\ sin\ \tilde{a}_{rs}}{sin\ \tilde{\lambda}_{rs}^{(\;)}} \right\}$$

As $A_{ri} = A_{rs} + \alpha_{sri}$, point i will be:

$$\tilde{\lambda}_i^{(\;)} = \lambda_r^o + arc\ cot \left\{ \frac{cot\ \tilde{a}_{ri} \,.\, cos\ \varphi_i^o - sin\ \varphi_r^o \,.\, cos\,(A_{rs}^o + \tilde{\alpha}_{sri})}{sin\,(A_{rs}^o + \tilde{\alpha}_{sri})} \right\}$$

$$\tilde{\varphi}_i^{(\;)} = arc\ cos \left\{ \frac{sin\,(A_{rs}^o + \tilde{\alpha}_{sri})\ sin\ \tilde{a}_{ri}}{sin\ \tilde{\lambda}_{ri}^{(\;)}} \right\}$$

No further extension is needed for the S-base, thus we find for spherical triangulation:

S-base : φ_r^o , λ_r^o $and\ A_{rs}^o$

Computation for point i gives: $(\varphi_i^{(r;s)}, \lambda_i^{(r;s)})$. The upper indices indicate the points of the S-base, and they have been separated by a semi-colon, because r and s have a different meaning for this base. In terms of chapter II we find for spherical coordinates :

see (2.4.a) $\{\ldots x_o^u \ldots\} \rightarrow \{\lambda_r^o\ , \varphi_r^o,\ A_{rs}^o\}$ (3.17)

$$\begin{pmatrix} \tilde{\lambda}_r^{(r;s)} \\ \tilde{\varphi}_r^{(r;s)} \\ \tilde{\lambda}_s^{(r;s)} \\ \tilde{\varphi}_s^{(r;s)} \\ \tilde{\lambda}_i^{(r;s)} \\ \tilde{\varphi}_i^{(r;s)} \end{pmatrix} = \begin{pmatrix} \lambda_r^o \\ \varphi_r^o \\ \lambda_r^o + \tilde{\lambda}_{rs}^{(r;s)} = \lambda_r^o + arc\ cot \left\{ \dfrac{cot\ \tilde{a}_{rs} \,.\, cos\ \varphi_r^o - sin\ \varphi_r^o \,.\, cos\ A_{rs}^o}{sin\ A_{rs}^o} \right\} \\ arc\ cos \left\{ \dfrac{sin\ A_{rs}^o \,.\, sin\ \tilde{a}_{rs}}{sin\ \tilde{\lambda}_{rs}^{(r;s)}} \right\} \\ \lambda_r^o + arc\ cot \left\{ \dfrac{cot\ \tilde{a}_{ri} \,.\, cos\ \varphi_r^o - sin\ \varphi_r^o \,.\, cos\,(A_{rs}^o + \tilde{\alpha}_{sri})}{sin\,(A_{rs}^o + \tilde{\alpha}_{sri})} \right\} \\ arc\ cos \left\{ \dfrac{sin\,(A_{rs}^o + \tilde{\alpha}_{sri}) \,.\, sin\ \tilde{a}_{ri}}{sin\ \tilde{\lambda}_{ri}^{(r;s)}} \right\} \end{pmatrix} \quad (3.17')$$

Linearization with respect to measurable quantities will give relationships like (2.6.a).

The approach of this section makes apparent some serious setbacks in the spherical computational model. The root of these lies in the way the spherical distance a_{rs} is estimated. In textbooks on geodesy the computation of spherical coordinates is based on the assumption

that a good estimate is available for this distance. This is obtained as follows:
The length of the base in a triangulation network is measured with very high accuracy. From this base the length of a side of the network can be derived. With modern equipment the length of such a side can be measured directly. The ratio of the computed or measured side and the radius of the sphere then gives the spherical length of the side.

The most critical assumption in this method is that distance measurement will be performed using the same length unit as the one in which the radius of the sphere has been expressed, hence no unknown length-scale factor appears in the computations. In this chapter, that assumption has not been made, and a length-scale factor appears in the expression relating s_{rs}, R and a_{rs}. Elimination of the length-scale factor leads to the expression for $ln\ v_{sri}$, linking up distance and angular measurements. From that expression it becomes clear that distance measurements do not lead to an estimator for a_{rs}. Then, only the cosine rule for spherical triangles gives such an estimator. In practice, however, that estimator will have very poor precision. A triangle with three sides of 100 km and three angles of 66.6691 grades gives for the linearized relationship given by the cosine rule:

$$\sin a_{rs} \,.\, \Delta \underline{a}_{rs} = \frac{\sin \alpha_{ris}}{\sin \alpha_{sri} \sin \alpha_{isr}} \Delta \underline{\alpha}_{ris} + (\cot \alpha_{isr} + \cos a_{rs} \cot \alpha_{sri})\, \Delta \underline{\alpha}_{sri} + $$
$$+ (\cot \alpha_{sri} + \cos a_{rs} \cot \alpha_{isr})\, \Delta \underline{\alpha}_{isr}$$

which is in this case:

$$0.0156 \,.\, \Delta \underline{a}_{rs} = 1.1545\ \Delta \underline{\alpha}_{ris} + 1.1545\ \Delta \underline{\alpha}_{sri} + 1.1545\ \Delta \underline{\alpha}_{isr}$$

For the case where the angles α have the same precision and if they are not correlated, we find :

$$\sigma_{a_{rs}} = 130 \,.\, \sigma_\alpha$$

if $\sigma_\alpha = 10^{-4}$ gr, then for $l_{rs} = R \,.\, a_{rs}$ we get :

$$\sigma_{l_{rs}} \approx 6400 \,.\, \sigma_{a_{rs}} = 1.3\ km$$

This example demonstrates that for the case where there is no better method available to estimate a_{rs}, the computation of spherical coordinates will lack precision. One should be very careful in using spherical geometry as the mathematical model for the adjustment of triangulation networks. It seems to be more appropriate to apply a three-dimensional model as Quee does [40]. Yet the spherical model is not without meaning as will be demonstrated in chapter V, where a criterion variance-covariance matrix is to be developed for extensive three-dimensional pointfields.

3.4 Epilogue to chapter III

The indices in (2.1) and (2.4) run over the values :

$$i = 1, \ldots\ldots, n \qquad u = n+1, \ldots\ldots, n+b \qquad r = 1, \ldots\ldots, n+b$$

In the first example, section 3.3.2, the shape of the planimetric triangle is defined by two form elements : v_{sri} and α_{sri}. So,in that case the index i is $i = 1,2$. Four base parameters were needed therefore : $u = 3, 4, 5, 6$. Three complex, that is six real functional relationships have been formulated in (3.16′), thus r is then $r = 1, \ldots.., 6$. Each new point added to (3.16′) requires two more form elements whereas the system (3.16′) will be extended by two more real relationships.

In the second example, the definition of the shape of the triangle required three form elements, hence in that case $i = 1, 2, 3$, whereas for computation of coordinates three base parameters were necessary : $u = 4, 5, 6$. In (3.17′) six functional relationships were formulated : $r = 1, \ldots, 6$. Each new point added to the system requires two more form elements, thus the set of relationships will be extended by two.

In planimetry the S–base consists of the four coordinates of two points. This allows a simple reduction of the S-matrix to make it non-singular, Baarda did so in [6]. The second example shows a different situation, there of course a reduction is also possible, but it doesn't have such a simple interpretation. A similar situation occurs in R_3 (see chapter IV); therefore we prefer to consider the S–matrix as singular.

It would be interesting to apply the line of thought of this chapter to the ellipsoidal model in geodesy. This application, however, is complicated by the fact that the formulae of ellipsoidic triangulation are not of a closed character (see [20]),and the fact that interpretation of measurable quantities in terms of form elements on the ellipsoid is somewhat problematic. One method to solve this problem uses provisional computations on an osculating sphere, from which the results are transformed to ellipsoidal coordinates. One should be very careful when applying such a transformation, because of the fact that according to section 3.3.3 one is free to adopt any (φ , λ)-system on the sphere, whereas on an ellipsoid the coordinate system should be related to the rotation axis; therefore the definition of the latter system will have less degrees of freedom. Not much attention has been paid to this problem in literature as yet, and textbooks on geodesy often assume a unique relationship between the two systems.

To find out whether a function of the observations is estimable, one should check whether it agrees with Def. 3.I.2 or not. Some authors on this subject check the estimability of functions by investigating the structure of the coefficient matrix in the Gauss–Markov model for estimation of parameters [11, 24, 42]. This model is known in geodesy as "Standard Problem II", or the adjustment based on observation-equations. If the coefficient matrix in this model leads to a singular matrix for the normal equations the parameters are said to be non-estimable. This approach is dangerous because singularity can be caused by different reasons. One reason can be that a coordinate system has not been defined properly. In that case it is correct to conclude that the parameters in the observation equations are non-estimable because there is no method to avoid this singularity. Another reason for the singularity can be a wrong design of the geodetic network, as in the case where one tries to solve a resection using points on the danger circle. In this case it would be wrong to call the parameters in the model non-estimable, because the singularity can be avoided by a better net structure. So, it is not sufficient to check the matrix of the normal equations for singularity, one should also check why this singularity occurs in order to avoid wrong conclusions.

It is possible, as for the spherical distance, that no good estimates can be obtained for some estimable quantities whatever the structure of the network is. Then the linking up of mathematical model and measuring procedure cannot be carried out properly. It may be that for practical reasons one tries to get away with this problem by means of a modification of the model, as in geodesy, where measured distances are assumed to be observed in the same length unit as the radius of the sphere. One should be careful, however, when applying such a modification, as it will lead to a conflict between the observations and the computing model if it has not been done correctly. In geodesy, for instance, the radius of the sphere can be chosen too large or too small. The effect is not necessarily felt at once, but it may show up when one wants to extend an existing network, or when one tries to connect a network to neighbouring ones.

A similar reasoning applies to the ellipsoidal model in geodesy, which is considered to be a refinement of the spherical model. Geodesists have experienced these problems indeed; therefore, different ellipsoids have been used, for which the choice of the parameters mainly depends on the geographical position of the network to be computed and the area to be covered by it. It is very likely that a further analysis of problems met in geometric geodesy will finally lead to the principal choice of three-dimensional Euclidean geometry as a computational model for triangulation purposes.

CHAPTER IV S–TRANSFORMATIONS IN THREE-DIMENSIONAL EUCLIDIAN SPACE

4.1 Some principles of quaternion algebra

In the three-dimensional euclidean space R_3 the measurable quantities for the terrestrial geodesist are, according to section 3.3.1, angles and length ratios. These are to be used as x^i-variates in the coordinate computations. The coordinates will be computed in a righthanded system, for which the S–base has to be defined.

To demonstrate in which way that definition follows from the method of coordinate computation, we should use an algebra which clearly shows the structure of these computations and which also describes the measuring procedure very well. Baarda based his choice for quaternion-algebra [2,6] on these criteria, Vermaat and Quee worked out his ideas in [40] and [43]. In this chapter we shall follow this choice.

A point i with coordinates

$$x_i\,,\ y_i\,,\ z_i$$

will be described by a quaternion:

$$q_i = 0 + x_i.i + y_i.j + z_i.k \tag{4.1}$$

with product rules :

$$\begin{aligned} i.i &= j.j = k.k = -1 \\ i.j &= -j.i = k \\ j.k &= -k.j = i \\ k.i &= -i.k = j \end{aligned}$$

Point j is:

$$q_j = 0 + x_j.i + y_j.j + z_j.k$$

Substraction gives:

$$q_{ij} = q_j - q_i = 0 + (x_j - x_i).i + (y_j - y_i).j + (z_j - z_i).k$$

For q_{ij} the conjugate quaternion is :

$$q_{ij}^T = 0 - (x_j - x_i).i - (y_j - y_i).j - (z_j - z_i).k$$

and the norm :

$$\begin{aligned} N\{q_{ij}\} &= q_{ij}\,.\,q_{ij}^T = q_{ij}^T\,.\,q_{ij} = l_{ij}^2 \\ &= (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2 \end{aligned}$$

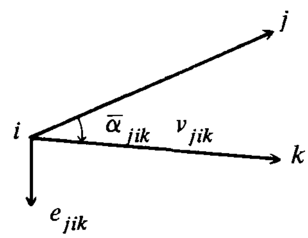
The inverse of q_{ij} with respect to multiplication is :

$$q_{ij}^{-1} = \frac{q_{ij}^T}{N\{q_{ij}\}}$$

where :

$$q_{ij}\,.\,q_{ij}^{-1} = q_{ij}^{-1}\,.\,q_{ij} = 1$$

Add point k with q_k and $q_{ik} = q_k - q_i$. The left division by q_{ij} is then:



$$Q_{jik} = q_{ik} \cdot q_{ij}^{-1} \tag{4.2.1}$$

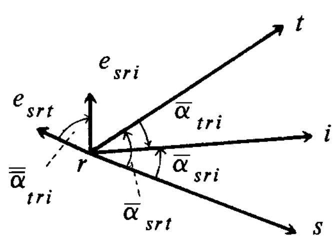
which can also be written as :

$$Q_{jik} = v_{jik} \cdot (\cos\bar{\alpha}_{jik} + e_{jik} \cdot \sin\bar{\alpha}_{jik}) \tag{4.2.2}$$

v_{jik} is the length ratio of side i, j and side i, k and $\bar{\alpha}_{jik}$ is the angle between these sides, whereas e_{jik} is the unit vector in the direction normal to plane j, i, k. This expression demonstrates in which way the measurable quantities can be entered in the coordinate computation when using quaternion algebra.

If Q_{jik} has been computed from observations, then the components of e_{jik} are given with respect to the i, j, k-system of the theodolite (see section 3.3.1), whereas if Q_{jik} has been computed from coordinates, then the components of e_{jik} are given with respect to coordinate axes. The transformation from the one system to the other can be made by means of an S-base.

4.2 The S-base in R_3



Points r, s, t, i are not all in one plane. According to section 3.3.1 the measurable quantities are the angles $\bar{\alpha}_{srt}$, $\bar{\alpha}_{sri}$ and $\bar{\alpha}_{tri}$ and the ratios of lengths $\ln v_{srt}$, $\ln v_{sri}$ and $\ln v_{tri}$.

By means of the sine and the cosine rules other angles and length ratios can be found, without using extra non-estimable parameters. The angle between the normal vectors of different planes e.g.: e_{sri} and e_{srt} etc., follows from

$$\cos\bar{\bar{\alpha}}_{tri} = \frac{\cos\bar{\alpha}_{irt} - \cos\bar{\alpha}_{srt} \cdot \cos\bar{\alpha}_{irs}}{\sin\bar{\alpha}_{srt} \cdot \sin\bar{\alpha}_{irs}} \tag{4.3}$$

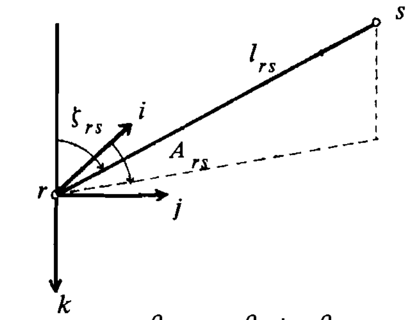
Thus the angles $\bar{\bar{\alpha}}$ are estimable too.

Computation of coordinates will start from point r. The coordinates of this point are not estimable, nor can they be computed from other variates. Therefore these coordinates belong to the S-base. For further computations we have :

$$q_r^o = 0 + x_r^o \cdot i + y_r^o \cdot j + z_r^o \cdot k \tag{4.4.1}$$

Point s can be obtained by :

$$\begin{aligned} q_s &= q_r + q_{rs} \\ &= q_r + l_{rs} \cdot (i \cdot \cos A_{rs} \cdot \sin\zeta_{rs} + j \cdot \sin A_{rs} \cdot \sin\zeta_{rs} + k \cdot \cos\zeta_{rs}) \end{aligned}$$

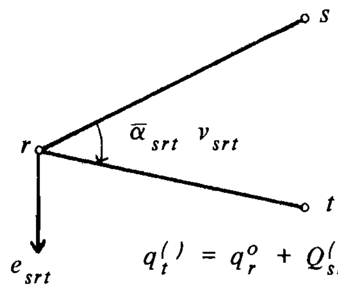


l_{rs} is the length of side r, s, A_{rs} is the angle, measured clockwise, from the i-axis in the i, j-plane and ζ_{rs} is the angle measured from the negative k-axis (see [40]). These quantities do not belong to the set of estimable quantities mentioned before, nor can they be computed from variates introduced earlier. Therefore they belong to the S–base. Thus computations make use of :

$$\begin{aligned} q_s^o &= q_r^o + q_{rs}^o \\ &= q_r^o + l_{rs}^o . (i.\cos A_{rs}^o . \sin\zeta_{rs}^o + j.\sin A_{rs}^o . \sin\zeta_{rs}^o + k.\cos\zeta_{rs}^o) \\ &= q_r^o + l_{rs}^o . e_{rs}^o \end{aligned} \tag{4.4.2}$$

Point t is now :

$$\begin{aligned} q_t &= q_r + q_{rt} = q_r + Q_{srt} . q_{rs} \\ &= q_r + v_{srt} . (\cos\bar{\alpha}_{srt} + e_{srt} . \sin\bar{\alpha}_{srt}) . q_{rs} \end{aligned}$$



e_{srt} is the unit vector normal to plane s, r, t and thus perpendicular to side r, s, so, only one directional component is undetermined. This component is not estimable and also not expressable as a function of variates introduced as yet. So it belongs to the S–base, therefore, we use e_{srt}^o. Then point t is :

$$\begin{aligned} q_t^{(\,)} &= q_r^o + Q_{srt}^{(\,)} . q_{rs}^o \\ &= q_r^o + v_{srt} . (\cos\bar{\alpha}_{srt} + e_{srt}^o . \sin\bar{\alpha}_{srt}) . q_{rs}^o \end{aligned} \tag{4.4.3}$$

$(\,)$ is indicing reserved to indicate the S–base. The S–base has seven elements so far, and we might expect that no further extension is necessary in R_3. To verify this, point i will be computed :

$$\begin{aligned} q_i^{(\,)} &= q_r^o + Q_{sri}^{(\,)} . q_{rs}^o \\ &= q_r^o + v_{sri} . (\cos\bar{\alpha}_{sri} + e_{sri}^{(\,)} . \sin\bar{\alpha}_{sri}) . q_{rs}^o \end{aligned} \tag{4.4.4}$$

e_{sri} is perpendicular to q_{rs}, which means that a rotation on q_{rs} has to be determined yet. This rotation is given by the angle between e_{sri} and e_{srt}, that is $\bar{\alpha}_{tri}$, which is estimable according to (4.3). So the S–base is complete. Then according to (4.4.1–3) :

$$\boxed{\text{S–base in } R_3 : \; q_r^o, \; q_s^o, \; e_{srt}^o} \tag{4.5}$$

Quantities computed with respect to this base will be denoted by :

$$q_i^{(r,s;t)}$$

The indices have been separated by a semi-colon because the role of t in this base is different from the role of r and s. The latter two have a similar role, therefore:

$$q_i^{(r,s;t)} = q_i^{(s,r;t)}$$

4.3 K-transformations in R_3

According to section 2.5, several sets of values can be introduced for the elements of the S-base. These sets define different K-systems, which are related by K-transformations (2.29). Values for the S-base :

$$q^o_{(1)_r}\,,\; q^o_{(1)_s}\,,\; e^o_{(1)_{srt}} \tag{4.6.1}$$

lead to (see (2.24) and (4.4.1–4)):

$$\left.\begin{aligned}
\tilde{q}^{(r,s;t)}_{(1)_r} &= q^o_{(1)_r}\\
\tilde{q}^{(r,s;t)}_{(1)_s} &= q^o_{(1)_s}\\
\tilde{q}^{(r,s;t)}_{(1)_t} &= q^o_{(1)_r} + \tilde{Q}^{(r,s;t)}_{(1)_{rst}} \cdot q^o_{(1)_{rs}}\\
\tilde{q}^{(r,s;t)}_{(1)_i} &= q^o_{(1)_r} + \tilde{Q}^{(r,s;t)}_{(1)_{sri}} \cdot q^o_{(1)_{rs}}
\end{aligned}\right| \tag{4.6.2}$$

Another set of values :

$$q^o_{(2)_r}\,,\; q^o_{(2)_s}\,,\; e^o_{(2)_{srt}} \tag{4.6.3}$$

leads to:

$$\left.\begin{aligned}
\tilde{q}^{(r,s;t)}_{(2)_r} &= q^o_{(2)_r}\\
\tilde{q}^{(r,s;t)}_{(2)_s} &= q^o_{(2)_s}\\
\tilde{q}^{(r,s;t)}_{(2)_t} &= q^o_{(2)_r} + \tilde{Q}^{(r,s;t)}_{(2)_{rst}} \cdot q^o_{(2)_{rs}}\\
\tilde{q}^{(r,s;t)}_{(2)_i} &= q^o_{(2)_r} + \tilde{Q}^{(r,s;t)}_{(2)_{rsi}} \cdot q^o_{(2)_{rs}}
\end{aligned}\right| \tag{4.6.4}$$

The transformation which relates these two systems should leave the measurable angles and length ratios invariant, so it must be a similarity transformation. In quaternion-algebra, this may be expressed as (see [14, 40, 43]):

$$q_{(2)_i} = \lambda_{21} \cdot p_{21} \cdot q_{(1)_i} \cdot p_{12} + d_{21} \tag{4.7}$$

In the $(r,s;t)$–system the transformation is (see section 2.5.2):

$$\boxed{\tilde{q}^{(r,s;t)}_{(2)_i} = \lambda^o_{21} \cdot p^o_{21} \cdot \tilde{q}^{(r,s;t)}_{(1)_i} \cdot p^o_{12} + d^o_{12}} \tag{4.7'}$$

The parameters in this transformation are:

λ_{21} = change of length scale

p_{21} = p^{-1}_{12} = rotation quaternion, three rotation elements

d_{21} = shift, three elements

The seven elements have to be computed from the values of the parameters of the S–base in (1) - and (2) - system. Therefore the upper index, ^{o}, has been used in (4.7′). The computation goes as follows: the length scale in system (1) is determined by the length of $q^o_{(1)_{rs}}$ and in system (2) by $q^o_{(2)_{rs}}$, so the change of scale is :

$$\lambda^o_{21} = \left(\frac{N\{q^o_{(2)_{rs}}\}}{N\{q^o_{(1)_{rs}}\}} \right)^{\frac{1}{2}} \tag{4.8}$$

The unit vectors (see (4.4.3)):

$$e^o_{(1)_{rs}},\ e^o_{(2)_{rs}} \text{ and } e^o_{(1)_{srt}},\ e^o_{(2)_{srt}}$$

give the key to the solution of the rotation elements. The rotation will be solved in two steps:

$$p^o_{12} = p^o_{11'} \cdot p^o_{1'2} \tag{4.9.1}$$

$p^o_{11'}$ transforms $e^o_{(1)_{rs}}$ into $e^o_{(2)_{rs}}$, that is a rotation on the vector normal to $e^o_{(1)_{rs}}$ and $e^o_{(2)_{rs}}$, over the angle $\tilde{\alpha}$ between these two. Here we consider these vectors as having components with respect to the same *i, j, k*–system. Thus the transformation is not interpreted as a base transformation, but as a point transformation. This leads to:

$$e^o_{(2)_{rs}} \cdot e^{o\,-1}_{(1)_{rs}} = \cos\bar{\alpha}^o_{11'} + e^o_{11'} \,.\, \sin\bar{\alpha}^o_{11'} \tag{4.9.2'}$$

$\bar{\alpha}^o_{11'}$ is the rotation angle and $e^o_{11'}$ the rotation unit vector, so:

$$p^o_{11'} = \cos\tfrac{1}{2}\bar{\alpha}^o_{11'} + e^o_{11'} \,.\, \sin\tfrac{1}{2}\bar{\alpha}^o_{11'} \tag{4.9.2''}$$

Then follows :

$$e^o_{(1')_{srt}} = p^o_{1'1} \cdot e^o_{(1)_{srt}} \cdot p^o_{11'} \tag{4.9.3'}$$

and

$$e^o_{(2)_{srt}} \cdot e^{o\,-1}_{(1')_{srt}} = \cos\bar{\alpha}^o_{1'2} + e^o_{1'2} \,.\, \sin\bar{\alpha}^o_{1'2} \tag{4.9.3''}$$

$\bar{\alpha}^o_{1'2}$ is the rotation angle from (1′) - to (2)- system and $e^o_{1'2}$ the according rotation vector, hence :

$$p^o_{1'2} = \cos\tfrac{1}{2}\bar{\alpha}^o_{1'2} + e^o_{1'2} \,.\, \sin\tfrac{1}{2}\bar{\alpha}^o_{1'2} \tag{4.9.3'''}$$

(4.9.3‴) and (4.9.2′) substituted in (4.9.1) give p^o_{12} . (4.9.1) and (4.8) and (4.7) for q^o_r give the shifts:

$$d^o_{21} = q^o_{(2)_r} - \lambda^o_{21} \cdot p^o_{21} \cdot q^o_{(1)_r} \cdot p^o_{12} \tag{4.10}$$

Now all seven parameters of the transformation have been expressed as a function of the elements of the S-base. This implies that the S-base of (4.5) and the K-transformation (4.7) satisfy (2.32.2). The difference equation of (4.7′) is now :

$$\tilde{\Delta} q^{(r,s;t)}_{(2)_i} = \lambda^o_{21} \cdot p^o_{21} \cdot \tilde{\Delta} q^{(r,s;t)}_{(1)_i} \cdot p^o_{12} \tag{4.11}$$

which corresponds to (2.34′).

4.4 S-transformations in R_3

Suppose the coordinates of points in a geodetic network have been computed with respect to an S-base (a, b; c) and that they should be transformed to base (r, s; t). Then we have to find the transformation :

$$\tilde{q}^{(a,b;c)}_{(1)_i} \longrightarrow \tilde{q}^{(r,s;t)}_{(2)_i} \tag{4.12}$$

Let :

$$q^{o\,(a,b;c)}_{(1)_i} = q^{o\,(r,s;t)}_{(2)_i} = q^o_i$$

In that case the index to indicate the K-system may be omitted :

$$\tilde{q}^{(a,b;c)}_i \longrightarrow \tilde{q}^{(r,s;t)}_i$$

with (4.12′)

$$q^{o\,(a,b;c)}_i = q^{o\,(r,s;t)}_i = q^o_i$$

In this chapter a simplified notation will be used :

(r) replaces ($r,s;t$)

(a) replaces ($a,b;c$)

then (4.12′) becomes :

$$\tilde{q}^{(a)}_i \longrightarrow \tilde{q}^{(r)}_i$$

$$q^{o\,(a)}_i = q^{o\,(r)}_i = q^o_i \tag{4.12″}$$

The derivation of the S-transformation will follow the method of section 2.5.3. Due to (4.7) we get for (2.29′) :

(2.29′.a) $\quad \tilde{q}^{(r)}_i = \tilde{\lambda}^{(ra)} \cdot \tilde{p}^{(ra)} \cdot \tilde{q}^{(a)}_i \cdot \tilde{p}^{(ar)} + \tilde{d}^{(ra)}$ $\quad$ a)

(4.13)

(2.29′.c) $\quad q^{o\,(r)}_i = \lambda^{o\,(ra)} \cdot p^{o\,(ra)} \cdot q^{o\,(a)}_i \cdot p^{o\,(ar)} + d^{o\,(ra)} = q^{o\,(a)}_i$ $\quad$ c)

The notation will be simplified more for the following derivations :

$$
\begin{aligned}
&\tilde{q}_i^{(r)} - q_i^{o(r)} = \Delta q_i^{(r)} && \tilde{q}_i^{(a)} - q_i^{o(a)} = \Delta q_i^{(a)} \\
&\tilde{\lambda}^{(ra)} - \lambda^{o(ra)} = \Delta\lambda && \tilde{p}^{(ra)} - p^{o(ra)} = \Delta p \\
&\tilde{p}^{(ar)} - p^{o(ar)} = \Delta p^{-1} && \tilde{d}^{(ra)} - d^{o(ra)} = \Delta d
\end{aligned}
\qquad (4.14')
$$

where :

$$p^{(ra)} \,.\, p^{(ar)} = 1$$

if :

$$p^{(ar)} = p \qquad \text{then} \qquad p^{(ra)} = p^{(ar)-1} = p^{-1}$$

from

$p \,.\, p^{-1} = 1$ follows :

$$\Delta(p \,.\, p^{-1}) = \Delta p \,.\, p^{-1} + p \,.\, \Delta p^{-1} = 0 \longrightarrow \Delta p^{-1} = -p^{-1} \,.\, \Delta p \,.\, p^{-1}$$

(4.13. c) leads to :

$$
\begin{aligned}
&\lambda^{o(ra)} = 1, \quad p^{o(ra)} = p^{o(ar)} = 1, \quad d^{o(ra)} = 0 \\
&q_i^{o(a)} = q_i^{o(r)} = q_i^o
\end{aligned}
\qquad (4.13.c)
$$

The difference between (4.13.a) and (4.13.c) is :

$$\Delta q_i^{(r)} = \Delta\lambda \,.\, p^{-1} \,.\, q_i^{(a)} \,.\, p + \lambda \,.\, \Delta p^{-1} \,.\, q_i^{(a)} \,.\, p + \lambda \,.\, p^{-1} \,.\, \Delta q_i^{(a)} \,.\, p + \lambda \,.\, p^{-1} \,.\, q_i^{(a)} \,.\, \Delta p + \Delta d$$

where second and higher order terms in the linearisation have been neglected. In this equation, we have:

$$\Delta\lambda = \frac{\lambda \,.\, \Delta\lambda}{\lambda} = \lambda \,.\, \Delta ln\lambda \,, \quad \Delta p^{-1} = -p^{-1} \,.\, \Delta p \,.\, p^{-1} \,, \quad \Delta p = p \,.\, p^{-1} \,.\, \Delta p$$

This gives for the difference equation :

$$\boxed{\Delta q_i^{(r)} = \Delta ln\lambda \,.\, q_i^o - p^{-1} \,.\, \Delta p \,.\, q_i^o + \Delta q_i^{(a)} + q_i^o \,.\, p^{-1} \,.\, \Delta p + \Delta d} \qquad (4.15.1)$$

The difference quantities of the transformation parameters have to be eliminated according to the method of section 2.5.2. Therefore we use the new S-base, which gives :

$$
\boxed{
\begin{aligned}
&\tilde{q}_r^{(r,s;t)} - q_r^o = \tilde{\Delta} q_r^{(r,s;t)} = \Delta q_r^{(r)} = 0 \\
&\tilde{q}_s^{(r,s;t)} - q_s^o = \tilde{\Delta} q_s^{(r,s;t)} = \Delta q_s^{(r)} = 0 \\
&\tilde{e}_{srt}^{(r,s;t)} - e_{srt}^o = \tilde{\Delta} e_{srt}^{(r,s;t)} = \Delta e_{srt}^{(r)\cdot} = 0
\end{aligned}
}
$$

Then (4.15.1) gives for point r :

$$\Delta q_r^{(r)} = 0 = \Delta ln\lambda \,.\, q_r^o - p^{-1} \Delta p q_r^o + \Delta q_r^{(a)} + q_r^o p^{-1} \Delta p + \Delta d$$

Subtraction from (4.15.1) eliminates Δd :

$$\Delta q_{ri}^{(r)} = \Delta q_i^{(r)} - \Delta q_r^{(r)} = \Delta q_i^{(r)} - 0$$
$$= \Delta q_i^{(r)} = \Delta ln\lambda \,.\, q_{ri}^o - p^{-1} \,.\, \Delta p \,.\, q_{ri}^o + \Delta q_{ri}^{(a)} + q_{ri}^o \,.\, p^{-1} \,.\, \Delta p \qquad (4.15.2)$$

where

$$q_{ri}^o = q_i^o - q_r^o \qquad \Delta q_{ri} = \Delta q_i - \Delta q_r$$

In a similar way :

$$\Delta q_{rs}^{(r)} = 0 = \Delta ln\lambda \,.\, q_{rs}^o - p^{-1} \,.\, \Delta p \,.\, q_{rs}^o + \Delta q_{rs}^{(a)} + q_{rs}^o \,.\, p^{-1} \,.\, \Delta p$$

Premultiplication by $Q_{sri}^o = q_{ri}^o \,.\, {q_{rs}^o}^{-1}$ in this expression results in :

$$Q_{sri}^o \,.\, \Delta q_{rs}^{(r)} = 0 = \Delta ln\lambda \,.\, q_{ri}^o - Q_{sri}^o \,.\, p^{-1} \,.\, \Delta p \,.\, q_{rs}^o + Q_{sri}^o \,.\, \Delta q_{rs}^{(a)} + q_{ri}^o \,.\, p^{-1} \,.\, \Delta p$$

Postmultiplication by ${Q_{sri}^o}^T = {q_{rs}^o}^{-1} \,.\, q_{ri}^o$ gives :

$$\Delta q_{rs}^{(r)} \,.\, {Q_{sri}^o}^T = 0 = \Delta ln\lambda \,.\, q_{ri}^o - p^{-1} \,.\, \Delta p \,.\, q_{ri}^o + \Delta q_{rs}^{(a)} \,.\, {Q_{sri}^o}^T + q_{rs}^o \,.\, p^{-1} \,.\, \Delta p \,.\, {Q_{sri}^o}^T$$

from these new expressions we find :

$$\Delta ln\lambda \,.\, q_{ri}^o + q_{ri}^o \,.\, p^{-1} \,.\, \Delta p = -Q_{sri}^o \,.\, \Delta q_{rs}^{(a)} + Q_{sri}^o \,.\, p^{-1} \,.\, \Delta p \,.\, q_{rs}^o$$

and

$$\Delta ln\lambda \,.\, q_{ri}^o - p^{-1} \,.\, \Delta p \,.\, q_{ri}^o = -\Delta q_{rs}^{(a)} \,.\, {Q_{sri}^o}^T - q_{rs}^o \,.\, p^{-1} \,.\, \Delta p \,.\, {Q_{sri}^o}^T$$

Substitution in (4.15.2) gives:

$$\Delta q_i^{(r)} = -Q_{sri}^o \,.\, \Delta q_{rs}^{(a)} + Q_{sri}^o \,.\, p^{-1} \,.\, \Delta p \,.\, q_{rs}^o - p^{-1} \,.\, \Delta p \,.\, q_{ri}^o + \Delta q_{ri}^{(a)} \qquad \text{a)}$$
$$(4.15.3)$$
$$\Delta q_i^{(r)} = -\Delta q_{rs}^{(a)} \,.\, {Q_{sri}^o}^T - q_{rs}^o \,.\, p^{-1} \,.\, \Delta p \,.\, {Q_{sri}^o}^T + \Delta q_{ri}^{(a)} + q_{ri}^o \,.\, p^{-1} \,.\, \Delta p \qquad \text{b)}$$

Write :

$$p^{-1} \,.\, \Delta p \,.\, q_{ri}^o = p^{-1} \,.\, \Delta p \,.\, q_{rs}^o \,.\, {Q_{sri}^o}^T \text{ and } q_{ri}^o \,.\, p^{-1} \,.\, \Delta p = Q_{sri}^o \,.\, q_{rs}^o \,.\, p^{-1} \,.\, \Delta p$$

Using this, the addition of (a) and (b) in (4.15.3) divided by 2 results in:

$$\Delta q_i^{(r)} = \Delta q_{ri}^{(a)} - \tfrac{1}{2} \,.\, (Q_{sri}^o \,.\, \Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)} \,.\, {Q_{sri}^o}^T) + \tfrac{1}{2} \,.\, Q_{sri}^o \,.\, [p^{-1} \,.\, \Delta p \,.\, q_{rs}^o + q_{rs}^o \,.\, p^{-1} \,.\, \Delta p]$$
$$- \tfrac{1}{2} \,.\, [p^{-1} \,.\, \Delta p \,.\, q_{rs}^o + q_{rs}^o \,.\, p^{-1} \,.\, \Delta p] \,.\, {Q_{sri}^o}^T$$

in $\Delta q_i^{(r)}$ the expressions within [ ] are scalars, hence :

$$\Delta q_i^{(r)} = \Delta q_{ri}^{(a)} - \tfrac{1}{2}.(Q_{sri}^o.\Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)}.Q_{sri}^{o^T}) + \tfrac{1}{2}.(Q_{sri}^o - Q_{sri}^{o^T}).[p^{-1}.\Delta p.q_{rs}^o + q_{rs}^o.p^{-1}.\Delta p] \tag{4.15.4}$$

A similar expression for point t is :

$$\Delta q_t^{(r)} = \Delta q_{rt}^{(a)} - \tfrac{1}{2}.(Q_{srt}^o.\Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)}.Q_{srt}^{o^T}) + \tfrac{1}{2}.(Q_{srt}^o - Q_{srt}^{o^T}).[p^{-1}.\Delta p.q_{rs}^o + q_{rs}^o.p^{-1}.\Delta p] \tag{4.15.4'}$$

hence :

$$[p^{-1}.\Delta p.q_{rs}^o + q_{rs}^o.p^{-1}.\Delta p] =$$

$$= 2.(Q_{srt}^o - Q_{srt}^{o^T})^{-1}.\left\{\Delta q_t^{(r)} - \Delta q_{rt}^{(a)} + \tfrac{1}{2}.(Q_{srt}^o.\Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)}.Q_{srt}^{o^T})\right\}$$

Substitution in (4.15.4) results in:

$$\begin{aligned}\Delta q_i^{(r)} = {} & \Delta q_{ri}^{(a)} - \tfrac{1}{2}.(Q_{sri}^o.\Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)}.Q_{sri}^{o^T}) \\ & + (Q_{sri}^o - Q_{sri}^{o^T}).(Q_{srt}^o - Q_{srt}^{o^T})^{-1}.\left\{\Delta q_t^{(r)} - \Delta q_{rt}^{(a)} + \tfrac{1}{2}.(Q_{srt}^o.\Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)}.Q_{srt}^{o^T})\right\}\end{aligned} \tag{4.15.5}$$

In the righthand side of (4.15.5) the term $\Delta q_t^{(r)}$ has still to be eliminated from the expression within { } . Therefore we use :

$$\tilde{q}_t^{(r)} = q_r^o + \tilde{Q}_{srt}^{(r)}.q_{rs}^o = q_r^o + \tilde{v}_{srt}.(\cos\tilde{\bar{\alpha}}_{srt} + e_{srt}^o.\sin\tilde{\bar{\alpha}}_{srt}).q_{rs}^o$$

and

$$q_t^o = q_r^o + Q_{srt}^o.q_{rs}^o = q_r^o + v_{srt}^o.(\cos\bar{\alpha}_{srt}^o + e_{srt}^o.\sin\bar{\alpha}_{srt}^o).q_{rs}^o$$

The difference between these equations is :

$$\begin{aligned}\Delta q_t^{(r)} &= \Delta Q_{srt}^{(r)}.q_{rs}^o \\ &= Q_{srt}^o.q_{rs}^o.\Delta \ln v_{srt} + v_{srt}^o.(-\sin\bar{\alpha}_{srt}^o + e_{srt}^o.\cos\bar{\alpha}^o{}_{srt}).q_{rs}^o.\Delta\bar{\alpha}_{srt} \\ &= q_{rt}^o.\Delta \ln v_{srt} + e_{srt}^o.Q_{srt}^o.q_{rs}^o.\Delta\bar{\alpha}_{srt} \\ &= q_{rt}^o.\Delta \ln v_{srt} + e_{srt}^o.q_{rt}^o.\Delta\bar{\alpha}_{srt}\end{aligned}$$

hence

$$\Delta q_t^{(r)} \perp e_{srt}^o \longrightarrow \Delta q_t^{(r)} \perp (Q_{srt}^o - Q_{srt}^{o^T})$$

and

$$(Q_{srt}^o - Q_{srt}^{o^T})^{-1}.\Delta q_t^{(r)}.(Q_{srt}^o - Q_{srt}^{o^T}) = -\Delta q_t^{(r)} \tag{4.16}$$

In (4.15.4') the term within [ ] is a scalar, therefore :

$$(Q^o_{srt} - Q^{o^T}_{srt})^{-1} . (Q^o_{srt} - Q^{o^T}_{srt}) . [p^{-1} . \Delta p . q^o_{rs} + q^o_{rs} . p^{-1} . \Delta p] . (Q^o_{srt} - Q^{o^T}_{srt})$$
$$= (Q^o_{srt} - Q^{o^T}_{srt}) . [p^{-1} . \Delta p . q^o_{rs} + q^o_{rs} . p^{-1} . \Delta p]$$

Hence (4.16) leads to :

$$½ . [\Delta q^{(r)}_t - (Q^o_{srt} - Q^{o^T}_{srt})^{-1} . \Delta q^{(r)}_t . (Q^o_{srt} - Q^{o^T}_{srt})]$$
$$= \Delta q^{(r)}_t = ½ . \{ [\Delta q^{(a)}_{rt} - ½ . (Q^o_{srt} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . Q^{o^T}_{srt})]$$
$$- (Q^o_{srt} - Q^{o^T}_{srt})^{-1} . [\Delta q^{(a)}_{rt} - ½ . (Q^o_{srt} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . Q^{o^T}_{srt})] . (Q^o_{srt} - Q^{o^T}_{srt}) \}$$

Substitution in (4.15.5) will lead to :

$$\boxed{\begin{aligned} \Delta q^{(r)}_i = {} & \Delta q^{(a)}_{ri} - ½ . (Q^o_{sri} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . Q^{o^T}_{sri}) \\ & - ½ . (Q^o_{sri} - Q^{o^T}_{sri}) . (Q^o_{srt} - Q^{o^T}_{srt})^{-1} . \{ [\Delta q^{(a)}_{rt} - ½ (Q^o_{srt} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . Q^{o^T}_{srt})] \\ & + (Q^o_{srt} - Q^{o^T}_{srt})^{-1} . [\Delta q^{(a)}_{rt} - ½ . (Q^o_{srt} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . Q^{o^T}_{srt})] . (Q^o_{srt} - Q^{o^T}_{srt}) \} \end{aligned}}$$

(4.15.6)

This is the complete S-transformation of a point i from (a)– to (r)–system. Write : $\Delta q^{(r)}_i = \Delta q^{(r,s;t)}_i$. Some rewriting of this expression gives the form which Baarda found independently in 1974.

$$\boxed{\begin{aligned} \Delta q^{(r,s)}_t = {} & \Delta q^{(a)}_t - ½ . (Q^o_{rst} . \Delta q^{(a)}_r + \Delta q^{(a)}_r . Q^{o^T}_{rst}) - ½ . (Q^o_{srt} . \Delta q^{(a)}_s + \Delta q^{(a)}_s . Q^{o^T}_{srt}) \\ \Delta q^{(r,s;t)}_i = {} & \Delta q^{(a)}_i - ½ . (Q^o_{rsi} . \Delta q^{(a)}_r + \Delta q^{(a)}_r . Q^{o^T}_{rsi}) - ½ . (Q^o_{sri} . \Delta q^{(a)}_s + \Delta q^{(a)}_s . Q^{o^T}_{sri}) \\ & - (Q^o_{rsi} - Q^{o^T}_{rsi}) . (Q^o_{rst} - Q^{o^T}_{rst})^{-1} . ½ . [\Delta q^{(r,s)}_t + (Q^o_{rst} - Q^{o^T}_{rst}) . \Delta q^{(r,s)}_t . (Q^o_{rst} - Q^{o^T}_{rst})^{-1}] \end{aligned}}$$

(4.15.6')

The derivation of (4.15.6) has been made without use of any specific knowledge of the S-base in the (a)–system. This agrees with what was said in the discussion of (2.18). So any (a)–system satisfying (4.12') will be transformed to $(r,s;t)$–system by (4.15.6 or 6').

Baarda developed, in [2], a matrix algebra which is isomorphic with the algebra of quaternions. So it is possible to translate the quaternion expression in (4.15.6) into a linear algebraic expression in which the difference quantities of the coordinates of a point in the new system are given as a linear function of the difference quantities of the coordinates in the old system.

We will use a short notation for (4.15.6) :

$$\boxed{\Delta q^{(r,s;t)} = S^{(r,s;t)} \{ \Delta q^{(a)}_i \}} \qquad (4.15.6'')$$

The symbol S gets the superscript $(r,s;t)$ to indicate a transformation to $(r,s;t)$–system (see the discussion of (2.19') and (2.21)).

4.5 Invariants to S-transformations in R_3

Estimable quantities should be invariant under S-transformations. So in R_3 we should have:

$$\left.\begin{array}{llcl} \text{a)} & \Delta lnv^{(r)}_{jik} & = & \Delta lnv^{(a)}_{jik} \\ \text{b)} & \Delta\bar{\alpha}^{(r)}_{jik} & = & \Delta\bar{\alpha}^{(a)}_{jik} \\ \text{c)} & \Delta\bar{\bar{\alpha}}^{(r)}_{jik} & = & \Delta\bar{\bar{\alpha}}^{(a)}_{jik} \end{array}\right\} \longrightarrow \left\{\begin{array}{lcll} \Delta lnv^{(\,)} & = & ln\tilde{v}^{(\,)} & - lnv^{o} \\ \Delta\bar{\alpha}^{(\,)} & = & \tilde{\bar{\alpha}}^{(\,)} & - \bar{\alpha}^{o} \\ \Delta\bar{\bar{\alpha}}^{(\,)} & = & \tilde{\bar{\bar{\alpha}}}^{(\,)} & - \bar{\bar{\alpha}}^{o} \end{array}\right. \qquad (4.17)$$

The cosine rule for plane triangles expresses $\bar{\alpha}_{jik}$ as a function of the length ratios, whereas (4.3) expresses $\bar{\bar{\alpha}}$ as a function of the angles $\bar{\alpha}$. Therefore it is sufficient to prove that lnv is invariant to S-transformations, then (4.17.b and c) are consequences of (4.17.a). The proof will be given for Δlnv.

(4.15.6) gives for coordinate differences between point j and i:

$$\begin{aligned} \Delta q^{(r)}_{ij} = \Delta q^{(r)}_{j} - \Delta q^{(r)}_{i} = \Delta q^{(a)}_{ij} - ½ . (q^{o}_{ij} . q^{o^{-1}}_{rs} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . q^{o^{-1}}_{rs} . q^{o}_{ij}) - \\ - (q^{o}_{ij} . q^{o^{-1}}_{rs} - q^{o^{-1}}_{rs} . q^{o}_{ij}) . (Q^{o}_{srt} - Q^{o^{T}}_{srt})^{-1} . \{\ldots\ldots\} \end{aligned} \qquad (4.18.1)$$

The derivation of (4.15.4) and (4.15.5) proves that the expression:

$$(Q^{o}_{srt} - Q^{o^{T}}_{srt})^{-1} . \{\ldots\ldots\ldots\}$$

is a scalar. Hence (4.18.1) premultiplied by $q^{o^{-1}}_{ij}$ becomes:

$$\begin{aligned} q^{o^{-1}}_{ij} . \Delta q^{(r)}_{ij} = q^{o^{-1}}_{ij} . \Delta q^{(a)}_{ij} - ½ . (q^{o^{-1}}_{rs} . \Delta q^{(a)}_{rs} + q^{o^{-1}}_{ij} . \Delta q^{(a)}_{rs} . q^{o^{-1}}_{rs} . q^{o}_{ij}) - \\ - ½ . (q^{o^{-1}}_{rs} - q^{o^{-1}}_{ij} . q^{o^{-1}}_{rs} . q^{o}_{ij}) . (Q^{o}_{srt} - Q^{o^{T}}_{srt})^{-1} . \{\ldots\ldots\ldots\} \end{aligned}$$

and postmultiplication results in:

$$\begin{aligned} \Delta q^{(r)}_{ij} . q^{o^{-1}}_{ij} = \Delta q^{(a)}_{ij} . q^{o^{-1}}_{ij} - ½ . (q^{o}_{ij} . q^{o^{-1}}_{rs} . \Delta q^{(a)}_{rs} . q^{o^{-1}}_{ij} + \Delta q^{(a)}_{rs} . q^{o^{-1}}_{rs}) - \\ - ½ . (q^{o}_{ij} . q^{o^{-1}}_{rs} . q^{o^{-1}}_{ij} - q^{o^{-1}}_{rs}) . (Q^{o}_{srt} - Q^{o^{T}}_{srt})^{-1} . \{\ldots\ldots\ldots\} \end{aligned}$$

as $q^{o}_{ij} . q^{o^{-1}}_{rs} . q^{o^{-1}}_{ij} = q^{o^{-1}}_{ij} . q^{o^{-1}}_{rs} . q^{o}_{ij}$, the addition of these two expressions gives:

$$\begin{aligned} q^{o^{-1}}_{ij} . \Delta q^{(r)}_{ij} + \Delta q^{(r)}_{ij} . q^{o^{-1}}_{ij} = q^{o^{-1}}_{ij} . \Delta q^{(a)}_{ij} + \Delta q^{(a)}_{ij} . q^{o^{-1}}_{ij} - ½ . (q^{o^{-1}}_{rs} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . q^{o^{-1}}_{rs}) - \\ - ½ . (q^{o}_{ij} . q^{o^{-1}}_{rs} . \Delta q^{(a)}_{rs} . q^{o^{-1}}_{ij} + q^{o^{-1}}_{ij} . \Delta q^{(a)}_{rs} . q^{o^{-1}}_{rs} . q^{o}_{ij}) \end{aligned} \qquad (4.18.2)$$

The last term on the righthand side is:

$$q^{o}_{ij} . q^{o^{-1}}_{rs} . \Delta q^{(a)}_{rs} . q^{o^{-1}}_{ij} + q^{o^{-1}}_{ij} . \Delta q^{(a)}_{rs} . q^{o^{-1}}_{rs} . q^{o}_{ij} = q^{o^{-1}}_{ij} . (q^{o^{-1}}_{rs} . \Delta q^{(a)}_{rs} + \Delta q^{(a)}_{rs} . q^{o^{-1}}_{rs}) . q^{o}_{ij}$$

with

$$q^{-1}_{rs} . \Delta q_{rs} + \Delta q_{rs} . q^{-1}_{rs} = 2 . \Delta ln\ l_{rs}$$

and similarly :

$$q_{ij}^{-1} \,.\, \Delta q_{ij} + \Delta q_{ij} \,.\, q_{ij}^{-1} = 2 \,.\, \Delta ln\ l_{ij}$$

hence (4.18.2) becomes :

$$\left.\begin{array}{l} q_{ij}^{o^{-1}} \,.\, \Delta q_{ij}^{(r)} + \Delta q_{ij}^{(r)} \,.\, q_{ij}^{o^{-1}} = q_{ij}^{o^{-1}} \,.\, \Delta q_{ij}^{(a)} + \Delta q_{ij}^{(a)} \,.\, q_{ij}^{o^{-1}} - (q_{rs}^{o^{-1}} \,.\, \Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)} \,.\, q_{rs}^{o^{-1}}) \\ \text{or} \\ \Delta ln\ l_{ij}^{(r)} = \Delta ln\ l_{ij}^{(a)} - \Delta ln\ l_{rs}^{(a)} \end{array}\right\} \quad (4.18.3)$$

In a similar way we obtain :

$$\Delta ln\ l_{ik}^{(r)} = \Delta ln\ l_{ik}^{(a)} - \Delta ln\ l_{rs}^{(a)}$$

(4.18.3) subtracted from the latter expression :

$$ln\ v_{jik}^{(r)} = ln\ l_{ik}^{(r)} - ln\ l_{ij}^{(r)} = ln\ l_{ik}^{(a)} - ln\ l_{ij}^{(a)} = ln\ v_{jik}^{(a)}$$

hence

$$ln\ v_{jik}^{(r)} = ln\ v_{jik}^{(a)} = ln\ v_{jik}$$

This completes the proof for (4.17.a) and implicitly for (4.17.b–c). Consequently length ratios and angles do not need a superscript to indicate the S-system. As these variates are invariant to K-transformations, the index for the K-system can be omitted too.

4.6 Epilogue to chapter IV

The S-transformation in R_3 is a generalisation of the one found by Baarda for R_2 (see [6]). Therefore it should be possible to obtain the latter as a special case of (4.15.6) or (4.15.6′). Let R_2 be the subspace of R_3 with :

$$V_k \{q_i\} = V_k \{q_r\} = V_k \{q_s\} = V_k \{q_t\} = 0$$

$V_k \{q\}$ is the k-component of q. Consequently :

$$q_i = x_i i + y_i j \quad \text{etc.}$$

This means that in the last term of (4.15.6) the expression within accolades is :

$$\{\,\cdots\cdots\cdots\,\} = 0$$

hence :

$$\Delta q_i^{(r)} = \Delta q_i^{(a)} - \tfrac{1}{2} \,.\, (Q_{sri}^{o} \,.\, \Delta q_{rs}^{(a)} + \Delta q_{rs}^{(a)} \,.\, Q_{sri}^{o^T})$$

or in this special case :

$$\Delta q_i^{(r)} = \Delta q_i^{(a)} - Q_{rsi}^{o} \,.\, \Delta q_r^{(a)} - Q_{sri}^{o} \,.\, \Delta q_s^{(a)}$$

If this expression is premultiplied by $-j$, then we get a new expression with a scalar- and a k-component, which is similar to the formulation in complex numbers of the S-transformation in R_2 (see [6] (2.5′)).

According to section 2.5.1 any set of values can be introduced for the elements of the S-base. For algebraic reasons there is, however, one restriction:

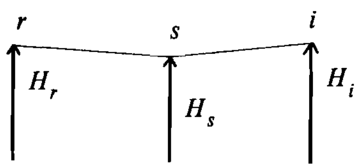
$$N\{q_{rs}\} = N\{q_s - q_r\} \neq 0 \tag{4.19}$$

the distance between point r and s should not be equal to zero, because in that case the ratio of lengths :

$$v_{sri} = \frac{l_{ri}}{l_{rs}}$$

is not defined. As distances define the metric of R_3 ([44] § 2),a distance $l_{rs} = 0$ implies that $q_r = q_s$, which means that there is no distinction between point r and point s. A similar restriction has to be made for the S-base in R_2 in section 3.3.2, but not for the S-base for spherical coordinates as the computation of the latter does not require any definition of scale. There, the metric of the space is defined by the spherical distance, which is an estimable quantity, so no distance can be part of the S-base. Yet there is a restriction for the choice of the base points r and s: one should not choose two points for which $a_{rs} = \pi$, because in that case the angle between the side r,s and any other side r,i will not be defined. Hence A_{rs}, the azimuth of side r,s, cannot be used to fix the orientation of the coordinate system. Restriction (4.19) gives no practical limitations for R_2 and R_3 as the measuring process will never lead to a result $l_{rs} = 0$. The situation is different in R_1, for measurements of height differences. If the height difference between points r and s is:

$$h_{rs} = H_s - H_r$$



one may be apt to eliminate the unit of length by means of the proportion:

$$v_{rsi} = \frac{h_{si}}{h_{sr}}$$

and use (H^o_r, H^o_s) as an S-base. Then for algebraic reasons we have the restriction:

$$H_s - H_r = h_{rs} \neq 0$$

This is conflicting with the measuring process, which may give as a result:

$$h_{rs} = 0$$

So the above approach cannot be applied, because it does not lead to a consistent computational model. The measuring procedure for height differences requires a different interpretation, i.e. Baarda [8] .

Another observation which should be made concerns the difference between photogrammetry and terrestrial geodesy.

The S-transformation (4.15.6) leaves angles and length ratios invariant, because analysis of the measuring process in terrestrial geodesy proved these to be measurable. In photogrammetry the measuring process is much more complicated. Each photoflight is unique in the sense that if a flight is repeated, it will not be possible to get the camera in the same position again for each exposure. When photo coordinates of image points are measured for different flights it will not be possible to define measurable quantities in the sense of def. 3.I.1. But photogrammetry has traditionally been based on the assumption that the photograph is a perfect perspective projection of the terrain. Under this assumption repetition of measurements of plate coordinates gives angles between light rays as measurable quantities. This mathematical model leads to the formation of stereomodels from pairs of photographs, and even to block formations.These stereomodels or photogrammetric blocks are then linear conformal mappings of the terrain, to which S-transformations, according to (4.15.6), can be applied. Many factors may, however, disturb the perspective projection, e.g.: refraction, lens distortion, inner orientation, lack of film flatness, processing of the film etc. Some of these effects can be taken care of simply, but others are not so easy to correct for. If the perspective projection has been disturbed, then consequently the stereomodels,or the block obtained from such photographs, will have a geometry which differs from the geometry of the real terrain, i.e.: angles and length ratios observed in the terrain are different from similar elements obtained from the block. One should act with special care then if a photogrammetrically determined point field has to be connected to one determined terrestrially. If the connection is made by means of an S-transformation, then the discrepancies between photogrammetric and terrestrial coordinates, at points not belonging to the S-base, cannot be simply considered as the misclosures of condition equations in an adjustment according to standard problem I (see [37]). Moreover, statistical tests are required to check whether these discrepancies are caused by systematic deformations of the photogrammetric block. If this is true then the block should be corrected for these deformations before or simultaneously with the final connection to ground control (see [35] and ref.).

CHAPTER V CRITERION MATRICES FOR LARGE NETWORKS

5.1 The comparison of covariance matrices

5.1.1 The inner precision of networks

In modern geodetic research, rules are formulated for the reconnaissance of geodetic networks. These rules are mainly based on requirements for the precision and reliability of the point determination. In this chapter we concentrate on the requirements for precision, for which we will follow the approach given by Baarda in [6]. He developed a 'Criterion variance-covariance matrix' for coordinates, which is invariant with respect to rotations of the coordinate system. For cartesian planimetric coordinates this means that the point and relative standard ellipses are circles. When designing a network one should check whether the real variance-covariance matrix (V.C. matrix) of the coordinates agrees sufficiently with the criterion matrix. Baarda performs this test by means of the general eigenvalue problem and he proves in [6] § 10 that such a test is valid only if both matrices have been computed with respect to the same S-base. An alternative proof will be given in section 5.1.2. It is based on the precision of the geometry of a network, in which respect we shall use the expression: 'Inner precision'.

Def. 5.I.

> The 'inner precision' of a point field is the V.C. matrix of all its form elements (modulo permutations)

In large point fields there are many form elements, so the order of their V.C. matrix will be large, whereas the rank will be considerably less than the order. As these large matrices are difficult to handle we formulate the auxiliary definition.

Def. 5.II.

> If (x^i) is a necessary and sufficient set of form elements to describe the geometry of a point field, then we say that the inner precision of this point field is given by the V.C. matrix:
>
> $(\sigma^{ij}) = (E\{(\tilde{x}^i - \underline{x}), \; (\tilde{x}^j - \underline{x}^j)^*\})$
>
> $i \; j = 1, \ldots\ldots, n$ = number of elements x^i
>
> $(x^j)^*$ is the transpose of (x^i)

For a planimetric point field of m points $n = 2m - 4$, for a spatial point field $n = 3m - 7$. All other form elements of the network can be found as a function of the elements of the set (x^i) in def. 5.II. So their V.C. matrix is found by means of the propagation of variances and covariances. In networks there are always more sets of form elements like (x^i), which describe the geometry completely. To each of these sets belongs a V.C. matrix which, like (σ^{ij}), gives the inner precision.

Of course all these matrices are interrelated, so we formulate:

Theorem (5.1.)

> If the V.C. matrix (σ^{ij}) of $(\underline{x}^i)$ satisfies Def. 5.II and if (σ^{kl}) of $(\underline{x}^k)$ is related to it by:
>
> $(\sigma^{kl}) = (\Lambda^k_i)(\sigma^{ij})(\Lambda^l_j)^*$
>
> with $\Lambda^k_i = \left.\dfrac{\partial x^k}{\partial x^i}\right|_{x^i_o \approx \tilde{x}^i}$ and rank $(\Lambda^k_i) = n$ (5.1)
>
> $i = 1, \ldots\ldots, n$
>
> then (σ^{kl}) gives the same inner precision as (σ^{ij})

This theorem considers variates x^k which can be written as functions of the variates x^i alone, e.g.: (x^k) can be any sufficient set of form elements. The formulation of the theorem allows however also the use of the V.C. matrix of coordinates, computed in any S–system, to give the inner precision of a point field.

The only restriction for the number of variates x^k is that there are at least n variates which are not interdependent, because rank $(\Lambda^k_i) = n$. So there may be more than n variates x^k but then their V.C. matrix will be singular. Here we could think of the 'inner coordinates with minimum trace' defined by Meissl [32]. He defines the inner precision of a point field by means of the trace of the V.C. matrix of these coordinates, which is invariant with respect to similarity transformations. In this way only a part of the available information about the precision has been used, whereas def. 5.I. refers to the full V.C. matrix of the form elements in the point field. This definition does not refer to any particular coordinate system. By means of def. 5.II. and theorem (5.1) it is possible however to use the full V.C. matrix of the coordinates in any S–system to analyse the inner precision of a point field. This will be done in the next section.

5.1.2 The general eigenvalue problem

Suppose a network has been measured twice. As a result of the first measurement we obtain the quantities $\underline{x}^i_1$, which form a necessary and sufficient set of angles and length ratios to describe the geometry of the network. The second measurement gives $\underline{x}^i_2$. Let :

$$E\{\underline{x}^i_1\} = E\{\underline{x}^i_2\} \text{ , or } \tilde{x}^i_1 = \tilde{x}^i_2$$

whereas

$$\underline{x}^i_1 \approx \underline{x}^i_2$$

Introduction of S–bases will make the computation of coordinates possible, e.g. (see (2.4) and (2.11')) :

first measurement :

$$\begin{aligned} &\text{a)} \quad (\tilde{y}_1^{(u)^r}) = (y^r\{\ldots.\tilde{x}^i_1\ldots.x^u_o\ldots.\}) \\ &\text{b)} \quad (\underline{y}_1^{(u)^r}) = (y^r\{\ldots.\underline{x}^i_1\ldots.x^u_o\ldots.\}) \\ &\text{c)} \quad (y_o^{(u)^r}) = (y^r\{\ldots.x^i_o\ldots.x^u_o\ldots.\}) \end{aligned} \tag{5.2.1}$$

Second measurement :

$$\begin{aligned} &\text{a)} \quad (\tilde{y}_2^{(p)^r}) = (y^r\{\ldots.\tilde{x}^i_2\ldots.x^p_o\ldots.\}) \\ &\text{b)} \quad (\underline{y}_2^{(p)^r}) = (y^r\{\ldots.\underline{x}^i_2\ldots.x^p_o\ldots.\}) \\ &\text{c)} \quad (y_o^{(p)^r}) = (y^r\{\ldots.x^i_o\ldots.x^p_o\ldots.\}) \end{aligned} \tag{5.2.2}$$

We choose (x^p_o) and (x^u_o) so that (see(2.15)) :

$$\begin{aligned} &(y_o^{(u)^r}) = (y_o^{(p)^r}) = (y^r_o) \\ &\text{then} \quad (\tilde{y}_1^{(u)^r}) \approx (\tilde{y}_2^{(p)^r}) \qquad \text{and} \quad (\underline{y}_1^{(u)^r}) \approx (\underline{y}_2^{(p)^r}) \end{aligned} \tag{5.2.3}$$

Hence the (u)- and the (p)-system have been computed in the same coordinate system (K-system). The index 1 or 2 for y^r indicates the measurement and it should not be mixed up with the index used in section 2.5, which indicated the K-system.

The precision of the measurements is given by the V.C. matrix of the variates x^i. This is :

$$(\sigma^{ij}) = (E\{(\tilde{x}^i - \underline{x}^i)(\tilde{x}^j - \underline{x}^j)^*\}) \tag{5.3}$$

$$i, j = 1, \ldots\ldots, n$$

the differences $\tilde{x}^i - \underline{x}^i$ are small.

This matrix gives the inner precision of the geometry of the network, so we get

for measurement 1 : (σ_1^{ij})

for measurement 2 : (σ_2^{ij})

These matrices will be used to compare both measurements, using the general eigenvalue problem ([6] § 8 ; [29] § 5 - 3):

$$\boxed{\begin{aligned} &\left|(\sigma_1^{ij}) - \lambda(\sigma_2^{ij})\right| = 0 \\ \text{or}\quad &\left|(\sigma_2^{ij}) - \mu(\sigma_1^{ij})\right| = 0 \end{aligned}} \qquad \begin{aligned}(5.4.1)\\(5.4.2)\end{aligned}$$

For these eigenvalues we can prove that ([29] § 5 - 3):

$$\mu_{max} = \frac{1}{\lambda_{min}} \quad \text{and} \quad \lambda_{max} = \frac{1}{\mu_{min}} \tag{5.5}$$

Measurement 1 will be considered as good as measurement 2 if (5.4.1) gives $\lambda_{max} \approx 1$ and if possible (5.4.2) gives $\mu_{max} \approx 1$. The choice of critical values for these eigenvalues will not be discussed here.

Measurement 1 is better than 2 if:

$$(5.4.1) \longrightarrow \lambda_{max} < 1 \quad \text{or} \quad (5.4.2) \longrightarrow \mu_{min} > 1 \quad (\text{see } (5.5)) \text{ and v.v.}$$

If both measurements have exactly the same precision, (5.4.1) and (5.4.2) should not lead to any preference and the results should be:

$$\lambda_{max} = \lambda_{min} = \mu_{max} = \mu_{min} = 1 \longrightarrow (\sigma_1^{ij}) = (\sigma_2^{ij}) \tag{5.6}$$

In geodetic practice one often prefers to make such a comparison by means of the precision of computed coordinates. One reason for this is that criterion matrices as developed in [6] can be formulated irrespective of the structure of the network. This means, according to (5.2.1-2) and (5.3), a comparison of:

$$(\sigma_1^{(u)^{rs}}) = (E\{(\tilde{y}_1^{(u)^r} - \underline{y}_1^{(u)^r}),(\tilde{y}_1^{(u)^s} - \underline{y}_1^{(u)^s})^*\})\ \underline{\text{say}}\ \overline{(\tilde{y}_1^{(u)^r} - \underline{y}_1^{(u)^r}),(\tilde{y}_1^{(u)^s} - \underline{y}_1^{(u)^s})^*} \tag{5.7.1}$$

and $$(\sigma_2^{(p)^{rs}}) = \overline{(\tilde{y}_2^{(p)^r} - \underline{y}_2^{(p)^r}),(\tilde{y}_2^{(p)^s} - \underline{y}_2^{(p)^s})^*} \tag{5.7.2}$$

$(y^s)^*$ is the transpose of (y^r)

according to (2.6) and (2.13):

$$(\tilde{y}_1^{(u)^r} - y_1^{(u)^r}) = (U_i^r)(\tilde{x}_1^i - \underline{x}_1^i) \longrightarrow (\tilde{y}_1^{(u)^s} - \underline{y}_1^{(u)^s})^* = (\tilde{x}_1^i - \underline{x}_1^j)^*(U_j^s)^* \qquad (5.8.1)$$

$(U_j^s)^*$ is the transpose of (U_i^r)

$$(\tilde{y}_2^{(p)^r} - \underline{y}_2^{(p)^r}) = (V_i^r)(\tilde{x}_2^i - \underline{x}_2^i) \longrightarrow (\tilde{y}_2^{(p)^s} - y_2^{(p)^s})^* = (\tilde{x}^j - \underline{x}^j)^*(V_j^s)^* \qquad (.5.8.2)$$

$(V_j^s)^*$ is the transpose of (V_i^r)

Thus instead of (5.4.1) we get:

$$\left| (\sigma_1^{(u)^{rs}}) - \lambda(\sigma_2^{(p)^{rs}}) \right| = 0 \qquad (5.9.1)$$

Introduce (5.7.1–2) and (5.8.1–2):

$$\left| (U_i^r)(\sigma_1^{ij})(U_j^s)^* - \lambda(V_i^r)(\sigma_2^{ij})(V_j^s)^* \right| = 0$$

The matrices in (5.9.1) are of order $(n+b) \times (n+b)$ and have rank n, therefore we will reduce them by taking $n \times n$ submatrices so that:

$$(\sigma_2^{(p)^{r_1 s_1}}) = (V_i^{r_1})(\sigma_2^{ij})(V_j^{s_1})^* \qquad r_1 = 1, \ldots\ldots\ldots, n$$

is non-singular. Then (5.9.1) reduces to:

$$\left| (\sigma_1^{(u)^{r_1 s_1}}) - \lambda(\sigma_2^{(p)^{r_1 s_1}}) \right| = 0 \qquad (5.9.2)$$

or

$$\left| (U_i^{r_1})(\sigma_1^{ij})(U_j^{s_1})^* - \lambda(V_i^{r_1})(\sigma_2^{ij})(V_j^{s_1})^* \right| = 0$$

$(V_i^{r_1})$ is non-singular, hence with $(\overline{V}_{r_1}^{i}) = (V_i^{r_1})^{-1}$:

$$\left| (V_{i'}^{r_1'}) \left\{ (\overline{V}_{r_1}^{i'})(U_i^{r_1})(\sigma_1^{ij})(U_j^{s_1})(\overline{V}_{s_1'}^{j'}) - \lambda(\sigma_2^{ij}) \right\} (V_{j'}^{s_1'}) \right| = 0 \qquad (5.9.3)$$

The comparison was originally based on the inner precision of the network, therefore (5.9.2 or 3) must not lead to a decision in favour of measurement 1 or 2 if:

$$(\sigma_1^{ij}) = (\sigma_2^{ij}) \qquad (5.10.1)$$

To get results similar to (5.6) the solution of (5.9.2) should be:

$$\lambda_{max} = \lambda_{min} = 1 \qquad (\sigma_1^{(u)^{r_1 s_1}}) = (\sigma_2^{(p)^{r_1 s_1}}) \qquad (5.10.2)$$

This is only true if in (5.9.3):

$$(\overline{V}^{i'}_{r_1})(U^{r_1}_i) = (\delta^{i'}_i) \longrightarrow (U^{r_1}_i) = (V^{r_1}_i) \tag{5.10.3}$$

This means that the (u)–system should be identical to the (p)–system. Hence to compare different measurements of a network, or to compare the precision of a network with a criterion matrix according to (5.9.1) and to get results similar to those of (5.4.3), the V.C. matrices of the coordinates should be computed with respect to a common S–base.
(5.9.1) should be:

$$\boxed{\left| (\sigma_1^{(p)^{rs}}) - \lambda (\sigma_2^{(p)^{rs}}) \right| = 0} \tag{5.11}$$

5.2 Pointfields with homogeneous and isotropic inner precision

The testing of the V.C. matrix of the coordinates in a pointfield requires a criterion matrix, which gives an ideal precision. The problem now is how this should be defined. In literature we find two examples of such a definition.

According to Grafarend [19,22] the coordinates in a pointfield should have an isotropic and homogeneous V.C. matrix, so it is invariant with respect to translations and rotations of the coordinate system. All points then have circular standard ellipses of the same size, whereas the size of the relative standard ellipses depends on the distance between the related points. The matrix of Grafarend has the 'Taylor–Karman structure', a special case is the 'chaotic structure' where relative standard ellipses are circles as well.

As the matrix of Grafarend refers to coordinates in an (a)–system, it is not clear how his definition of an ideal precision can be related to operational S–systems. The latter have a specified S–base, whereas the size of the point standard ellipses will increase with the distance from the point to the base. So it is not possible to realise a homogenous V.C. matrix in practice.

Baarda knew this problem, that is why his definition of an ideal precision refers to coordinates in an S–system [6, 7]. He requires that point and relative standard ellipses are circles. The radius of such a circle is a function of the relative positioning of the points with respect to the S–base. When formulated for an (a)–system the matrix has the chaotic structure mentioned before. Baarda's definition, though referring to coordinates in an S–system, is valid for the complex plane only. It is yet an open question whether it can be extended for large three-dimensional pointfields. In that case namely the circular standard ellipses should be replaced by spheres, but is that realistic in a purely geometric setting?

Experience with photogrammetric blocks and geodetic networks shows that the vertical relative positioning of points is less precise than their horizontal relative positioning, and is to a great extent stochastically independent of it. For large pointfields, the earth has approximately the shape of a sphere, and in that case, the criteria for the precision of the positioning of points on the sphere can be developed independently from the criteria for the determination of spherical heights.

If the earth is considered to have an ellipsoidal shape, then networks can, with some approximation, be computed on an osculating sphere with the tangent point in the centre of the network.

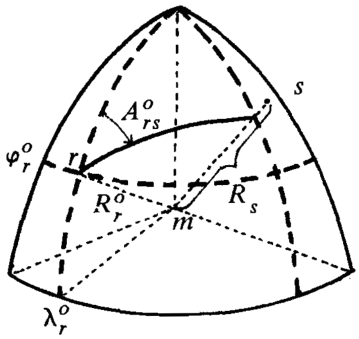
For the positioning of a point, i, on the sphere we compute the coordinates φ_i and λ_i as in section 3.3.3 and for the vertical positioning the radius R_i. For the computation of spherical coordinates, an S–base is required. If the base points are r and s, then the length scale for computing R_i will be defined by means of R^o_r. This S-base on the sphere is equivalent to an S–base in R_3 which consists of the points m = the centre of the sphere, r = a point on the sphere,and s = a point on or near the surface of the sphere. This S–system cannot be used in practice, because the point m will never be part of a real network. So we have in fact an (a)-system which will be used to design a criterion matrix. The spherical coordinates and the coordinate computation in R_3 by means of quaternions, are linked by the following definitions:

centre of the sphere m : $q^o_m = 0$

point r : $q^o_r = q^o_m + q^o_{mr}$

$$= q^o_m + R^o_r(0 + i\cos\varphi^o_r \cos\lambda^o_r + j\cos\varphi^o_r\sin\lambda^o_r + k\sin\varphi^o_r)$$

R^o_r is an approximate value of the radius at r (5.12.1)



point s : $q^{(m,r;s)}_s = Q^{(m,r;s)}_{rms} \cdot q^o_{mr} + q^o_m$ (5.12.2)

where : $Q^{(m,r;s)}_{rms} = \dfrac{R_s}{R^o_r}(\cos\bar{\alpha}_{rms} + e^o_{rms}\sin\bar{\alpha}_{rms})$

with $\bar{\alpha}_{rms} = a_{rs}$

and $e^o_{rms} = e_{rms}(\varphi^o_r, \lambda^o_r, A^o_{rs})$

because $e^o_{rms} \perp$ plane r,m,s

Point i in this system is:

$$q^{(r,m;s)}_i = Q^{(r,m;s)}_{rmi}\, q^o_{mr} + q^o_m = v_{rmi}(\cos a_{ri} + e^{(r,m;s)}_{rmi}\sin a_{ri}) \cdot q^o_{mr} + q^o_m$$

with $v_{rmi} = \dfrac{R_i}{R^o_r}$

hence

$$\dot{q}^{(r,m;s)}_i = q^{(r,m;s)}_{mi} + q^o_m = R^{(r,m;s)}_i \cdot e^{(r,m;s)}_{mi} + q^o_m$$

So the vertical positioning in the $(r,m;s)$-system is determined by

$$R^{(r,m;s)}_i = v_{rmi} \cdot R^o_r$$

here, point r is the datum point and we will write shortly:

$$R^{(r)}_i = v_{rmi} \cdot R^o_r \quad , \quad \ln R^{(r)}_i - \ln R^o_r = \ln v_{rmi}$$

"Horizontal" positioning of i is determined by:

$$e_i^{(r,m;s)} = e\{\varphi_i^{(r,m;s)},\ \lambda_i^{(r,m;s)}\}$$

Differential equations at i are:

$$q_{mi}^{-1}\ \Delta\underline{q}_{mi}^{(r,m;s)} = q_{mi}^{-1}\ \Delta\underline{q}_i^{(r,m;s)} = \Delta ln\ \underline{R}_i^{(r)} + e_{mi}^{-1}\ \Delta\underline{e}_{mi}^{(r,m;s)}$$

the coefficients of the difference quantities must be computed in a closed system of approximate values. In this expression is:

$$\Delta ln\ \underline{R}_i^{(r)} = \Delta ln\ \underline{v}_{rmi}$$

and $\quad e_{mi}^{-1}\ \Delta\underline{e}_{mi}^{(r,m;s)} = e'_{mi}\cos\varphi_i\ \Delta\underline{\lambda}_i^{(r,m;s)} + e''_{mi}\ \Delta\underline{\varphi}_i^{(r,m;s)}$

see Quee [40]:

$$e' = (0, \cos\lambda\ \sin\varphi, \sin\lambda\ \sin\varphi, -\cos\varphi)$$

$$e'' = (0, -\sin\lambda, \cos\lambda, 0)$$

The design of a criterion matrix over the sphere can now refer to the differential variates:

For "horizontal" positioning:

$$(\cos\varphi_i\ \Delta\underline{\lambda}_i^{(r,m;s)},\ \Delta\underline{\varphi}_i^{(r,m;s)}) \overset{\text{say}}{=} (\Delta\underline{u}_i^{(r,m;s)},\ \Delta\underline{v}_i^{(r,m;s)})$$

and for "vertical" positioning:

$$\Delta ln\ \underline{R}_i^{(r)}$$

The spherical coordinates can also be written as (see section 3.3.3):

$$(\Delta\underline{u}_i^{(r;s)},\ \Delta\underline{v}_i^{(r;s)})$$

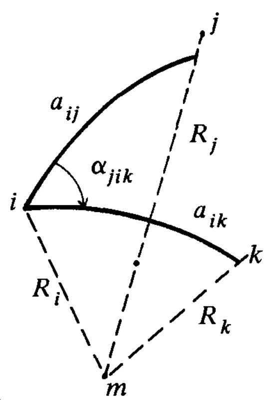
From the characteristics of the present S–system it becomes clear that we, unlike Baarda in [6], are not able to construct a criterion matrix which gives circular standard ellipses for our horizontal positioning. Hence we cannot construct an isotropic matrix for spherical coordinates. But this is not necessary, because in the sections 5.1.1 - 2 we saw that the precision of a point field is determined by its inner precision. So the construction of a criterion matrix should be based on requirements for the precision of the form elements of a network. These are:

$\Delta ln\ \underline{v}_{rmi}$ for vertical positioning

$\Delta\underline{a}_{ri}$ and $\Delta\underline{\alpha}_{sri}$ for horizontal positioning

In an ideal case the precision of these elements should only be dependent on the relative positioning of the points to which they refer and not on their location in the point field. In this respect we will talk about a point field with a "homogeneous and isotropic inner" precision:

Def. 5.III



A geodetic point field over a sphere has a "homogeneous and isotropic" inner precision if:

- for side a_{ij}

$$\sigma_{a_{ij},\, a_{ij}} = 2d^2_{ij} = 2d^2_{ji} = 2f\{a_{ij}\} = 2f\{a_{ji}\}$$

- for angle α_{jik}

$$\sigma_{\alpha_{jik},\, \alpha_{jik}} = G_{jik} = G_{kij} = g\{a_{ij},\, a_{jk},\, a_{ki}\}$$

- for ratio of lengths v_{imj}

$$\sigma_{ln\, v_{imj},\, ln\, v_{imj}} = 2k^2_{ij} = 2h\{a_{ij}\} = 2h\{a_{ji}\}$$

So according to this definition the functions k^2_{ij}, d^2_{ij} and G_{jik} are independent of the location and orientation of triangle j, i, k, that is, they are independent of e.g. the spherical coordinates (φ_i, λ_i) of point i and the azimuth A_{ij} of side i, j.

For testing the V.C. matrix of a network, a criterion matrix will be used which gives a homogeneous and isotropic inner precision.

For the design of a criterion matrix we will assume that a_{ij} and $ln\, v_{imj}$ are not correlated, i.e. the relative horizontal positioning over the sphere is not considered to be correlated to the relative positioning in the vertical sense. So besides def. 5.III we have:

$$\boxed{\sigma_{a_{ij},\, ln\, v_{imj}} = 0} \qquad (5.13)$$

5.3 A criterion matrix for point fields over the sphere

The structure of two matrices will be given in this section, one for spherical coordinates and one for spherical heights. These two matrices can be developed independent of each other because of assumption (5.13). The fact that a matrix is developed for spherical coordinates may seem to be in conflict with the conclusions of section 3.3.3. It was stated there that such coordinates could not be defined with sufficient precision. One should keep in mind however that the spherical coordinates only serve as an (a)-system in which a matrix, which has the characteristics of def. 5.III, can easily be generated. In section 5.4 we will see how the two matrices can be transformed to an operational S-system.

5.3.1 A criterion matrix for spherical coordinates

The design of a criterion matrix for spherical coordinates will be based on the discussion in section 5.2 which resulted in def. 5.III. Hence from the matrix we should be able to obtain:

$$\sigma_{a_{ij}, a_{ij}} \underline{\underline{\text{say}}}\ \overline{a_{ij}, a_{ij}} = 2d^2_{ij} = 2d^2_{ji} \qquad (5.14.1)$$

$$\sigma_{\alpha_{jik}, \alpha_{jik}} \underline{\underline{\text{say}}}\ \overline{\alpha_{jik}, \alpha_{jik}} = G_{jik} = G_{kij} \qquad (5.14.2)$$

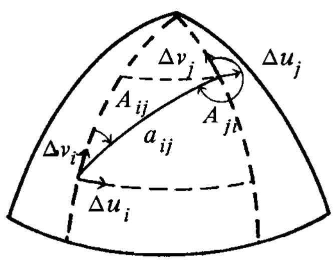
where d^2_{ij} and G_{jik} are independent of position and orientation of side a_{ij} and triangle j,i,k. We will first consider $\overline{a_{ij},a_{ij}}$. Definition 5.III stated:

$$\overline{a_{ij},a_{ij}} = 2d^2_{ij} \quad \text{with} \quad d^2_{ij} = f\{a_{ij}\}$$

Three restrictions will be made for $f\{a_{ij}\}$ which seem reasonable in the view of modern research on the precision of geodetic networks [9,12]:

$$\left.\begin{array}{ll} - f\{0\} = 0 & \text{a)} \\ - f\{a\} \text{ is continuous and monotonous nondecreasing for } 0 \leqslant a \leqslant \pi & \text{b)} \\ - f\{a\} = f\{2\pi - a\} & \text{c)} \end{array}\right\} \qquad (5.15)$$

For the derivation of a matrix which satisfies (5.14.1–2) we make use of the isometric coordinates introduced in section 5.2:



$$\Delta u_i = \cos\varphi_i \, . \, \Delta\lambda_i$$

$$\Delta v_i = \Delta\varphi_i$$

The linearised relationship between side a_{ij} and the coordinates of the points i and j is:

$$\Delta a_{ij} = -\sin A_{ij}\Delta u_i - \cos A_{ij}\Delta v_i - \sin A_{ji}\Delta u_j - \cos A_{ji}\Delta v_j \qquad (5.16)$$

The variance–covariance matrix for the isometric coordinates is constructed of submatrices such as:

$$\begin{pmatrix} \overline{u_i u_i} & \overline{u_i v_i} \\ \overline{v_i u_i} & \overline{v_i v_i} \end{pmatrix} \ \underline{\underline{\text{say}}} \ (\sigma_{ii})$$

$$\begin{pmatrix} \overline{u_i u_j} & \overline{u_i v_j} \\ \overline{v_i u_j} & \overline{v_i v_j} \end{pmatrix} \ \underline{\underline{\text{say}}} \ (\sigma_{ij}) \qquad (5.17)$$

If we write:

$$(-\sin A_{ij} - \cos A_{ij})\,(\sigma_{ii}) \begin{pmatrix} -\sin A_{ij} \\ -\cos A_{ij} \end{pmatrix} \ \underline{\underline{\text{say}}} \ c^{(i,j)}_{ii}$$

$$(-\sin A_{ij} - \cos A_{ij})\,(\sigma_{ij}) \begin{pmatrix} -\sin A_{ji} \\ -\cos A_{ji} \end{pmatrix} \ \underline{\underline{\text{say}}} \ - c^{(i,j)'}_{ij} \qquad (5.18.1)$$

$$(-\sin A_{ji} - \cos A_{ji})\,(\sigma_{jj}) \begin{pmatrix} -\sin A_{ji} \\ -\cos A_{ji} \end{pmatrix} \ \underline{\underline{\text{say}}} \ c^{(i,j)}_{jj}$$

then we get from (5.14.1), (5.16) and (5.18.1):

$$\overline{a_{ij}, a_{ij}} = c^{(i,j)}_{ii} - 2c^{(i,j)}_{ij} + c^{(i,j)}_{jj} = 2d^2_{ij} \tag{5.18.2}$$

From (5.15.a) follows:

$$\lim_{j \to i} \overline{a_{ij}, a_{ij}} = 0 \tag{5.19.1}$$

Hence with

$$\lim_{j \to i} c^{(i,j)}_{jj} = c^{(i,j)}_{ii} \tag{5.19.2}$$

(5.18.2) and (5.19.1) lead to:

$$\lim_{j \to i} c^{(i,j)}_{ij} = c^{(i,j)}_{ii} \tag{5.19.3}$$

Because of (5.18.2) and (5.19.3):

$$c^{(i,j)}_{ij} = ½(c^{(i,j)}_{ii} + c^{(i,j)}_{jj}) - d^2_{ij} \tag{5.19.4}$$

The coordinate computation on the sphere implies that only for the points s and r of the S–base one finds:

$$c^{(r,s)}_{rr} = 0, \quad c^{(r,s)}_{rs} = 0 \text{ and } c^{(r,s)}_{ss} = 2d^2_{rs}$$

in general one wants that for points not belonging to the S–base:

(σ_{ii}) is nonsingular

(σ_{ij}) is nonsingular

The following derivations will be based on this nonsingularity. (5.18.2) will satisfy def. 5.III if $c^{(i,j)}_{ii}, c^{(i,j)}_{jj}$ and $c^{(i,j)}_{ij}$ are independent of (φ_i, λ_i) and (φ_j, λ_j) and thus independent of A_{ij} and A_{ji}.

- <u>Ad $c^{(i,j)}_{ii}$ and $c^{(i,j)}_{jj}$</u>

The precision of the coordinates of a point in an S–system should only depend on its relative positioning with respect to the S–base. In that case, its precision is not effected by points in its vicinity, hence (σ_{ii}) should be independent of A_{ij}. This restriction for $c^{(i,j)}_{ii}$ would then be satisfied by the choice:

$$(\sigma_{ii}) = \begin{pmatrix} c_{ii} & 0 \\ 0 & c_{ii} \end{pmatrix} \text{ then } c^{(i,j)}_{ii} = c_{ii} \tag{5.20.1}$$

and similarly

$$(\sigma_{jj}) = \begin{pmatrix} c_{jj} & 0 \\ 0 & c_{jj} \end{pmatrix} \text{ then } c^{(i,j)}_{jj} = c_{jj} \tag{5.20.2}$$

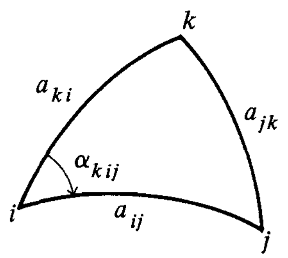
These matrices give circular standard ellipses, but in section 5.2 we stated that such a solution was not possible in an S-system on the sphere. Therefore the introduction of (5.20.1 and 2) means that we first develop a matrix in an (a)-system on the sphere. This matrix can be transformed later to an S-system on the sphere by means of an S-transformation which will be given in section 5.4.

- Ad $c_{ij}^{(i,j)}$

For the case where matrix (σ_{ij}) is nonsingular, the restriction for $c_{ij}^{(i,j)}$ will be satisfied if:

$$(\sigma_{ij}) = -\begin{pmatrix} sinA_{ij} & -cosA_{ij} \\ cosA_{ij} & sinA_{ij} \end{pmatrix} \begin{pmatrix} c_{ij}^{11} & c_{ij}^{12} \\ c_{ij}^{21} & c_{ij}^{22} \end{pmatrix} \begin{pmatrix} sinA_{ji} & cosA_{ji} \\ -cosA_{ji} & sinA_{ji} \end{pmatrix} \tag{5.21}$$

where the elements of the central matrix are independent of A_{ij} and A_{ji}.
We assume that they are a function of a_{ij}.
We will use (5.14.2) to work out (5.21) in more detail:



In triangle k, i, j is:

$$cos\alpha_{kij} = \frac{cos(a_{jk}) - cos(a_{ij})cos(a_{ik})}{sin(a_{ij})sin(a_{ik})}$$

The linearisation of this relationship gives:

$$\Delta\alpha_{kij} = k_1 \Delta a_{ik} + k_2 \Delta a_{kj} + k_3 \Delta a_{ij}$$

where k_1, k_2, k_3 are functions of a_{ij}, a_{jk}, a_{ki} and α_{kij} only.

Hence:

$$\overline{\alpha_{kij}, \alpha_{kij}} = G_{kij} = (k_1 \;\; k_2 \;\; k_3) \overline{\begin{pmatrix} a_{ik} \\ a_{kj} \\ a_{ji} \end{pmatrix}, (a_{ik} \;\; a_{kj} \;\; a_{ji})} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \tag{5.22}$$

If all the symbolic products $\overline{a_{ij}, a_{ik}}$ are independent of the location and orientation of triangle i, j, k then that will be the case for G_{kij} as well.

According to (5.16) and (5.17) is:

$$\overline{a_{ij}, a_{ik}} = (sinA_{ij} \;\; cosA_{ij})(\sigma_{ii}) \begin{pmatrix} sinA_{ik} \\ cosA_{ik} \end{pmatrix} + (sinA_{ij} \;\; cosA_{ij})(\sigma_{ik}) \begin{pmatrix} sinA_{ki} \\ cosA_{ki} \end{pmatrix}$$

$$+ (sinA_{ji} \;\; cosA_{ji})(\sigma_{ji}) \begin{pmatrix} sinA_{ik} \\ cosA_{ik} \end{pmatrix} + (sinA_{ji} \;\; cosA_{ji})(\sigma_{jk}) \begin{pmatrix} sinA_{ki} \\ cosA_{ki} \end{pmatrix}$$

With (5.20.1-2) and (5.21) expressed for other sides too we get:

$$\overline{a_{ij}, a_{ik}} = \cos\alpha_{kij} \cdot c_{ii} - \cos\alpha_{kij} \cdot c^{11}_{ik} + \sin\alpha_{kij} \cdot c^{21}_{ik}$$

$$- \cos\alpha_{kij} \cdot c^{11}_{ji} - \sin\alpha_{kij} \cdot c^{12}_{ji} - (\cos\alpha_{ijk} \cos\alpha_{jki} c^{11}_{jk} - \sin\alpha_{ijk} \sin\alpha_{jki} c^{22}_{jk})$$

$$- (\sin\alpha_{ijk} \cos\alpha_{jki} c^{21}_{jk} - \cos\alpha_{ijk} \sin\alpha_{jki} c^{12}_{jk})$$

and with (5.19.4)

$$\overline{a_{ij}, a_{ik}} = \cos\alpha_{kij} (d^2_{ik} + d^2_{ij} - \tfrac{1}{2}(c_{kk} + c_{jj})) - (\cos\alpha_{ijk} \cos\alpha_{jki} c^{11}_{jk} - \sin\alpha_{ijk} \sin\alpha_{jki} c^{22}_{jk})$$

$$+ \sin\alpha_{kij} \cdot (c^{21}_{ik} - c^{12}_{ji}) - (\sin\alpha_{ijk} \cos\alpha_{jki} c^{21}_{jk} - \cos\alpha_{ijk} \sin\alpha_{jki} c^{12}_{jk}) \qquad (5.23)$$

The first term on the righthand side of equation (5.23) shows that $\overline{a_{ij}, a_{ik}}$ is dependent on the position of point j and k, because of the term $\tfrac{1}{2}(c_{kk} + c_{jj})$. This effect will be eliminated by the choice:

$$\boxed{c^{22}_{jk} = \cos(a_{jk})\, c^{11}_{jk}} \qquad (5.24)$$

The second term on the righthand side of (5.23) is then:

$$(\cos\alpha_{ijk} \cos\alpha_{jki} - \sin\alpha_{ijk} \sin\alpha_{kji} \cos(a_{jk}))\, c^{11}_{jk} = - \cos\alpha_{kij} c^{11}_{jk}$$

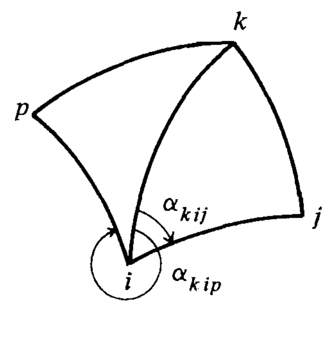
$$= \cos\alpha_{kij} (d^2_{jk} - \tfrac{1}{2}(c_{kk} + c_{jj}))$$

substitution in (5.23) gives:

$$\overline{a_{ij}, a_{ik}} = \cos\alpha_{kij} (d^2_{ik} + d^2_{ij} - d^2_{jk}) \qquad (5.23')$$

$$+ \sin\alpha_{kij} (c^{21}_{jk} - c^{12}_{ji}) - (\sin\alpha_{ijk} \cos\alpha_{jki} c^{21}_{jk} - \cos\alpha_{ijk} \sin\alpha_{jki} c^{12}_{jk})$$

G_{kij} in (5.22) should be independent of position and orientation of triangle k, i, j. This means that it should be independent of the choice of origin and orientation for the coordinate system on the sphere. But then G_{kij} should also be independent of the choice for a right- or a lefthanded coordinate system. This can be interpreted as follows:
Extend triangle k, i, j with a triangle k, i, p such that the figure becomes symmetric with respect to side $i, k,$ then:



$$\alpha_{kip} = 2\pi - \alpha_{kij}$$

$$\alpha_{ipk} = 2\pi - \alpha_{ijk}$$

$$\alpha_{pki} = 2\pi - \alpha_{jki}$$

The invariance of G_{kij} means for the symmetric figure:

$$G_{kip} = G_{jik} = G_{kij} \tag{5.25.1}$$

G_{kip} can be developed according to (5.22). Then (5.25.1) is true if:

$$\overline{a_{ip}, a_{ik}} = \overline{a_{ij}, a_{ik}} \tag{5.25.2}$$

Because of the symmetry:

$$a_{ij} = a_{ip} \longrightarrow d_{ij}^2 = d_{ip}^2$$

$$a_{kp} = a_{kj} \longrightarrow d_{kp}^2 = d_{kj}^2$$

and according to (5.21):

$$c_{ip}^{\cdot\cdot} = c_{ij}^{\cdot\cdot}$$

$$c_{pk}^{\cdot\cdot} = c_{jk}^{\cdot\cdot}$$

With these relationships we find in triangle p, i, k:

$$\overline{a_{ip}, a_{ik}} = \cos\alpha_{kij}\,(d_{ik}^2 + d_{ij}^2 - d_{jk}^2) +$$

$$- \sin\alpha_{kij}\,(c_{jk}^{21} - c_{ji}^{12}) + (\sin\alpha_{ijk}\cos\alpha_{jki}\,c_{jk}^{21} - \cos\alpha_{ijk}\sin\alpha_{jki}\,c_{jk}^{12})$$

If we compare this expression with (5.23′) than it becomes clear that (5.25.2) and hence (5.25.1) will be satisfied if:

$$\boxed{c_{pq}^{12} = c_{pq}^{21} = 0 \quad \text{for all } p, q} \tag{5.26}$$

(5.25) and (5.26) substituted in (5.21) give a solution for (σ_{ij}):

$$\boxed{(\sigma_{ij}) = - \begin{pmatrix} \sin A_{ij} & -\cos A_{ij} \\ \cos A_{ij} & \sin A_{ij} \end{pmatrix} \begin{pmatrix} c_{ij}^{11} & 0 \\ 0 & \cos(a_{ij})c_{ij}^{11} \end{pmatrix} \begin{pmatrix} \sin A_{ij} & \cos A_{ij} \\ -\cos A_{ij} & \sin A_{ij} \end{pmatrix}} \tag{5.27.1}$$

(5.19.2) will be satisfied if:

$$\boxed{c_{ii} = c_{jj} = d^2} \tag{5.28.1}$$

Hence:

$$\boxed{(\sigma_{ii}) = \begin{pmatrix} d^2 & 0 \\ 0 & d^2 \end{pmatrix}} \tag{5.27.2}$$

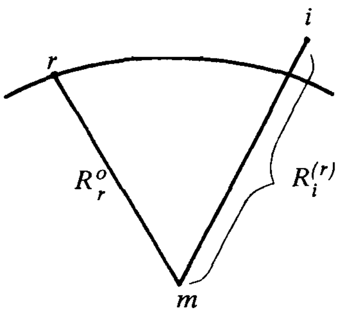
and consequently (see (5.19.4)):

$$\boxed{c^{(i,j)}_{ij} = c^{11}_{ij} = d^2 - d^2_{ij}} \tag{5.28.2}$$

The combination of (5.27.1 and 2) gives the criterion variance-covariance matrix for the points i and j:

$$\begin{array}{c|cc:cc} & u_i & v_i & u_j & v_j \\ \hline u_i & (\sigma_{ii}) & & (\sigma_{ij}) & \\ v_i & & & & \\ \hdashline u_j & (\sigma_{ji}) & & (\sigma_{jj}) & \\ v_j & & & & \end{array} \tag{5.29}$$

5.3.2 *A criterion matrix for vertical positioning over the sphere*



In this section the derivations will refer to the $(r,m;s)$-system of (5.12.1-2). The vertical positioning of points is then determined by:

$$ln\, R^{(r)}_i = ln\, v_{rmi} + ln\, R^o_r$$

The relative positioning of i with respect to r can also be written as:

$$ln\, R^{(r)}_i - ln\, R^o_r = ln\, v_{rmi}$$

For points j and r this is:

$$ln\, R^{(r)}_j - ln\, R^o_r = ln\, v_{rmj}$$

and thus for points j and i:

$$ln\, R^{(r)}_j - ln\, R^{(r)}_i = ln\, v_{rmj} - ln\, v_{rmi} = ln\, v_{imj}$$

According to def. 5.III the precision of this relative positioning should be:

for points j and i:

$$\sigma_{ln\, v_{imj},\, ln\, v_{imj}} = 2k^2_{ij}$$

for points i and r:

$$\sigma_{ln\, v_{rmi},\, ln\, v_{rmi}} = 2k^2_{ri}$$

and for points j and r:

$$\sigma_{ln\, v_{rmj},\, ln\, v_{rmj}} = 2k^2_{rj}$$

Now,

$$\sigma_{ln\, v_{imj},\, ln\, v_{imj}} = \sigma_{ln\, v_{rmj},\, ln\, v_{rmj}} - 2.\sigma_{ln\, v_{rmj},\, ln\, v_{rmi}} + \sigma_{ln\, v_{rmi},\, ln\, v_{rmi}}$$

hence

$$\boxed{\sigma_{ln\, v_{rmj},\, ln\, v_{rmi}} = \sigma_{ln\, R_j^{(r)},\, ln\, R_i^{(r)}} = k_{rj}^2 + k_{ri}^2 - k_{ij}^2}$$

For the vertical positioning we find then with respect to datum point r the symmetric matrix:

$$\begin{array}{c|ccc} & ln\, R_s^{(r)} & ln\, R_i^{(r)} & ln\, R_j^{(r)} \\ \hline ln\, R_s^{(r)} & 2k_{rs}^2 & k_{rs}^2 + k_{ri}^2 - k_{si}^2 & k_{rs}^2 + k_{rj}^2 - k_{sj}^2 \\ ln\, R_i^{(r)} & '' & 2k_{ri}^2 & k_{ri}^2 + k_{rj}^2 - k_{ij}^2 \\ ln\, R_j^{(r)} & '' & '' & 2k_{rj}^2 \end{array} \qquad (5.30)$$

Here too it seems to be reasonable to state for the covariance function:

$$k_{ij}^2 = h\{a_{ij}\}$$

$$\left.\begin{array}{l} -\ h\{0\} = \{0\} \\ -\ h\{a\} \text{ is continuous and monotonous non decreasing for } 0 \leqslant a \leqslant \pi \\ -\ h\{a\} = h\{2\pi - a\} \end{array}\right\} \qquad (5.31)$$

According to the discussion of (5.20.1–2) the matrices (5.27.1–2) have been formulated for an (a)–system on the sphere, so an S–transformation is required to obtain a criterion matrix for spherical coordinates computed in a proper S–system. Matrix (5.30) has been formulated, however, with respect to the datumpoint r, so if (5.29) is transformed to base $(r;s)$ then the two matrices can be combined to a three–dimensional criterion matrix in the $(r,m;s)$–system of section 5.2.

The matrices have been developed so that they have a structure which satisfies def. 5.III. This characteristic of the matrices will not be effected by S–transformations. The fact that the matrices have such a structure is, however, not sufficient to make them useful as criterion matrices.

They should be positive definite as well, and whether that is true depends on the choice of the functions d_{ij}^2 and k_{ij}^2. This problem will be discussed in section 5.5.

5.4 S–transformations on the sphere

To transform matrix (5.29) from the (a)–system to the $(r;s)$–system, the S–transformation for spherical coordinates should be derived. This is possible, but the problem can also be tackled in another way, which seems more elegant in the scope of this publication.

The relationship between spherical coordinates and quaternions has been given in (5.12.1–2). For the case where we are only interested in the spherical coordinates of the points, these relationships can be simplified by considering the radius to all points as a constant: $R_i = R_r^o$. For a point i in the (a)–system of section 5.3.1, this gives:

$$\bar{q}_{mi}^{(a)} \stackrel{\text{say}}{=} R_r^o\,(0 + i\,.\,\cos\varphi_i^{(a)}\,.\,\cos\lambda_i^{(a)} + j\,.\,\cos\varphi_i^{(a)}\,.\,\sin\lambda_i^{(a)} + k\,.\,\sin\varphi_i^{(a)}) \quad (5.31.1)$$

If an appropriate choice is made for approximate values of these variates, then we get for difference quantities:

$$\left.\begin{aligned}\Delta\bar{q}_{mi}^{(a)} &= R_r^o\,(0 - i\,.\,\sin\lambda_i^o + j\,.\,\cos\lambda_i^o + k\,.\,0)\;\Delta u_i^{(a)} + \\ &\quad R_r^o\,(0 - i\,.\,\sin\varphi_i^o\cos\lambda_i^o - j\,.\,\sin\varphi_i^o\sin\lambda_i^o + k\,.\,\cos\varphi_i^o)\;\Delta v_i^{(a)} \\ &\text{with } \Delta u_i^{(a)} = \cos\varphi_i^o\,\Delta\lambda_i^{(a)} \text{ and } \Delta v_i^{(a)} = \Delta\varphi_i^{(a)}\end{aligned}\right| \quad (5.31.2)$$

In this (a)-system the centre of the sphere is fixed with

$$\bar{q}_m^{(a)} = 0 \quad \text{and} \quad \Delta\bar{q}_m^{(a)} = 0$$

hence

$$\Delta\bar{q}_{mi}^{(a)} = \Delta\bar{q}_i^{(a)}$$

According to section 5.2 the $(r;s)$-system for spherical coordinates is equivalent to the $(r,\ m;s)$-system for quaternions. Therefore the transformation from the (a)-system to the $(r;s)$-system, expressed for quaternions, is (see (4.15.6")):

$$\Delta\bar{q}_i^{(r,m;s)} = S^{(r,m;s)}\{\Delta\bar{q}_i^{(a)}\} \quad (5.32)$$

So matrix (5.29) can be transformed to the $(r;s)$-system by means of (5.31.2) and (5.32). The variates $\Delta\bar{q}_i^{(a)}$ are functions of $\Delta u_i^{(a)}$ and $\Delta v_i^{(a)}$ only. So, if matrix (5.29) has been designed properly, the V.C.matrix for $\Delta\bar{q}_i^{(a)}$ has a rank equal to 2n for a pointfield of n points over the sphere. The matrix for the variates $\Delta\bar{q}_i^{(r,m;s)}$ will then have a rank equal to 2n–3, because points r and s belong now to the S–base.

In the $(r,m;s)$-system it is possible to link the criterion matrix for spherical coordinates with the matrix (5.30) for spherical heights. For the positioning of points we should use then:

$$q_{mi}^{(r,m;s)} = \frac{R_i^{(r,m;s)}}{R_r^o}\;.\;\bar{q}_{mi}^{(r,m;s)}$$

$R_i^{(r,m;s)}$ is equal to $R_i^{(r)}$ in section 5.3.2. With an appropriate choice for approximate values for these variates, the following difference equations can be obtained:

$$\Delta q_i^{(r,m;s)} = q_{mi}^o\,\Delta \ln R_i^{(r,m;s)} + \frac{R_i^o}{R_r^o}\,\Delta\bar{q}_i^{(r,m;s)}$$

The variates $R_i^{(r,m;s)}$ and the spherical coordinates are not correlated, according to (5.13), hence the criterion matrix will give no correlations between $R_i^{(r,m;s)}$ and $\bar{q}_i^{(r,m;s)}$. In this combined criterion matrix the submatrix for the variates $R_i^{(r,m;s)}$ has, according to section 5.3.2, a rank equal to n–1, for a pointfield of n points over the sphere. The submatrix for the variates $\bar{q}_i^{(r,m;s)}$ has rank = 2n–3, so the rank of the total matrix is 3n–4. The fact that there is only a rank deficiency of four is due to point m, the centre of the sphere, being a part of the S–base. As this point can never be part of a real geodetic pointfield, the $(r,m;s)$-system is in fact an (a)-system. To be able to test the precision of a geodetic network, we have to transform the criterion matrix to an operational S-system, from which the base $(r,s;t)$ belongs completely to the network.

Hence we have to compute:

$$\Delta q_i^{(r,s;t)} = S^{(r,s;t)} \{ \Delta q_i^{(r,m;s)} \}$$

The criterion matrix in this new system will have a rank = $3n - 7$.

5.5 On the positive definiteness of the criterion matrices

5.5.1 Positive definite matrices

The criterion matrices for spherical coordinates and heights should be positive definite. We call the real matrix H of the order $n \times n$, positive definite if:

$$\text{for all real } \Lambda^* = (\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \neq 0 : \Lambda^* H \Lambda > 0 \tag{5.33}$$

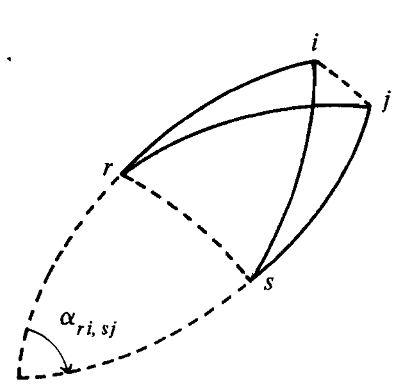
Whether this is true depends on the choice of the covariance functions for these matrices. For the matrix for spherical coordinates this function has two components: a constant part d^2 and a component d^2_{ij} which depends on the sidelength a_{ij}. However, the constant part d^2 occurs only when the matrix is formulated in an (a)-system, but as soon as it is transformed to an $(r;s)$-system, d^2 will disappear. As the criterion matrix will only be used in an $(r;s)$-system, it is, in principle, sufficient to investigate which conditions d^2_{ij} should fulfil to assure that the matrix is positive definite. Matrix (5.30) for heights has been developed in the (r)-system. The positive definiteness of this matrix depends only on k^2_{ij}, so we should also check which conditions this function should fulfil.

If a real matrix H, of order $n \times n$, is positive definite then there is a real matrix Γ of rank n such that

$$H = \Gamma^* \Gamma$$

This property will be used to establish the conditions to be fulfilled by d^2_{ij} and k^2_{ij}.

5.5.2 A decomposition of the criterion matrix for spherical coordinates



In the $(r;s)$-system φ^o_r, λ^o_r and A^o_{rs} are fixed. The other coordinates of the points in the figure can be expressed as functions of these parameters and the sides in the network. Hence the V.C. of the coordinates is a function of the V.C. matrix of the sides. Therefore we should check under which conditions the latter matrix is positive definite. If we use only a sufficient set of sides to compute the coordinates of the points s, i and j, then we get the V.C. matrix:

$$\begin{pmatrix} \overline{a_{rs}, a_{rs}} & \overline{a_{rs}, a_{ri}} & \overline{a_{rs}, a_{rj}} & \overline{a_{rs}, a_{si}} & \overline{a_{rs}, a_{sj}} \\ & \overline{a_{ri}, a_{ri}} & \overline{a_{ri}, a_{rj}} & \overline{a_{ri}, a_{si}} & \overline{a_{ri}, a_{sj}} \\ & & \overline{a_{rj}, a_{rj}} & \overline{a_{rj}, a_{si}} & \overline{a_{rj}, a_{sj}} \\ \text{symmetric} & & & \overline{a_{si}, a_{si}} & \overline{a_{si}, a_{sj}} \\ & & & & \overline{a_{sj}, a_{sj}} \end{pmatrix} \tag{5.34}$$

The elements of this matrix are (see (5.16) and (5.27.1)):

$$\overline{a_{ri}, a_{sj}} = (\sin A_{ri} \quad \cos A_{ri}) \left\{ (\sigma_{rs}) \begin{pmatrix} \sin A_{sj} \\ \cos A_{sj} \end{pmatrix} + (\sigma_{rj}) \begin{pmatrix} \sin A_{js} \\ \cos A_{js} \end{pmatrix} \right\}$$

$$+ (\sin A_{ir} \quad \cos A_{ir}) \left\{ (\sigma_{is}) \begin{pmatrix} \sin A_{sj} \\ \cos A_{sj} \end{pmatrix} + (\sigma_{ij}) \begin{pmatrix} \sin A_{js} \\ \cos A_{js} \end{pmatrix} \right\}$$

Further elaboration gives with (5.27.1) and (5.28.2):

$$\overline{a_{ri}, a_{sj}} = \cos\alpha_{ri,sj} \, (d_{is}^2 + d_{rj}^2 - d_{rs}^2 - d_{ij}^2) \stackrel{\text{say}}{=\!=} \cos\alpha_{ri,sj} \quad D_{ri,sj}$$

$\alpha_{ri,sj}$ is the angle at the intersection of the large circles containing the sides a_{ri} and a_{sj} (see the figure). This is the angle between the planes m, r, i and m, s, j respectively, m is the centre of the sphere.

Let $n_{ri}^T = (n_{ri}^1 \;\; n_{ri}^2 \;\; n_{ri}^3)$ be a normal vector of unit length to plane m, r, i and let n_{sj} be a similar normal vector to plane m, s, j. The inproduct of these vectors is $n_{ri}^T \, . \, n_{sj}$ = $\cos\overline{\alpha}_{ri,sj}$, so we can write:

$$\overline{a_{ri}, a_{sj}} = (n_{ri}^1 \;\; n_{ri}^2 \;\; n_{ri}^3) \begin{pmatrix} D_{ri,sj} & 0 & 0 \\ 0 & D_{ri,sj} & 0 \\ 0 & 0 & D_{ri,sj} \end{pmatrix} \begin{pmatrix} n_{sj}^1 \\ n_{sj}^2 \\ n_{sj}^3 \end{pmatrix} \tag{5.35}$$

One should notice that if $a_{ri} = \pi$ then r, m and i are on a straight line and n_{ri} is not defined. The covariance $\overline{a_{ri}, a_{sj}}$ can then only be evaluated if a specific direction for side a_{ri} has been chosen.

With (5.28.2) $D_{ri,sj}$ can be written as:

$$D_{ri,sj} = c_{ij}^{11} + c_{rs}^{11} - c_{is}^{11} - c_{rj}^{11} \tag{5.36}$$

The covariance function c_{ij}^{11} is defined as a function of a_{ij}

$$\boxed{c_{ij}^{11} = c\{a_{ij}\} \qquad 0 \leqslant a \leqslant 2\pi} \tag{5.37}$$

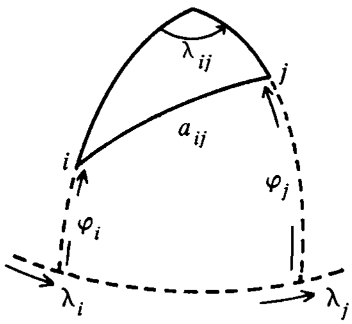
it can be expanded by the series (see [31]):

$$\boxed{c\{a\} = \sum_{n=0}^{\infty} \beta_n \cos(n.a)} \tag{5.37'}$$

The function $cos(n.a_{ij})$ can be decomposed in trigonometric functions of the spherical coordinates of the points i and j:

$$cos(n.a_{ij}) = sin(n.\varphi_i)\, sin(n.\varphi_j) + cos(n.\varphi_i) \,.\, cos(n.\varphi_j) \,.\, cos(n.\lambda_i) \,.\, cos(n.\lambda_j)$$

$$+ cos(n.\varphi_i) \,.\, cos(n.\varphi_j) \,.\, sin(n.\lambda_i) \,.\, sin(n.\lambda_j)$$



If in (5.37′) all $\beta_n \geqslant 0$ then we can define a vector:

$$\bar{c}_i = \begin{pmatrix} \beta_o \\ \beta_1 \; sin\varphi_i \\ \beta_1 \; cos\varphi_i \; cos\lambda_i \\ \beta_1 \; cos\varphi_i \; sin\lambda_i \\ \vdots \\ \beta_n \; cos(n.\varphi_i) \\ \beta_n \; cos(n.\varphi_i) \,.\, cos(n.\lambda_i) \\ \beta_n \; cos(n.\varphi_i) \,.\, sin(n.\lambda_i) \end{pmatrix} \qquad n \longrightarrow \infty$$

With such vectors it is possible to write c_{ij}^{11} as an inproduct:

$$\boxed{\begin{array}{l} c_{ij}^{11} = \bar{c}_i^{\,*} \,.\, \bar{c}_j \\ \text{only if all } \beta_n \geqslant 0 \end{array}} \tag{5.38}$$

In this notation

$$c_{ii}^{11} = \bar{c}_i^{\,*} . \bar{c}_i = \sum_{n=0}^{\infty} \beta_n cos(n.a_{ii}) = \sum_{n=0}^{\infty} \beta_n$$

and $$2d_{ij}^2 = (\bar{c}_j^{\,*} - \bar{c}_i^{\,*}) . (\bar{c}_j - \bar{c}_i) = c_{ii}^{11} + c_{jj}^{11} - 2c_{ij}^{11}$$

$$= 2\sum_{n=0}^{\infty} \beta_n - 2\sum_{n=0}^{\infty} \beta_n \; cos(n.a_{ij})$$

This is consistent with (5.15) because $d_{ij}^2 \longrightarrow 0$ for $a_{ij} \longrightarrow 0$.

Now (5.36) can be factorised:

$$D_{ri,sj} = (\bar{c}_i^{\,*} - \bar{c}_r^{\,*}) . (\bar{c}_j - \bar{c}_s) \;\underline{\underline{\text{say}}}\; \bar{c}_{ri}^{\,*} . \bar{c}_{sj}$$

and (5.35) can be decomposed according to:

$$\overline{a_{ri} . a_{sj}} = (n_{ri}^1 \;\; n_{ri}^2 \;\; n_{ri}^3) \begin{pmatrix} \bar{c}_{ri}^{\,*} & 0 & 0 \\ 0 & \bar{c}_{ri}^{\,*} & 0 \\ 0 & 0 & \bar{c}_{ri}^{\,*} \end{pmatrix} \begin{pmatrix} \bar{c}_{sj} & 0, & 0 \\ 0 & \bar{c}_{sj} & 0 \\ 0 & 0 & \bar{c}_{sj} \end{pmatrix} \begin{pmatrix} n_{sj}^1 \\ n_{sj}^2 \\ n_{sj}^3 \end{pmatrix} \;\underline{\underline{\text{say}}}$$

$$= n_{ri}^{*} \,.\, \bar{\bar{c}}_{ri}^{\,*} \,.\, \bar{\bar{c}}_{sj} \,.\, n_{sj}$$

hence we obtain for (5.34):

$$\text{matrix (5.34)} = \begin{pmatrix} n^*_{rs} & & & & \\ & n^*_{ri} & & 0 & \\ & & n^*_{rj} & & \\ & 0 & & n^*_{si} & \\ & & & & n^*_{sj} \end{pmatrix} \begin{pmatrix} \bar{\bar{c}}^*_{rs} \\ \bar{\bar{c}}^*_{ri} \\ \bar{\bar{c}}^*_{rj} \\ \bar{\bar{c}}^*_{si} \\ \bar{\bar{c}}^*_{sj} \end{pmatrix} (\bar{\bar{c}}_{rs}\ \bar{\bar{c}}_{ri}\ \bar{\bar{c}}_{rj}\ \bar{\bar{c}}_{si}\ \bar{\bar{c}}_{sj}) \begin{pmatrix} n_{rs} & & & & \\ & n_{ri} & & 0 & \\ & & n_{rj} & & \\ & 0 & & n_{si} & \\ & & & & n_{sj} \end{pmatrix} \qquad (5.34')$$

This matrix can be extended for each point added to the system. If the number of points in the network is np then the rank of the first and the last matrices in the product is $2 \times np - 3$. If in (5.37′) an infinite number of $\beta_n > 0$ then the rank of the second and the third matrices is $3 \times (np - 1)$ because $\bar{\bar{c}}_{si} = \bar{\bar{c}}_{ri} - \bar{\bar{c}}_{rs}$. Consequently the rank of matrix (5.34) is then $(2 \times np - 3)$. If only a finite number, k, of coefficients $\beta_n > 0$, then the maximum rank of the two central matrices is:

$$\text{max. rank} = \text{minimum}\,(3 \times (np - 1), 9 \times k)$$

then there will be a rank deficiency in (5.34) if np is too large, e.g.: if only $\beta_1 > 0$ then the max. rank of (5.34) is 9. This leads to the conclusion that:

> To generate a positive definite criterion matrix which has a full rank for any size of network, the expansion (5.37′) must have an infinite number of coefficients $\beta_n > 0$ and no $\beta_n < 0$. (5.39)

Only under this condition c^{11}_{ij} can be used as a covariance function.

5.5.3 A decomposition of the criterion matrix for spherical heights

The decomposition of the criterion matrix for spherical heights can be tackled in a similar way. First we define the function G_{ij} as:

$$2\,G_{ij} = 2\,k^2 - 2\,k^2_{ij} = G_{ii} + G_{jj} - 2\,k^2_{ij} \qquad \text{with } G_{ii} = G_{jj} = k^2$$

G_{ij} can be expanded by the series:

$$\boxed{G_{ij} = \sum_{n=0}^{\infty} \gamma_n \cos(n.a_{ij})}$$

If we define $\bar{G}_i$ and $\bar{G}_{ri}$ similar to $\bar{c}_i$ and $\bar{c}_{ri}$, then G_{ij} can be decomposed according to:

$$\boxed{G_{ij} = \bar{G}^*_i \,.\, \bar{G}_j}$$

only if all $\gamma_n \geqslant 0$

Then the elements of (5.30) are:

$$k^2_{rs} + k^2_{ri} - k^2_{si} = k^2 + G_{si} - G_{ri} - G_{rs} = \bar{G}^*_{rs} \,.\, \bar{G}_{ri}$$

so matrix (5.30) can be decomposed as follows:

$$\text{matrix } (5.30) = \begin{pmatrix} \bar{G}^*_{rs} \\ \bar{G}^*_{ri} \\ \bar{G}^*_{rj} \end{pmatrix} (\bar{G}_{rs} \ \bar{G}_{ri} \ \bar{G}_{rj})$$

This matrix can be extended for each point added to the system. If np is the number of points and k is the number of coefficients $\gamma_n > 0$ then the rank of the matrix is:

$$\text{max. rank } (5.30) = \text{min. } (np - 1, 3 \times k)$$

so only if k is infinite a matrix of full rank can always be guaranteed and hence:

conclusion (5.39) is similarly valid for the coefficients γ_n in the series expansion of G_{ij}	(5.39')

5.5.4 On the choice of a covariance function

For the criterion matrix in the complex plane Baarda proposes for d^2_{ij}:

$$d^2_{ij} = \sum_p c_p \, l^p_{ij} \qquad c_p \geqslant 0 \qquad 0 \leqslant p < 2 \qquad ([6](15, \text{a}, 25))$$

p is not necessarily an integer.

l_{ij} is the distance between the points i and j in the plane, the coefficients c_p characterise the precision of the network. The restrictions for p are necessary in order to obtain a criterion matrix which is positive definite. For the sphere a similar choice can be made for d^2_{ij} and k^2_{ij}. The following developments will only be made for d^2_{ij}, but they are equally valid for k^2_{ij}.

For the distance function in d^2_{ij} we can either use the spherical distance a_{ij} or the length of the chord l_{ij}. For the further developments we will consider the case that one of the terms in the series given by Baarda is dominant, so that the function d^2_{ij}, with sufficient approximation, can be written as:

for spherical distance

a) $d^2_{ij} = \gamma_q \, a^q_{ij} \qquad 0 \leqslant a \leqslant \pi$

$\quad d^2_{ij} = \gamma_q \, (2\pi - a_{ij})^q \qquad \pi < a \leqslant 2\pi \qquad \gamma_q > 0$

for the chord

b) $d^2_{ij} = \gamma_q \, l^q_{ij} \qquad 0 \leqslant a \leqslant 2\pi \qquad \gamma_q > 0$ (5.40)

Because of (5.15) for both cases we have $q \geqslant 0$. In the following sections we will check what further restrictions there are for q.

5.5.4.1 The spherical distance

If the spherical distance is used, the covariance function (5.37) becomes:

$$c\{a\} = d^2 - \gamma_q \, a^q \text{ for } 0 \leqslant a \leqslant \pi, \qquad c\{a\} = d^2 - \gamma_q \, (2\pi - a)^q \text{ for } \pi < a \leqslant 2\pi$$

it is now to be seen for which values of q the expansion (5.37') fulfils (5.39).

Let:

$$t = \cos a$$

Then the spherical distance is:

$$a = \arccos t = \frac{\pi}{2} - \left(t + \frac{1}{2} \cdot \frac{t^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{t^5}{5} + \ldots\right) \quad \underline{\underline{\text{say}}} \quad \frac{\pi}{2}(1 - s)$$

and hence we get:

$$a^q = \left(\frac{\pi}{2}\right)^q \left(1 - qs + \frac{q(q-1)}{2!} s^2 - \frac{q(q-1)(q-2)}{3!} s^3 + \cdots\right) \tag{5.41}$$

s is a power series in t, therefore a^q and $c\{a\}$ are power series in t as well, so we can write:

$$c\{a\} = d^2 - \gamma_q a^q \quad \underline{\underline{\text{say}}} \quad \sum_{n=0}^{\infty} \alpha_n . t^n \tag{5.42}$$

t^n can be expressed as a binomial series:

$$t^n = \cos^n a = \frac{1}{2^n} (e^{ia} + e^{-ia})^n =$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{i(n-k)a} . e^{-ika} = \frac{1}{2^n} \sum_{k=0}^{n} e^{i(n-2k)a}$$

which means that:

$$t^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (\cos(n-2k).a + i \sin(n-2k).a)$$

the imaginary part of this sum vanishes, so:

$$t^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos(n-2k).a$$

So indeed the covariance function can be written as:

$$c\{a\} = \sum_{n=0}^{\infty} \alpha_n . t^n = \sum_{n=0}^{\infty} \beta_n \cos(n.a)$$

In this expansion all $\beta_n \geqslant 0$ if all $\alpha_n \geqslant 0$. From (5.42) it follows that all $\alpha_n \geqslant 0$ if $\frac{d^2}{\gamma_q} \geqslant \left(\frac{\pi}{2}\right)^q$ and if all the coefficients of the terms containing s in (5.41) are $\leqslant 0$. The latter condition is fulfilled if $0 \leqslant q \leqslant 1$. Then (5.40.a) satisfies (5.38). The condition for q is a sufficient one, now we will check whether it is necessary. Towards that aim we compute β_2:

$$\beta_2 = \frac{1}{2\pi} \int_0^{2\pi} c(a).\cos 2a.da = \frac{1}{2\pi}\left[\int_0^{\pi} (d^2 - \gamma_q a^q).\cos 2a.da + \right.$$

$$\left. + \int_{\pi}^{2\pi} (d^2 - \gamma_q .(2\pi - a)^q).\cos 2a.da\right]$$

$$= \frac{\gamma_q}{2\pi} \left[\int_0^{\pi} - a^q . \cos 2a.da + \int_{\pi}^{2\pi} -(2\pi - a)^q . \cos 2a.da\right]$$

Hence:

$$\beta_2 = \frac{\gamma_q}{\pi} \int_0^{\pi} - a^q . \cos 2a . da = \frac{\gamma_q}{\pi} \int_0^{\frac{\pi}{4}} - \varphi(a) . \cos 2a . da$$

$$\text{with } \varphi\{a\} = a^q + (\pi - a)^q - (\tfrac{\pi}{2} + a)^q - (\tfrac{\pi}{2} - a)^q$$

if $q < 0$ then $\varphi\{a\} > 0$ and hence $\beta_2 < 0$
if $0 \leqslant q \leqslant 1$ then $\varphi\{a\} \leqslant 0$ and hence $\beta_2 \geqslant 0$
if $q > 1$ then $\varphi\{a\} > 0$ and hence $\beta_2 < 0$

So the conclusion is that (5.40.a) satisfies (5.38) only if $0 \leqslant q \leqslant 1$ and $\frac{d^2}{\gamma_q} \geqslant (\frac{\pi}{2})^q$. The value $q = 0$ leads to the situation where all $\beta_n = 0$ for $n \geqslant 1$ in which case the criterion matrix will be singular. The value $q = 1$ gives $\beta_n = 0$ for all even $n \geqslant 2$. For some particular point configurations in the network $q = 1$ may lead to a singular matrix, so it is safe to choose $q < 1$. So if the spherical distance is used, a positive definite matrix will be generated by:

$$\boxed{\left.\begin{array}{ll} d_{ij}^2 = \gamma_q a_{ij}^q & 0 \leqslant a_{ij} \leqslant \pi \\ d_{ij}^2 = \gamma_q (2\pi - a_{ij})^q & \pi < a_{ij} \leqslant 2\pi \end{array}\right\} \gamma_q > 0 \, , \, 0 < q < 1} \tag{5.43}$$

5.5.4.2 The chord

On a sphere with radius R the relationship between the chord l_{ij} and the spherical distance a_{ij} is:

$$l_{ij}^2 = 2R^2 \, (1 - \cos a_{ij})$$

This expression introduced in (5.40.b) gives the covariance function:

$$c\{a_{ij}\} = d^2 - d_{ij}^2 = d^2 - \gamma_q l^q = d^2 - \gamma_q 2^p R^q (1 - \cos a_{ij})^p , \; p = \frac{q}{2}$$

or short:

$$c\{a\} = d^2 - \zeta_p (1 - t)^p \qquad \text{with } t = \cos a \, , \quad \zeta_p = \gamma_q 2^p R^q$$

A binomial expansion gives:

$$c\{a\} = d^2 - \zeta_p \cdot (1 - p \cdot t + p \cdot \frac{(p-1)}{2!} \cdot t^2 - \frac{p \cdot (p-1) \cdot (p-2)}{3!} . t^3 + \ldots)$$

$$= \sum_{n=0}^{\infty} \alpha_n t^n \tag{5.44}$$

From the previous section we know that all coefficients of the expansion (3.37') are $\beta_n \geqslant 0$ if all $\alpha_n \geqslant 0$. The latter condition will be satisfied if $d^2 \geqslant \zeta_p$ and $0 \leqslant p \leqslant 1$. To check whether these conditions are necessary, β_2 will be computed:

$$\beta_2 = \frac{1}{2\pi} \int_0^{2\pi} c\{a\} \cdot \cos(2 \cdot a) \cdot da = \frac{1}{2\pi} \int_0^{2\pi} (d^2 - \zeta_p (1 - \cos a)^p) . \cos(2a) . da$$

$$= -\frac{\zeta_p}{2\pi} \int_0^{2\pi} (1 - \cos a)^p \cdot \cos(2a) - da = -\frac{\zeta_p}{\pi} \int_0^{2\pi} (1 - \cos a)^p \cdot \cos(2a) \, da$$

The last integral is equivalent to:

$$\beta_2 = -\frac{\zeta_p}{\pi}\int_0^{\frac{\pi}{4}} [(1-\cos a)^p + (1-\cos(\pi-a))^p - (1-\cos(\tfrac{\pi}{2}-a))^p - (1-\cos(\tfrac{\pi}{2}-a))^p].\cos 2a.\,da =$$

$$\underline{\underline{\text{say}}}\ -\frac{\zeta}{\pi}\int_0^{\frac{\pi}{4}} \varphi\{a\}.\cos 2a.\,da$$

Hence:

$$\varphi\{a\} = (1-\cos a)^p + (1+\cos a)^p - (1-\sin a)^p - (1+\sin a)^p$$

on the interval $0 \leqslant a \leqslant \frac{\pi}{4}$ is $\cos a \geqslant \sin a$, therefore we find:

if $p < 0$ then $\varphi\{a\} > 0$ and hence $\beta_2 < 0$

if $0 \leqslant p \leqslant 1$ then $\varphi\{a\} \leqslant 0$ and hence $\beta_2 \geqslant 0$

if $p > 1$ then $\varphi\{a\} > 0$ and hence $\beta_2 < 0$

Further we find that if $p = 0$ then all $\beta_n = 0$ for $n \geqslant 1$ and if $p = 1$ then all $\beta_n = 0$ for $n \geqslant 2$, so these values lead to a singular criterion matrix. Therefore we should choose $0 < p < 1$ which means $0 < q < 2$. So if the chord is used,a positive definite criterion matrix will be generated by:

$$\boxed{d^2_{ij} = \gamma_q l^q_{ij} \qquad \gamma_q > 0 \qquad 0 < q < 2} \tag{5.45}$$

A further discussion of the function d^2_{ij} will follow in the last section.

5.6 Epilogue to chapter V

The construction of the criterion matrix in this chapter gave rise to several questions, which will be mentioned now. These problems will require some more attention in the future.

The limitations of section 5.5 leave quite a lot of freedom for the choice of a covariance function for the criterion matrices. This gives the possibility to classify point fields with respect to their precision. The functions d^2_{ij} and k^2_{ij} should be chosen so that they generate a matrix which has in some respects similar characteristics as the real V.C. matrix of a point field.

Baarda proposes in [6] that the criterion matrix gives at the same time an upperbound for the precision of the point field. This means that variances derived from it should always be larger than or equal to variances derived from the real V.C. matrix.

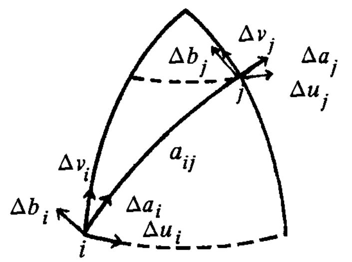
If one decides to use a function of the simple form (5.40) then a choice has to be made between (a) and (b). For networks covering small areas there is hardly any difference, but for large networks the difference becomes apparent in the conditioning of the matrix. Provisional computations showed that the chord gives a better conditioned matrix than the spherical distance, which can be understood from sections 5.5.4.1 and 5.5.4.2.

Another advantage of the chord is that it is a distance in three-dimensional space, so it can be used in a spherical coordinate system as well as in a three-dimensional coordinate system. For the case where it is necessary, it seems a minor step from the chord to the real distance between points. A third point, which is of less importance, is that (5.40.b) automatically takes care of the symmetry of d^2_{ij} with respect to $a = \pi$.

The condition $q > 0$, found in the previous sections, agrees with the constraint $\lim_{j \to i} d^2_{ij} = 0$. This means that the function d^2_{ij} cannot contain a constant term. Yet in photogrammetric blocks the variances of distances consist of a constant term and a term depending on the bridging distance between points. This is not a serious conflict if it is not required that the coefficients c_q are constant for the whole range of l_{ij}. Then, it is only in the neighbourhood of $l_{ij} = 0$ that $q > 0$ is necessary, whereas for any region $l_{ij} > \epsilon\ (\epsilon > 0)$ one could introduce a term with $q = 0$. So it should be investigated whether, and under which restrictions, it is possible to introduce different values for the coefficients c_q, for different regions for l_{ij}.

The condition $q < 1$ in (5.43) means that d^2_{ij} should be a concave function of a_{ij}, whereas $q < 2$ in (5.45) means that $\sqrt{d^2_{ij}}$ should be a concave function of l_{ij}. The latter restriction seems to agree with experience in photogrammetric blocks and planimetric geodetic networks, where such a behaviour of the variances of distances has indeed been found.

A closer look at (5.27.1) may lead to a better understanding of the structure of (φ_{ij}). This matrix can be written as the symbolic product:



$$(\sigma_{ij}) = \begin{pmatrix} \overline{u_i, u_j} & \overline{u_i, v_j} \\ \overline{v_i, u_j} & \overline{v_i, v_j} \end{pmatrix}$$

the (u, v) coordinate system can be replaced by an (a, b) system where for point i:

Δa_i is counted in the direction of the tangent to a_{ij} at i

Δb_i is perpendicular to Δa_i

and in point j:

Δa_j is counted in the direction of the tangent to a_{ij} at j

Δb_j is perpendicular to Δa_j.

The relation between the (u, v) and the (a, b) systems is:

$$\begin{pmatrix} \Delta u_i \\ \Delta v_i \end{pmatrix} = \begin{pmatrix} \sin A_{ij} & -\cos A_{ij} \\ \cos A_{ij} & \sin A_{ij} \end{pmatrix} \begin{pmatrix} \Delta a_i \\ \Delta b_i \end{pmatrix}$$

and

$$\begin{pmatrix} \Delta u_j \\ \Delta v_j \end{pmatrix} = -\begin{pmatrix} \sin A_{ji} & -\cos A_{ji} \\ \cos A_{ji} & \sin A_{ji} \end{pmatrix} \begin{pmatrix} \Delta a_j \\ \Delta b_j \end{pmatrix}$$

so we find for (σ_{ij}):

$$(\sigma_{ij}) = \begin{pmatrix} \overline{u_i, u_j} & \overline{u_i, v_j} \\ \overline{v_i, u_j} & \overline{v_i, v_j} \end{pmatrix} = -\begin{pmatrix} \sin A_{ij} & -\cos A_{ij} \\ \cos A_{ij} & \sin A_{ij} \end{pmatrix} \begin{pmatrix} \overline{a_i, a_j} & \overline{a_i, b_j} \\ \overline{b_i, a_j} & \overline{b_i, b_j} \end{pmatrix} \begin{pmatrix} \sin A_{ji} & \cos A_{ji} \\ -\cos A_{ji} & \sin A_{ji} \end{pmatrix}$$

This expression compared with (5.27.1) gives:

for the components along side a_{ij}: $\overline{a_i, a_j} = c_{ij}^{11}$

for components normal to side a_{ij}: $\overline{b_i, b_j} = cos(a_{ij})c_{ij}^{11}$

cross-correlation: $\overline{a_i, b_j} = \overline{b_i, a_j} = 0$

So the central matrix in (5.27.1) is independent of the choice of the coordinate system. The link with the actual system is made by the left- and righthand matrices in the formula for (σ_{ij}).

An important question is how matrix (5.29) is related to the criterion matrix in the complex plane [6] (15.48). In networks covering only small areas the lengths of the sides are small compared to the radius R of the sphere. This can be considered as the limiting case $R \longrightarrow \infty$, which gives:

$$\lim_{R\to\infty} A_{ij} = \pi - A_{ji}$$

$$\lim_{R\to\infty} a_{ij} = 0 \qquad \lim_{R\to\infty} cos(a_{ij})c_{ij}^{11} = c_{ij}^{11}$$

then with (5.27.1) and (5.28.2):

$$\lim_{R\to\infty} (\sigma_{ij}) = \begin{pmatrix} d^2 - d_{ij}^2 & 0 \\ 0 & d^2 - d_{ij}^2 \end{pmatrix}$$

In combination with (5.27.2) this gives for (5.29) a matrix with a structure similar to [6] (15.48). So if proper definitions are given for the transitions:

$$R\,\Delta u \longrightarrow \Delta y$$

$$R\,\Delta v \longrightarrow \Delta x$$

it will be possible to connect local or regional networks to national or even continental networks.

If the elements of (5.29) are rewritten in a proper way:

$$\begin{aligned} \overline{u_i, u_j} &= -(sinA_{ij} \,.\, sinA_{ji} + cosA_{ij} \,.\, cosA_{ji} \,.\, cosa_{ji}) \,.\, c_{ij}^{11} \\ &= -(cosA_{ij} \,.\, cosA_{ji} + sinA_{ij} \,.\, sin\,A_{ji}) \,.\, cosa_{ij} \,.\, c_{ij}^{11} \\ &\quad + (-c_{ij}^{11} + cosa_{ij} \,.\, c_{ij}^{11}) \,.\, sinA_{ij} \,.\, sinA_{ji} \end{aligned}$$

and similar for other elements and if we write:

$$cos(a_{ij})c_{ij}^{11} = \Psi\{a_{ij}\}$$

$$c_{ij}^{11} = \Omega\{a_{ij}\}$$

it appears that the expressions for the elements of (5.29) have a structure which is comparable with the Taylor–Karman structure as given by Grafarend for cartesian coordinates [19, 22]. The limit $R \longrightarrow \infty$ then gives the 'chaotic structure': $\Psi\{a_{ij}\} \longrightarrow \Omega\{a_{ij}\}$. This is a consequence of choice (5.24), an alternative choice for c^{22}_{jk} is:

$$c^{22}_{jk} = cos\{a_{jk}\}\, c^{11}_{jk} + g\{a_{jk}\}$$

It is not yet known what the characteristics of $g\{a\}$ should be, but most likely they should be similar to those formulated for $f\{a\}$ in (5.15). If $g\{a\}$ does not vanish for $R \longrightarrow \infty$ then this limit will lead to the Taylor–Karman structure for planimetric networks. For this new choice it has to be considered again which conditions c^{11}_{jk} and c^{22}_{jk} should both fulfil to generate a positive definite matrix. It is not clear as yet what advantage the Taylor–Karman structure has over the chaotic structure. In planimetric networks the latter appeared to be very useful, therefore choice (5.24) seems to be preferable.

Earlier in this epilogue we stated that the matrix of section 5.3 made it possible to design a consistent system for the analysis of the precision of networks ranging from continental to local scale. The question is when to replace the matrix for the complex plane by the matrix for the sphere, i.e.: when does the curvature of the earth have a significant effect on the precision of the computed coordinates. This has not been investigated yet in detail. Such an investigation could be based on the coordinates of points in a conformal mapping. Then two matrices can be computed, firstly matrix [6] (15.48) applied directly to the mapped points and secondly the matrix derived from the matrix for spherical coordinates. Comparison of the two will show for what size of networks the differences will be significant. Some tentative computations gave the impression that for networks up to 500 km by 500 km the difference is not large and that it grows slowly with the size for larger networks. But more exact computations should still be made.

For very large networks the earth is considered to have an ellipsoidal shape. Coordinates are then computed in an ellipsoidal system which even may be corrected for discrepancies between geoid and ellipsoid. It will be interesting therefore to develop a criterion matrix for such coordinates. But what was said about the difference between planimetric and spherical coordinates gives rise to the thought that for geodetic practice the difference between the matrix for spherical and the matrix for ellipsoidal coordinates will be negligible. This of course can only be verified after developing the matrix for the ellipsoid and establishing the relation between spherical and ellipsoidal coordinates in a way similar to the relation between the sphere and the complex plane.

The crucial point in the connection of local or regional planimetric coordinate systems with national or continental curvilinear systems is the definition of the latter. The difficulties indicated in sections 3.3 and 3.4 may rise serious problems, so that instead of using two-dimensional coordinate systems one should rather restrict oneself to three-dimensional systems.

The discussion in section 5.2 pointed out that the criterion matrix developed in this paper only represents one of the possible solutions. Another possibility is a criterion matrix which is isotropic and homogeneous throughout R_3. Baarda did preliminary studies on this approach, but no results have been published as yet. Grafarend proposed the Taylor–Karman structure for such matrices [19, 22],but as he did not refer to S–systems there is no link yet with measured networks. In the future, experience should show what the difference between the various solutions means for practice.

It appears that many questions are still open for future research. This paper took up the line pointed out by Baarda in [6] and tried to give a key for working out his concepts in some new directions, so that a framework could be constructed for further explorations.

M. Molenaar
1974 - 1980

References

1	Ackermann, F., et al.	Aerotriangulation mit unabhängigen Modellen. Bildmessung und Luftbildwesen 38/1970 Heft 4
2	Baarda, W.	Puntsbepaling in R_3 met quaternionen, 1962. Quaterniontheorie 1964–65, Unpublished papers
3	Baarda, W.	Puntsbepalingsystemen in het platte vlak. Lecture notes of the geodetic institute T.H.–Delft, 1966–69
4a	Baarda, W.	Statistical concepts in geodesy
4b	Baarda, W.	A generalization of the concept 'Strength of a Figure'. Netherlands Geodetic Commission Vol.2, Nr. 4, 1967
5	Baarda, W.	A Testing Procedure for Use in Geodetic Networks. Id. Vol. 2, Nr. 5, 1968
6	Baarda, W.	S-Transformations and Criterion Matrices. Id. Vol. 5, Nr. 1, 1973
7	Baarda, W.	Measures for the accuracy of geodetic networks. International Symposium on Optimization of Design and Computation of Control Networks, Sopron, Hungary, 1977
8	Baarda, W.	A Connection between Geometric and Gravimetric Geodesy: a first sketch. Netherlands Geodetic Commission Vol. 6, No. 4, 1979
9	Bartelme, N., and P. Meissl	Strength Analysis of Distance Networks. Mitteilungen der geodätischen Institute der Technischen Hochschule Graz, Folge 15, 1974
10	Baeschlin, C.F.	Lehrbuch der Geodäsie. Orell Füssli Verlag, Zürich, 1948
11	Bjerhammar, A.	Theory of Errors and Generalised Matrix Inverses. Elsevier Scientific Publishing Company, Amsterdam-London-New York, 1973
12	Borre, K., and P. Meissl	Strength Analysis of Levelling Type Networks. Report No. 50 of the Danish Geodetic Institute, 1974
13	Bouloucos, T.C.	Pre-adjustment Gross Error Detection in Photogrammetric Independent Model Blocks, M.Sc.-thesis, ITC, Enschede, 1978
14	Brand, L.	Vector and Tensor Analysis. Wiley and Sons, New York
15	Clark, D.	Plane and Geodetic Surveying. Constable & Co. Ltd., London, 4. ed., 1951
16	Förstner, W.	Die Suche nach groben Fehlern in photogrammetrischen Lageblöcken. Deutsche Geodätische Kommission, Reihe C, Heft Nr. 240 (1970)
17	Förstner, W.	On the Internal and External Reliability of Photogrammetric Coordinates. ASP–Meeting, Washington, March 1979
18	Förstner, W.	The Theoretical Reliability of Photogrammetric Coordinates. Presented Paper Commission III, ISP Hamburg 1980
19	Grafarend, E.	Genauigkeitsmasse geodätischer Netze. Deutsche Geodätische Commission, Reihe A, Heft Nr. 73 (1972)

20 Grafarend, E. — Attempts for a Unified Theory of Geodesy. Bolletino di Geodesia e Scienze Affini, No. 2, 1973

21 Grafarend, E. — Second Order Design of Geodetic Nets. Zeitschrift für Vermessungswesen No. 4, 1975

22 Grafarend, E. — Geodetic Applications of Stochastic Processes. Physics of the Earth and Planetary Interiors, No. 12, 1976

23 Grafarend, E., et al. — Optimal Design of Geodetic Nets, 2. Journal of Geophysical Research Vol. 78, No. 26, 1973

24 Grafarend, E., et al. — Unbiased Free Net Adjustment. Survey Review No. 171, 1974, Vol. XXII

25 Grafarend, E., et al. — Equivalence of Estimable Quantities and Invariants in Geodetic Networks. Zeitschrift für Vermessungswesen No. 11, 1976

26 Grün, A. — Accuracy, Reliability and Statistics in Close-Range Photogrammetry; Symposium Comm. V, ISP Stockholm, 1978

27 Grün, A. — Progress in Photogrammetric Point Determination by Compensation of Systematic Errors and Detection of Gross Errors. Symposium Comm. III, ISP Moscow 1978

28 Grün, A. — Internal Reliability Models for Aerial Bundle Systems. Presented Paper Comm. III, ISP Congress Hamburg, 1980

29 Hilbert, D., und R. Courant — Methoden der mathematischen Physik I. Springer Verlag, Berlin, Heidelberg, New York, 1968

30 Hout, C.M.A. van den — The Anblock Method of Planimetric Block Adjustment. Photogrammetria 21/1966, pp 171–178

31 Hwei, P. — Fourier Analysis. Schaum's Outline Series. McGraw-Hill Book Company, New York, 1970

32 Meissl, P. — Die innere Genauigkeit eines Punkthaufens. Österr. Zeitschrift für Vermessungswesen, 1962, pp 159–164, pp 186–194

33 Meissl, P. — Über die Verformungsfehler eines Systems von endlich viel Punkten. Id. 1964, pp 105–109

34 Molenaar, M. — Error Detection in Planimetric Block Adjustment. ITC Journal 1976–2

35 Molenaar, M., et al. — Essay on Empirical Accuracy Studies in Aerial Triangulation. ITC Journal 1978–1

36 Molenaar, M., et al. — Preadjustment Error Detection in Independent Model Blocks. ITC Journal 1978–3

37 Molenaar, M. — S-Transformations and Artificial Covariance Matrices in Photogrammetry. Presented Paper Comm. III, ISP Hamburg 1980

38 Moritz, H. — Advanced Physical Geodesy. Herbert Wichmann Verlag, Karlsruhe, 1980. Abacus Press, Tunbridge Wells, Kent

39 Neleman, R.C. — Het opsporen van foute waarnemingen in een 2-dimensionale blokvereffening (met onafh. mod.). Nederlands Geodetisch Tijdschrift 8, No. 8, 1978.

40 Quee, H. — Dissertatie. In preparation (1981?)

41 Rao, C.R., and S.K. Mitra — Generalised Inverse of Matrices and its Applications. Wiley & Sons, New York, London, Sydney, Toronto, 1971

42 Sjöberg, L. — Unbiased Estimation of Parameters. Zeitschrift für Vermessungswesen No. 11, 1976

43 Vermaat, E. — Driehoek en Viervlak. Afstudeerscriptie Lab. voor Geodesie, T.H.–Delft, 1970

44 Yosida, K. — Functional Analysis. Springer Verlag, Berlin, Heidelberg, New York, 1971