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THE ADJUSTMENT OF PRIMARY DIRECTION MEASUREMENTS WITH SPECIAL REFERENCE TO CIRCLE TESTING METHODS

by

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CONTENTS

INTRODUCTION AND SUMMARY

The methods for testing circles applied up to now can be arranged in two groups. The first group consists of the methods which have the object to determine the deviations in a limited number of specified graduation lines, evenly spaced on the circle. Originally these methods were applied to circles before they were mounted, but recently modifications of the observational programmes have also been used to test circles already built-in [2], [3], [4]. In the programme developed by BRUNS [1, p. 221] a specified number of standard angles is measured. The magnitude of these angles is taken equal to a multiple of the interval between the graduation lines to be investigated. The design of the programme is such, that equal precision is obtained in the deviations of all these graduation lines. The above testing methods are possibly useful when the theodolite is used for technical projects where often angles of a specified magnitude must be set out. They loose, however, much of their efficiency if the quality of the graduation as a whole must be reviewed. The methods developed for that purpose, and to which the name of HEUVELINK is closely connected, are classed under the second group. The methods of this second group are based on the assumption that the actual deviation in the graduation lines can be considered as a combination of a regular periodic- and a random component.

The determination of this regular part by means of a Fourier series is an essential part of them. The original observational programme after HEUVELINK [5] is more than once supplemented or completely replaced by other programmes and the statistical interpretation of the results has also changed in the course of years.

HEUVELINK restricted himself to the determination of the first three periods by using one standard angle. In [6] WERMANN demonstrated that more standard angles should be observed when the investigation is aimed at the determination of more periods.

WIERSMA [7] suggested the use of two standard angles and advised to realize these two angles by three collimators and to combine the observations of the three directions into one set instead of observing independently the three possible angles. WIERSMA, like most authors on this subject, applied the Fourier series in its asymmetrical form, i.e. with phase angles. The introduction of these phase angles leads however to non-linear model relations.

The mathematical and statistical elaboration then looses its elegance and becomes quite complex. That was the main reason that little attention was paid to WIERSMA's suggestions until ROELOFS published his "bundle of rays" method [8]. This method, where four directions are combined into one set, has successfully been applied. According to HUSTI [9] the results are very promising.

After this fundamental new approach by ROELOFS, it seemed attractive to extend the number of four standard directions to an arbitrary number of m standard directions and to compare the Schreiber and Bessel method of direction measurement. In the Schreiber method all possible combinations of two directions, which cannot be supplemented to a full round, are observed independently whereas in the Bessel method all directions are observed in one set. By posing the problem of circle testing in this way it can easily be seen that in fact there are no essential differences between the design and the elaboration of a testing programme and a normal field programme. The difference only appears in the objective of these programmes. In the former case one is interested in the parameters which describe the deviations in the circle graduation whereas the mean directions are of no importance, in the latter case it is just the other way round as least square estimations of the directions are the only thing wanted.

The computational elaboration in a model that includes the parameters representing the regular deviation in the circle graduation, is rather unwieldy and can hardly be executed on a desk calculating-machine. That is the main reason why up to now the field measurements have always been adjusted on a simplified model that does not take into account the regular deviation of the graduation. Another reason is that least square estimates of the directions are the same for both models so there is no need to bother about the determination of the parameters representing this regular deviation. A less favourable implication, however, of the use of such a simplified model is that the estimate of the variance factor, being the variance of the observational variate with unit weight, looses its value as a measure of precision. In the case large values for the variance factor are found, the observer might be tempted to pass an unfavourable judgement on the theodolite, whereas it is possible that such large values are caused by the occurrence of a large regular deviation of the graduated circle that had been left out of consideration. This implication and the analogy between normal field programmes and special circle testing programmes have led to the development of ALGOL programmes for the Bessel and Schreiber method in such a way that these programmes can be used for both purposes.

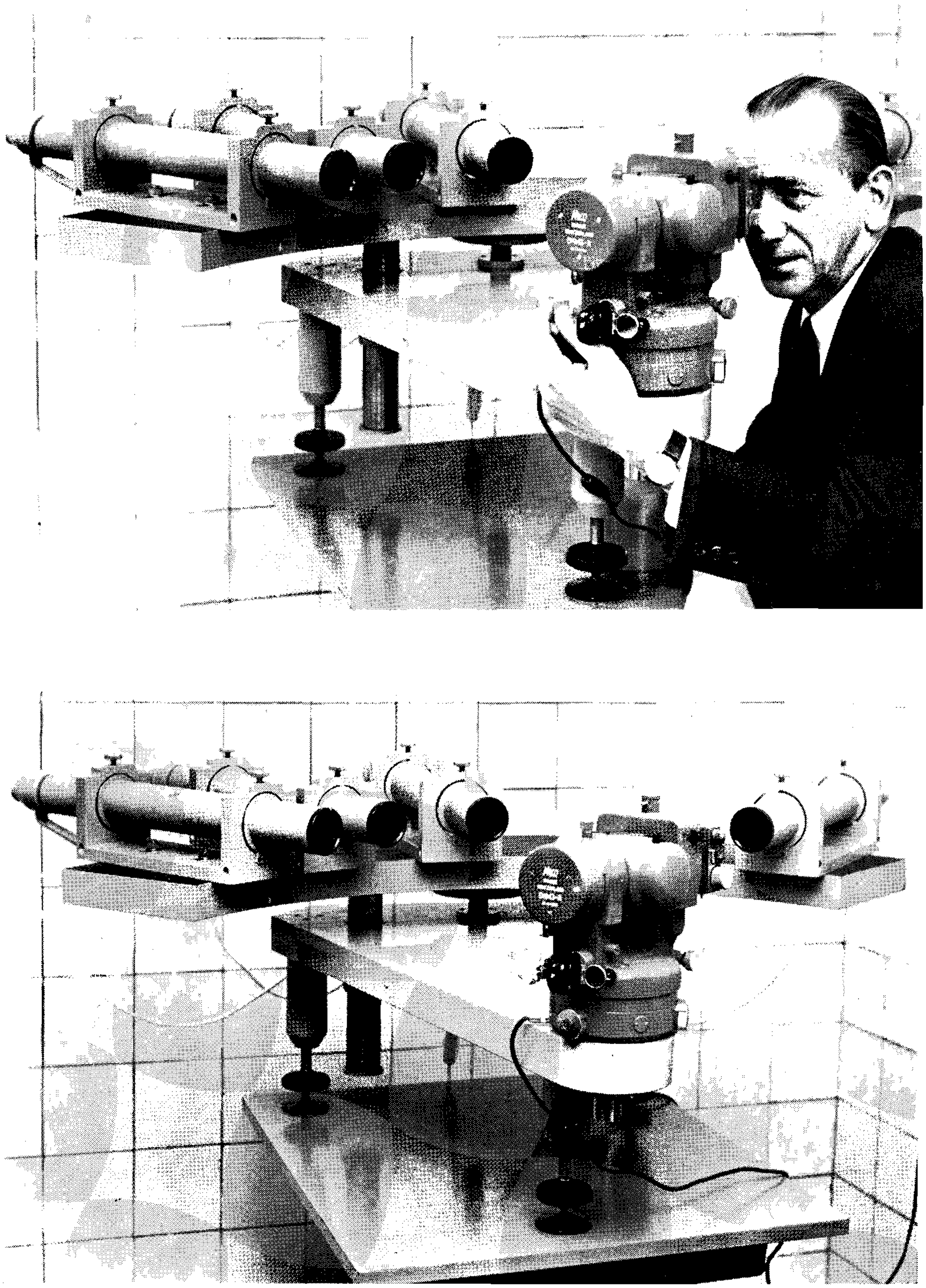
The use of a general procedure developed at the Computing Centre of the Geodetic Institute of the Delft University of Technology for the elaboration of the 2nd standard problem of the adjustment theory (the method of observation equations), was considered first. On closer examination, however, it appeared that the non-diagonal elements of the inverse matrix of coefficients of the normal equations $(\bar{g}^{\alpha\beta})$ – the algorithm and notation as given by BAARDA [10] are used throughout this publication – are zero whereas simple analytic expressions could be derived for the diagonal elements.

As also for the vector of reciprocal unknown variates $(\underline{y}_\beta)$ simple analytic expressions could be found, it was obvious to write a special ALGOL programme based on these properties. The simple form of the matrices $(\bar{g}^{\alpha\beta})$ and $(\underline{y}_\beta$)is due to the orthogonality relations that exist between the trigonometric functions. The derivation of some of these relations is given in section 1.

In the sections 2 and 3 the adjustment of the Bessel and Schreiber programmes is dealt with *in extenso*. The results can be summarized as follows. Let m directions $r[i]$ be observed in n circle positions. Let r be the circle reading, $2\pi/p$ the period of the trigonometric functions and $a[1,p]$ and $a[2,p]$ the amplitudes arranged in two one-dimensional arrays. The regular part Rr of the deviation in the graduation lines is then given by the Fourier series:

$$Rr = \sum_{p=1}^{p} a\,[1,p] \cos pr + a\,[2,p] \sin pr$$

For the Bessel and Schreiber method the following weight coefficients are obtained respectively:



Breithaupt circle testing device after ROELOFS

For the direction $r[i]$:

$$\frac{1}{n} \quad \text{and} \quad \frac{2}{mn}$$

and for the parameters $a[1,p]$, $a[2,p]$:

$$\frac{m}{2n}\left(\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\sin^2\tfrac{1}{2}p(r[j]-r[i])\right)^{-1}$$

and:

$$\frac{1}{n}\left(\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\sin^2\tfrac{1}{2}p(r[j]-r[i])\right)^{-1}$$

The ratio of these weight coefficients is $m/2$ and the time needed for a Schreiber programme is $(m-1)$ times the time needed for a Bessel programme. These facts demonstrate the efficiency of the Bessel programme for $m > 2$, although it should be remembered that such a programme is not always feasible, neither in the field nor in a laboratory (cf. the mechanical test method with the Askania apparatus [11]).

For $m = 2$ there is no difference between the Bessel and Schreiber method. A circle testing programme carried out on two directions therefore can be elaborated with both computational programmes.

In section 4 some remarks are made especially referring to the aspect of circle testing whereas in section 5 the ALGOL programmes for the elaboration of the Bessel and Schreiber method with directives for their use are given.

In the latter section also a few examples are worked out to elucidate the theory. The worked-out examples refer to laboratory tests. A comparitive investigation of a number of primary theodolites, based on extensive trial measurements in the field, will be carried out in the near future.

1. SOME ORTHOGONALITY RELATIONS OF THE TRIGONOMETRIC FUNCTIONS SIN $p\phi$, COS $q\phi$

For all integers p and q and provided that both p and q are not zero, the following relations hold:

$$\left.\begin{aligned}
\int_0^{2\pi} \cos p\phi \cos q\phi \, d\phi &= \tfrac{1}{2}\int_0^{2\pi} \cos(p+q)\phi \, d\phi + \tfrac{1}{2}\int_0^{2\pi} \cos(p-q)\phi \, d\phi = 0, \pi \\
\int_0^{2\pi} \sin p\phi \sin q\phi \, d\phi &= \tfrac{1}{2}\int_0^{2\pi} \cos(p-q)\phi \, d\phi - \tfrac{1}{2}\int_0^{2\pi} \cos(p+q)\phi \, d\phi = 0, \pi \\
\int_0^{2\pi} \sin p\phi \cos q\phi \, d\phi &= \tfrac{1}{2}\int_0^{2\pi} \sin(p+q)\phi \, d\phi + \tfrac{1}{2}\int_0^{2\pi} \sin(p-q)\phi \, d\phi = 0
\end{aligned}\right\} \tag{1}$$

where the value π is taken for $p = q$.

Because of these properties the system of trigonometric functions $\sin p\phi$, $\cos q\phi$ is called orthogonal in the interval 2π. Similar relations exist for finite sums if the arguments for the trigonometric functions are equally distributed over the interval 2π.

From De Moivre's relations:

$$\begin{pmatrix} \cos p\phi \\ \sin p\phi \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{ip\phi} \\ e^{-ip\phi} \end{pmatrix}$$

follows:

$$\begin{pmatrix}
\sum_{n=1}^{\mathrm{n}} \cos p(n-1)\frac{2\pi}{\mathrm{n}} \cos q(n-1)\frac{2\pi}{\mathrm{n}} \\
\sum_{n=1}^{\mathrm{n}} \sin p(n-1)\frac{2\pi}{\mathrm{n}} \sin q(n-1)\frac{2\pi}{\mathrm{n}} \\
\sum_{n=1}^{\mathrm{n}} \sin p(n-1)\frac{2\pi}{\mathrm{n}} \cos q(n-1)\frac{2\pi}{\mathrm{n}}
\end{pmatrix} = \tfrac{1}{4}\begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 \\
-i & -i & i & i
\end{pmatrix}\begin{pmatrix}
\sum_{n=1}^{\mathrm{n}} e^{i(p+q)(n-1)\frac{2\pi}{\mathrm{n}}} \\
\sum_{n=1}^{\mathrm{n}} e^{i(p-q)(n-1)\frac{2\pi}{\mathrm{n}}} \\
\sum_{n=1}^{\mathrm{n}} e^{i(-p+q)(n-1)\frac{2\pi}{\mathrm{n}}} \\
\sum_{n=1}^{\mathrm{n}} e^{i(-p-q)(n-1)\frac{2\pi}{\mathrm{n}}}
\end{pmatrix} \tag{2}$$

Let us consider:

$$\sum_{n=1}^{\mathrm{n}} e^{im(n-1)\frac{2\pi}{\mathrm{n}}},$$

in which m may be any integer. As the individual terms in this sum form a geometric progression, their sum is:

$$\sum_{n=1}^{\mathrm{n}} e^{im(n-1)\frac{2\pi}{\mathrm{n}}} = \frac{e^{im2\pi}-1}{e^{im\frac{2\pi}{\mathrm{n}}}-1} = 0, \mathrm{n} \tag{3}$$

The value $\mathbf{n}$ is taken only if m is a multiple of $\mathbf{n}$. In that case both nominator and denominator are zero and the sum is $\mathbf{n}$ because the terms of the series are all equal to one. In all other cases only the nominator is zero, which means that the value of the sum is zero. Consequently, substituting (3) in (2) the following table can be constructed.

	$p+q \neq c\mathbf{n}$ $p-q \neq c\mathbf{n}$	$p+q \neq c\mathbf{n}$ $p-q = c\mathbf{n}$	$p+q = c\mathbf{n}$ $p-q \neq c\mathbf{n}$	$p+q = c\mathbf{n}$ $p-q = c\mathbf{n}$
$\sum_{n=1}^{\mathbf{n}} \cos p(n-1)\frac{2\pi}{\mathbf{n}} \cos q(n-1)\frac{2\pi}{\mathbf{n}} =$	0	$\mathbf{n}/2$	$\mathbf{n}/2$	$\mathbf{n}$
$\sum_{n=1}^{\mathbf{n}} \sin p(n-1)\frac{2\pi}{\mathbf{n}} \sin q(n-1)\frac{2\pi}{\mathbf{n}} =$	0	$\mathbf{n}/2$	$-\mathbf{n}/2$	0
$\sum_{n=1}^{\mathbf{n}} \sin p(n-1)\frac{2\pi}{\mathbf{n}} \cos q(n-1)\frac{2\pi}{\mathbf{n}} =$	0	0	0	0

(4)

From the last two columns it is apparent that not all sums of products of trigonometric functions with different values for p and q are zero, for instance for the combination: $\mathbf{n} = 10, p = 7, q = 3$.

However, limiting the range for p, q to all positive integers less than $\mathbf{n}/2$, the last two columns can be left out of consideration, whereas from the first two columns follows:

$$\left.\begin{array}{l} \sum_{n=1}^{\mathbf{n}} \cos p(n-1)\frac{2\pi}{\mathbf{n}} \cos q(n-1)\frac{2\pi}{\mathbf{n}} \\ \sum_{n=1}^{\mathbf{n}} \sin p(n-1)\frac{2\pi}{\mathbf{n}} \sin q(n-1)\frac{2\pi}{\mathbf{n}} \end{array}\right\} = \left\{\begin{array}{l} 0 \text{ for } p \neq q \\ \mathbf{n}/2 \text{ for } p = q \end{array}\right.$$
$$\sum_{n=1}^{\mathbf{n}} \sin p(n-1)\frac{2\pi}{\mathbf{n}} \cos q(n-1)\frac{2\pi}{\mathbf{n}} = 0 \tag{5}$$

The relations (5) for a finite summation are similar to the relations (1) for an integral.

Let:

$$r[n,a] = r[1,a] + (n-1)\frac{2\pi}{\mathbf{n}}$$

$$r[n,b] = r[1,b] + (n-1)\frac{2\pi}{\mathbf{n}}$$

$$r[n,c] = r[1,c] + (n-1)\frac{2\pi}{\mathbf{n}}$$

be three sets of $\mathbf{n}$ arguments equally distributed over the interval 2π. The first arguments in the sets are indicated by:

$$r[1,a],\ r[1,b] \text{ and } r[1,c]$$

Again limiting the range for p, q to all positive integers less than $\mathbf{n}/2$, the following relations can now successively be derived. The last values on the right hand side of the equations refer to $p = q$.

$$\left.\begin{array}{l} \left.\begin{array}{l} \sum\limits_{n=1}^{\mathbf{n}} \cos pr\,[n, a] \cos qr\,[n, b] \\ \sum\limits_{n=1}^{\mathbf{n}} \sin pr\,[n, a] \sin qr\,[n, b] \end{array}\right\} = 0, \frac{\mathbf{n}}{2} \cos p(r\,[1, a] - r\,[1, b]) \\ \sum\limits_{n=1}^{\mathbf{n}} \sin pr\,[n, a] \cos qr\,[n, b] \quad = 0, \frac{\mathbf{n}}{2} \sin p\,(r\,[1, a] - r\,[1, b]) \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \sum\limits_{n=1}^{\mathbf{n}} (\cos pr\,[n, a] - \cos pr\,[n, c])(\cos qr\,[n, b) - \cos qr\,[n, c]) \\ \sum\limits_{n=1}^{\mathbf{n}} (\sin\, pr\,[n, a] - \sin\, pr\,[n, c])(\sin\, qr\,[n, b] - \sin\, qr\,[n, c]) \end{array}\right\} = \\ = 0, \mathbf{n}\{\sin^2\tfrac{1}{2}p(r\,[1, a] - r\,[1, c]) + \sin^2\tfrac{1}{2}p(r\,[1, b] - r\,[1, c]) - \sin^2\tfrac{1}{2}p(r\,[1, a] - r\,[1, b])\} \end{array}\right\} \tag{7}$$

$$\sum_{n=1}^{\mathbf{n}} (\sin pr\,[n, a] - \sin pr\,[n, c])(\cos qr\,[n, b] - \cos qr\,[n, c]) = 0 \tag{8}$$

and finally:

$$\left.\begin{array}{l} \sum\limits_{n=1}^{\mathbf{n}} (\cos pr\,[n, a] - \cos pr\,[n, b])^2 \\ \sum\limits_{n=1}^{\mathbf{n}} (\sin\, pr\,[n, a] - \sin\, pr\,[n, b])^2 \end{array}\right\} = 0, 2\mathbf{n}\{\sin^2\tfrac{1}{2}p(r\,[1, a] - r\,[1, b])\} \tag{9}$$

2 THE ADJUSTMENT OF A BESSEL PROGRAMME

In this paper a particular direction, set or programme is indicated by a running index given in italic type whereas the same letter is used in bold type when indicating the upper limit of the range of the running index in question.

Thus:

$$
\begin{aligned}
s &= 1, \ldots, \mathbf{s} \\
n &= 1, \ldots, \mathbf{n} \\
n1 &= 1, \ldots, \mathbf{n1} \\
n2 &= 1, \ldots, \mathbf{n2} \\
i, j &= 1, \ldots, \mathbf{m}
\end{aligned}
$$

In the elaboration of the adjustment use will be made of the notation and algorithm as given by BAARDA in [10].

Suppose that $\mathbf{m}$ directions have been observed in a number of circle positions regularly distributed over $2\pi/z$, where z takes the values 1 or 2, depending on whether a single or diametrical circle reading device has been fitted into the theodolite.

Suppose that the whole observational programme is divided into $\mathbf{s}$ partial programmes and let each partial programme consist of $\mathbf{n}$ circle positions. The total number of circle positions thus amounts to $\mathbf{sn}$. The introduction of partial programmes is necessary to eliminate the collimation- and azimuth error (in case of field measurements) or to avoid observational programmes that last too long (in case of special circle testing measurements).

A programme for testing circles is usually designed in such a way that each partial programme itself can be considered as a complete circle testing programme. That means that the following sequence of circle positions is used:

$$c[s,n] = c[s,1] + \frac{n-1}{\mathbf{n}} \cdot \frac{2\pi}{z}$$

$$c[s,1] = c[1,1] + \frac{s-1}{\mathbf{s} \cdot \mathbf{n}} \cdot \frac{2\pi}{z}$$

hence:

$$c[s,n] = c[1,1] + \left(n - 1 + \frac{s-1}{\mathbf{s}}\right) \frac{2\pi}{\mathbf{n}z}$$

A field programme usually has the following sequence of circle positions:

$$c[s,n] = c[s,1] + \frac{n-1}{\mathbf{s} \cdot \mathbf{n}} \cdot \frac{2\pi}{z}$$

$$c[s,1] = c[1,1] + \frac{s-1}{\mathbf{s}} \cdot \frac{2\pi}{z}$$

hence:

$$c[s,n] = c[1,1] + \left(s-1+\frac{n-1}{\mathbf{n}}\right)\frac{2\pi}{\mathbf{sz}}$$

It should be noted that the algorithm as will be given in this section can only be applied to the first mentioned programme. The second programme results in a matrix $(\bar{g}^{\alpha\beta})$ with no zero elements. Therefore the first programme is also recommended for the measurements in the field which means only a slight modification of the usual observational programme.

In circle testing programmes, in contrast with the field measurements, all directions are more than once observed in each circle position. Such an increase of the number of observations in each circle position is not only needed to obtain a higher precision in the parameters to be determined but also to give the observer the possibility to verify the stability of the bundle of rays during the observational period. Consequently in each circle position the bundle of rays is observed in clockwise and anticlockwise order. The number of sets that are observed immediately after each other, need not necessarily be limited to two. Let the total number of sets be denoted by **n1**. After the bundle of rays is observed in all circle positions, the observational programme thus obtained, is repeated in the reversed order of these circle positions. The total number of repetitions, denoted by **n2**, need not necessarily be two.

The design of the above sketched observational programme enables the observer to divide the adjustment into three phases. The first two phases refer to the averaging of the different observations in the same circle position whereas in the third phase the relation between these means and the circle positions is analysed. It will be clear that this last phase covers the whole adjustment if the observational programme concerns the usual field measurement.

Let a single observation of direction i, in the partial programme s, observed in the circle position n, in the sets with indices $n1$ and $n2$ be denoted by:

$$r[s,n,n2,n1,i]$$

In the system of weights this observation is given unit weight whereas the variance factor σ^2 is introduced as an unknown.

The angles that are obtained when all directions are referred to the first direction, are denoted by:

$$l4[s,n,n2,n1,i,1] = r[s,n,n2,n1,i] - r[s,n,n2,n1,1]$$

The addition of a suffix to the kernel l serves to distinguish the angle in the different phases of the adjustment.

For shortness sake the indices between the brackets will be omitted henceforth when no loss of clarity is to be feared.

The following three phases in the adjustment problem can be distinguished.

Phase I

If **n1** > 1, the number of correction equations, according to the 2nd standard problem, that can be formed in a first phase is $\mathbf{s}\cdot\mathbf{n}\cdot\mathbf{n1}\cdot\mathbf{n2}\cdot(\mathbf{m}-1)$. These equations – written in the variates, hence with an underscore – are:

$$\underline{l4}[s,n,n2,n1,i,1]+\underline{\varepsilon 4}[s,n,n2,n1,i,1] = \underline{l3}[s,n,n2,i,1]$$

The matrix of weight coefficients (g^{ij}) of the observation variates $\underline{l4}$ is composed of $\mathbf{s}\cdot\mathbf{n}\cdot\mathbf{n2}\cdot\mathbf{n1}$ submatrices $(g^{ij})_s$, which are connected along the principal diagonal. These submatrices of rank $(\mathbf{m}-1)$ have the elements 2 resp. 1 for $i=j$ resp. $i \neq j$ $(i,j = 2,\ldots,\mathbf{m})$

The submatrices $(\bar{g}_{ji})_s$ of the matrix of weights have the elements $(\mathbf{m}-1)/\mathbf{m}$ and $-(1/\mathbf{m})$ for $i=j$ and $i \neq j$ respectively.

According to the algorithm of the 2nd standard problem:

$$(\underline{y}^{\alpha\cdot\mathrm{I}}) \equiv \underline{l3} = \frac{1}{\mathbf{n1}}\sum \underline{l4}$$

$$(\bar{g}^{\alpha\beta\cdot\mathrm{I}}) = \frac{1}{\mathbf{n1}}(g^{ij});\ (g_{\beta\alpha\cdot\mathrm{I}}) = \mathbf{n1}(\bar{g}_{ji});$$

the shifting variate becomes:

$$\underline{E}^{\mathrm{I}} = \frac{\mathbf{m}-1}{\mathbf{m}}\sum\ldots\sum\sum_{i=2}^{\mathbf{m}}(\underline{l3}[i]-\underline{l4}[i])^2 - \frac{2}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}(\underline{l3}[i]-\underline{l4}[i])(\underline{l3}[j]-\underline{l4}[j])$$

and the number of redundant observations:

$$b^{\mathrm{I}} = \mathbf{s}\cdot\mathbf{n}\cdot\mathbf{n2}\cdot(\mathbf{n1}-1)(\mathbf{m}-1)$$

Phase II

If **n2** > 1, the number of correction equations, according to the 2nd standard problem, that can be formed in a second phase is $\mathbf{s}\cdot\mathbf{n}\cdot\mathbf{n2}\cdot(\mathbf{m}-1)$.

These equations are:

$$\underline{l3}[s,n,n2,i,1]+\underline{\varepsilon 3}[s,n,n2,i,1] = \underline{l2}[s,n,i,1]$$

with the matrix of weight coefficients of the observation variates $\underline{l3}$: $(\bar{g}^{\alpha\beta\cdot\mathrm{I}})$.

The following formulas are found:

$$(\underline{y}^{\alpha\cdot\mathrm{II}}) \equiv \underline{l2} = \frac{1}{\mathbf{n2}}\sum \underline{l3}$$

$$(\bar{g}^{\alpha\beta\cdot\mathrm{II}}) = \frac{1}{\mathbf{n2}}(\bar{g}^{\alpha\beta\cdot\mathrm{I}}),\ \ (g_{\beta\alpha\cdot\mathrm{II}}) = \mathbf{n2}(g_{\beta\alpha\cdot\mathrm{I}})$$

$$\frac{1}{\mathbf{n1}}\cdot\underline{E}^{\mathrm{II}} = \frac{\mathbf{m}-1}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}}(\underline{l2}[i]-\underline{l3}[i])^2 - \frac{2}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}(\underline{l2}[i]-\underline{l3}[i])(\underline{l2}[j]-\underline{l3}[j])$$

$$b^{\mathrm{II}} = \mathbf{s}\cdot\mathbf{n}(\mathbf{n2}-1)(\mathbf{m}-1)$$

The test on the stability of the directions is executed by computing a value of the statistic:

$$\underline{F}_{b^{\mathrm{I}},b^{\mathrm{II}}} = \frac{(\hat{\underline{\sigma}}^{\mathrm{II}})^2}{(\hat{\underline{\sigma}}^{\mathrm{I}})^2} \quad \text{with: } (\hat{\underline{\sigma}}^{\mathrm{I}})^2 = \frac{\underline{E}^{\mathrm{I}}}{b^{\mathrm{I}}} \quad \text{and} \quad (\hat{\underline{\sigma}}^{\mathrm{II}})^2 = \frac{\underline{E}^{\mathrm{II}}}{b^{\mathrm{II}}},$$

and using the right-hand tail critical region of this Fisher distribution with a careful choice of the significance level α [10, p. 16].

Phase III

Assuming that the departure of the actual circle graduation from an idealized one can be described by the Fourier series:

$$\underline{R}(r) = \sum \underline{a}[1, zp] \cos zpr + \underline{a}[2, zp] \sin zpr$$

the following number of $\mathbf{s} \cdot \mathbf{n} \cdot (\mathbf{m}-1)$ condition equations according to the 2nd standard problem can be formed:

$$\begin{aligned} \underline{l2}[s,n,i,1] + \underline{\varepsilon 2}[s,n,i,1] = \underline{l1}[s,i,1] &+ \sum \underline{a}[1,zp]\{\cos zpr[s,n,i] - \cos zpr[s,n,1]\} + \\ &+ \sum \underline{a}[2,zp]\{\sin zpr[s,n,i] - \sin zpr[s,n,1]\} \end{aligned}$$

with the matrix of weight coefficients of the observation variates $\underline{l2}$: $(\bar{g}^{\alpha\beta \cdot \mathrm{II}})$
and $\quad i = 2, \ldots, \mathbf{m}$
$\quad p = 1, \ldots, \mathbf{p}$

The coefficients of $\underline{a}[1, zp]$ and $\underline{a}[2, zp]$ may be regarded as non-stochastic.

The elaboration of the adjustment is given in the diagrams shown on pp. 16–20. In these diagrams the non-essential quantity z has been omitted.

Vector $(\underline{x}^i)$

$$\boxed{\begin{array}{c}
\underline{l2}[1,1,2,1] \\ \vdots \\ \underline{l2}[1,1,i,1] \\ \vdots \\ \underline{l2}[1,1,\mathbf{m},1] \\ \vdots \\ \\ \vdots \\ \underline{l2}[s,n,2,1] \\ \vdots \\ \underline{l2}[s,n,i,1] \\ \vdots \\ \underline{l2}[s,n,\mathbf{m},1] \\ \vdots \\ \\ \vdots \\ \underline{l2}[\mathbf{s},\mathbf{n},2,1] \\ \vdots \\ \underline{l2}[\mathbf{s},\mathbf{n},i,1] \\ \vdots \\ \underline{l2}[\mathbf{s},\mathbf{n},\mathbf{m},1]
\end{array}}$$

Vector $(\underline{y}^{\alpha\cdot\mathrm{III}})$

$$\boxed{\begin{array}{l}
\underline{l1}[1,2,1] \\ \quad\vdots \\ \underline{l1}[1,i,1] \\ \quad\vdots \\ \underline{l1}[1,\mathbf{m},1] \\ \quad\vdots \\ \\ \quad\vdots \\ \underline{l1}[s,2,1] \\ \quad\vdots \\ \underline{l1}[s,i,1] \\ \quad\vdots \\ \underline{l1}[s,\mathbf{m},1] \\ \quad\vdots \\ \\ \quad\vdots \\ \underline{l1}[\mathbf{s},2,1] \\ \quad\vdots \\ \underline{l1}[\mathbf{s},i,1] \\ \quad\vdots \\ \underline{l1}[\mathbf{s},\mathbf{m},1] \\ \\ \underline{a}[1,1] \\ \underline{a}[2,1] \\ \quad\vdots \\ \underline{a}[1,p] \\ \underline{a}[2,p] \\ \quad\vdots \\ \underline{a}[1,\mathbf{p}] \\ \underline{a}[2,\mathbf{p}]
\end{array}}$$

Matrix $(a_\alpha{}^i)$

$$
\begin{array}{ccc|ccc|ccc|ll|ll}
1 & & & & & & & & & \cos r[1,1,2] - \cos r[1,1,1] & & \cos pr[1,1,2] - \cos r[1,1,1] \\
 & 1 & & & & & & & & \cos r[1,1,i] - \cos r[1,1,1] & & \cos pr[1,1,i] - \cos r[1,1,1] \\
 & & 1 & & & & & & & \cos r[1,1,\mathbf{m}] - \cos r[1,1,1] & & \cos pr[1,1,\mathbf{m}] - \cos r[1,1,1] \\
\hline
 & & & & & & & & & & & \\
\hline
 & & & 1 & & & & & & \cos r[s,n,2] - \cos r[s,n,1] & & \cos pr[s,n,2] - \cos pr[s,n,1 \\
 & & & & 1 & & & & & \cos r[s,n,i] - \cos r[s,n,1] & & \cos pr[s,n,i] - \cos pr[s,n,1 \\
 & & & & & 1 & & & & \cos r[s,n,\mathbf{m}] - \cos r[s,n,1] & & \cos pr[s,n,\mathbf{m}] - \cos pr[s,n,1 \\
\hline
 & & & & & & & & & & & \\
\hline
 & & & & & & 1 & & & \cos r[\mathbf{s},\mathbf{n},2] - \cos r[\mathbf{s},\mathbf{n},1] & & \cos pr[\mathbf{s},\mathbf{n},2] - \cos pr[\mathbf{s},\mathbf{n},1 \\
 & & & & & & & 1 & & \cos r[\mathbf{s},\mathbf{n},i] - \cos r[\mathbf{s},\mathbf{n},1] & & \cos pr[\mathbf{s},\mathbf{n},i] - \cos pr[\mathbf{s},\mathbf{n},1 \\
 & & & & & & & & 1 & \cos r[\mathbf{s},\mathbf{n},\mathbf{m}] - \cos r[\mathbf{s},\mathbf{n},1] & & \cos pr[\mathbf{s},\mathbf{n},\mathbf{m}] - \cos pr[\mathbf{s},\mathbf{n},1
\end{array}
$$

Matrix $\frac{1}{\mathbf{n1}\cdot\mathbf{n2}}(a_\beta^{j*}\bar{g}_{ji})$ — the matrix can be divided in $\mathbf{s}\cdot\mathbf{n}$ submatrices; the submatrix with running indices s,n is given below.

$$
\begin{array}{c|c}
\begin{array}{c} \frac{\mathbf{m}-1}{\mathbf{m}} \\ \vdots \\ -\frac{1}{\mathbf{m}} \\ \vdots \\ -\frac{1}{\mathbf{m}} \end{array} & \begin{array}{c} - \\ \vdots \\ \frac{\mathbf{m}-}{\mathbf{m}} \\ \vdots \\ - \end{array} \\
\hline
\cos r[s,n,2] - \cos r[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} (\cos r[s,n,i] - \cos r[s,n,1]) & \cos r[s,n,i] - \cos r[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} \\
\sin r[s,n,2] - \sin r[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} (\sin r[s,n,i] - \sin r[s,n,1]) & \sin r[s,n,i] - \sin r[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} \\
\vdots & \vdots \\
\cos pr[s,n,2] - \cos pr[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} (\cos pr[s,n,i] - \cos pr[s,n,1]) & \cos pr[s,n,i] - \cos pr[s,n,1] - \frac{1}{\mathbf{m}} \\
\sin pr[s,n,2] - \sin pr[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} (\sin pr[s,n,i] - \sin pr[s,n,1]) & \sin pr[s,n,i] - \sin pr[s,n,1] - \frac{1}{\mathbf{m}} \\
\vdots & \vdots \\
\cos \mathbf{p}r[s,n,2] - \cos \mathbf{p}r[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} (\cos \mathbf{p}r[s,n,i] - \cos \mathbf{p}r[s,n,1]) & \cos \mathbf{p}r[s,n,i] - \cos \mathbf{p}r[s,n,1] - \frac{1}{\mathbf{m}} \\
\sin \mathbf{p}r[s,n,2] - \sin \mathbf{p}r[s,n,1] - \frac{1}{\mathbf{m}} \sum_{i=2}^{\mathbf{m}} (\sin \mathbf{p}r[s,n,i] - \sin \mathbf{p}r[s,n,1]) & \sin \mathbf{p}r[s,n,i] - \sin \mathbf{p}r[s,n,1] - \frac{1}{\mathbf{m}}
\end{array}
$$

$r[1,1,1]$ $r[1,1,1]$ $r[1,1,1]$	$\sin pr[1,1,2]\; - \sin pr[1,1,1]$ $\sin pr[1,1,i]\; - \sin pr[1,1,1]$ $\sin pr[1,1,\mathbf{m}] - \sin pr[1,1,1]$		$\sin \mathbf{p}r[1,1,2]\; - \sin \mathbf{p}r[1,1,1]$ $\sin \mathbf{p}r[1,1,i]\; - \sin \mathbf{p}r[1,1,1]$ $\sin \mathbf{p}r[1,1,\mathbf{m}] - \sin \mathbf{p}r[1,1,1]$
$pr[s,n,1]$ $pr[s,n,1]$ $pr[s,n,1]$	$\sin pr[s,n,2]\; - \sin pr[s,n,1]$ $\sin pr[s,n,i]\; - \sin pr[s,n,1]$ $\sin pr[s,n,\mathbf{m}] - \sin pr[s,n,1]$		$\sin \mathbf{p}r[s,n,2]\; - \sin \mathbf{p}r[s,n,1]$ $\sin \mathbf{p}r[s,n,i]\; - \sin \mathbf{p}r[s,n,1]$ $\sin \mathbf{p}r[s,n,\mathbf{m}] - \sin \mathbf{p}r[s,n,1]$
$pr[\mathbf{s},\mathbf{n},1]$ $pr[\mathbf{s},\mathbf{n},1]$ $pr[\mathbf{s},\mathbf{n},1]$	$\sin pr[\mathbf{s},\mathbf{n},2]\; - \sin pr[\mathbf{s},\mathbf{n},1]$ $\sin pr[\mathbf{s},\mathbf{n},i]\; - \sin pr[\mathbf{s},\mathbf{n},1]$ $\sin pr[\mathbf{s},\mathbf{n},\mathbf{m}] - \sin pr[\mathbf{s},\mathbf{n},1]$		$\sin \mathbf{p}r[\mathbf{s},\mathbf{n},2]\; - \sin \mathbf{p}r[\mathbf{s},\mathbf{n},1]$ $\sin \mathbf{p}r[\mathbf{s},\mathbf{n},i]\; - \sin \mathbf{p}r[\mathbf{s},\mathbf{n},1]$ $\sin \mathbf{p}r[\mathbf{s},\mathbf{n},\mathbf{m}] - \sin \mathbf{p}r[\mathbf{s},\mathbf{n},1]$

$-\frac{1}{\mathbf{m}}$ $\vdots$ $\frac{\mathbf{m}-1}{\mathbf{m}}$ $\vdots$ $-\frac{1}{\mathbf{m}}$		$-\frac{1}{\mathbf{m}}$ $\vdots$ $-\frac{1}{\mathbf{m}}$ $\vdots$ $\frac{\mathbf{m}-1}{\mathbf{m}}$
$-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\cos r[s,n,i]-\cos r[s,n,1])$ $-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\sin r[s,n,i]-\sin r[s,n,1])$ $\vdots$ $1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\cos pr[s,n,i]-\cos pr[s,n,1])$ $1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\sin pr[s,n,i]-\sin pr[s,n,1])$ $\vdots$ $1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\cos \mathbf{p}r[s,n,i]-\cos \mathbf{p}r[s,n,1]$ $1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\sin \mathbf{p}r[s,n,i]-\sin \mathbf{p}r[s,n,1])$		$\cos r[s,n,\mathbf{m}]-\cos r[s,n,1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\cos r[s,n,i]-\cos r[s,n,1])$ $\sin r[s,n,\mathbf{m}]-\sin r[s,n,1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\sin r[s,n,i]-\sin r[s,n,1])$ $\vdots$ $\cos pr[s,n,\mathbf{m}]-\cos pr[s,n,1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\cos pr[s,n,i]-\cos pr[s,n,1])$ $\sin pr[s,n,\mathbf{m}]-\sin pr[s,n,1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\sin pr[s,n,i]-\sin pr[s,n,1])$ $\vdots$ $\cos \mathbf{p}r[s,n,\mathbf{m}]-\cos \mathbf{p}r[s,n,1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\cos \mathbf{p}r[s,n,i]-\cos \mathbf{p}r[s,n,1])$ $\sin \mathbf{p}r[s,n,\mathbf{m}]-\sin \mathbf{p}r[s,n,1]-\frac{1}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}(\sin \mathbf{p}r[s,n,i]-\sin \mathbf{p}r[s,n,1])$

Matrix **n1·n2** ($g^{\alpha\beta \cdot \mathrm{II}}$)

$$
\left|\begin{array}{c|c|c|c}
\begin{matrix} 2 & 1 & 1 & 1 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \dots & 2 \end{matrix} & 0 & 0 & 0 \\
\hline
0 & \begin{matrix} 2 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \dots & 2 \end{matrix} & 0 & \\
\hline
0 & 0 & \ddots & 0 \\
\hline
0 & 0 & 0 & \begin{matrix} 2 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \dots & 2 \end{matrix}
\end{array}\right|
$$

Matrix $\frac{1}{\mathbf{n1} \cdot \mathbf{n2}} (g_{\beta\sigma \cdot \text{II}})$

$$\left[\begin{array}{c|c|c}
\begin{matrix} \frac{\mathbf{m}-1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \cdots & -\frac{1}{\mathbf{m}} \\ -\frac{1}{\mathbf{m}} & \frac{\mathbf{m}-1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \cdots & -\frac{1}{\mathbf{m}} \\ -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \frac{\mathbf{m}-1}{\mathbf{m}} & \cdots & -\frac{1}{\mathbf{m}} \\ \vdots & \vdots & \vdots & & \vdots \\ -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \cdots & \frac{\mathbf{m}-1}{\mathbf{m}} \end{matrix} & 0 & 0 \\
\hline
0 & \ddots & 0 \\
\hline
0 & 0 & \begin{matrix} \frac{\mathbf{m}-1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \cdots & -\frac{1}{\mathbf{m}} \\ -\frac{1}{\mathbf{m}} & \frac{\mathbf{m}-1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \cdots & -\frac{1}{\mathbf{m}} \\ -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \frac{\mathbf{m}-1}{\mathbf{m}} & \cdots & -\frac{1}{\mathbf{m}} \\ \vdots & \vdots & \vdots & & \vdots \\ -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & -\frac{1}{\mathbf{m}} & \cdots & \frac{\mathbf{m}-1}{\mathbf{m}} \end{matrix}
\end{array}\right]$$

Matrix $\frac{1}{\mathbf{n1}\cdot\mathbf{n2}}(g_{\beta\alpha\cdot\mathrm{III}})$

$$
\left(\begin{array}{cccc:c:cccc:cccc}
\frac{\mathbf{n(m-1)}}{\mathbf{m}} & -\frac{\mathbf{n}}{\mathbf{m}} & \cdots & -\frac{\mathbf{n}}{\mathbf{m}} & & & & & & & & & \\
-\frac{\mathbf{n}}{\mathbf{m}} & \frac{\mathbf{n(m-1)}}{\mathbf{m}} & \cdots & -\frac{\mathbf{n}}{\mathbf{m}} & & & & & & & & & \\
\vdots & \vdots & & \vdots & & & & & & & & & \\
-\frac{\mathbf{n}}{\mathbf{m}} & -\frac{\mathbf{n}}{\mathbf{m}} & \cdots & \frac{\mathbf{n(m-1)}}{\mathbf{m}} & & & & & & & & & \\
\hdashline
 & & & & & & & & & & & & \\
\hdashline
 & & & & & \frac{\mathbf{n(m-1)}}{\mathbf{m}} & -\frac{\mathbf{n}}{\mathbf{m}} & \cdots & -\frac{\mathbf{n}}{\mathbf{m}} & & & & \\
 & & & & & -\frac{\mathbf{n}}{\mathbf{m}} & \frac{\mathbf{n(m-1)}}{\mathbf{m}} & \cdots & -\frac{\mathbf{n}}{\mathbf{m}} & & & & \\
 & & & & & \vdots & \vdots & & \vdots & & & & \\
 & & & & & -\frac{\mathbf{n}}{\mathbf{m}} & -\frac{\mathbf{n}}{\mathbf{m}} & \cdots & \frac{\mathbf{n(m-1)}}{\mathbf{m}} & & & & \\
\hdashline
 & & & & & & & & & \frac{\mathbf{2ns}}{\mathbf{m}}\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}(r[j]-r[i]) & & & \\
 & & & & & & & & & & \frac{\mathbf{2ns}}{\mathbf{m}}\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}p(r[j]-r[i]) & & \\
 & & & & & & & & & & & \frac{\mathbf{2ns}}{\mathbf{m}}\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}p(r[j]-r[i]) & \\
 & & & & & & & & & & & & \frac{\mathbf{2ns}}{\mathbf{m}}\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}\mathbf{p}(r[j]-r[i])
\end{array}\right)
$$

Vector $\frac{1}{\mathbf{n1}\cdot\mathbf{n2}}(\underline{y}_{\beta\cdot\mathrm{III}})$

$$
\begin{array}{c}
\sum_{n=1}^{\mathbf{n}} l2\underline{[1,n,2,1]} - \frac{1}{\mathbf{m}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} l2\underline{[1,n,i,1]} \\
\vdots \\
\sum_{n=1}^{\mathbf{n}} l2\underline{[s,n,i,1]} - \frac{1}{\mathbf{m}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} l2\underline{[s,n,i,1]} \\
\vdots \\
\sum_{n=1}^{\mathbf{n}} l2\underline{[\mathbf{s},n,\mathbf{m},1]} - \frac{1}{\mathbf{m}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} l2\underline{[\mathbf{s},n,i,1]} \\
\sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} (\cos r[s,n,i] - \cos r[s,n,1])\, l2\underline{[s,n,i,1]} - \frac{1}{\mathbf{m}} \sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} \left(\sum_{i=2}^{\mathbf{m}} \cos r[s,n,i] - \cos r[s,n,1] \right) l2\underline{[s,n,i,1]} \\
\vdots \\
\vdots \\
\sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} (\cos pr[s,n,i] - \cos pr[s,n,1])\, l2\underline{[s,n,i,1]} - \frac{1}{\mathbf{m}} \sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} \left(\sum_{i=2}^{\mathbf{m}} \cos pr[s,n,i] - \cos pr[s,n,1] \right) l2\underline{[s,n,i,1]} \\
\sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} (\sin pr[s,n,i] - \sin pr[s,n,1])\, l2\underline{[s,n,i,1]} - \frac{1}{\mathbf{m}} \sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} \left(\sum_{i=2}^{\mathbf{m}} \sin pr[s,n,i] - \sin pr[s,n,1] \right) l2\underline{[s,n,i,1]} \\
\vdots \\
\vdots \\
\sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} (\sin \mathbf{p}r[s,n,i] - \sin \mathbf{p}r[s,n,1])\, l2\underline{[s,n,i,1]} - \frac{1}{\mathbf{m}} \sum_{s=1}^{\mathbf{s}} \sum_{n=1}^{\mathbf{n}} \sum_{i=2}^{\mathbf{m}} \left(\sum_{i=2}^{\mathbf{m}} \sin \mathbf{p}r[s,n,i] - \sin \mathbf{p}r[s,n,1] \right) l2\underline{[s,n,i,1]}
\end{array}
$$

A further explanation of some of the diagrams is given below. As to the matrix $(g_{\beta\alpha\cdot\mathrm{III}})$ it can easily be seen that for **p** less **n**/2, the non-diagonal elements of the $(2p,2p)$ submatrix are zero, as these elements are summations of products of the form as given in (7) for $p \neq q$ or as given in (8).

From the general expression $(g_{\beta\alpha}) = (a^j_\beta)^*(\bar{g}_{ji})(a^i_\alpha)$ it follows that the elements on the principal diagonal of the $(2\mathbf{p},2\mathbf{p})$ submatrix of matrix $(1/\mathbf{n}1\cdot\mathbf{n}2)(g_{\beta\alpha\cdot\mathrm{III}})$ are, for $p = 1, \ldots, \mathbf{p}$:

$$\sum\ldots\sum_{i=2}^{\mathbf{m}}(\cos zpr[s,n,i] - \cos zpr[s,n,1])^2 +$$
$$-\frac{1}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}}(\cos zpr[s,n,i] - \cos zpr[s,n,1])\left(\sum_{i=2}^{\mathbf{m}}\cos zpr[s,n,i] - \cos zpr[s,n,1]\right)$$

and:

$$\sum\ldots\sum_{i=2}^{\mathbf{m}}(\sin zpr[s,n,i] - \sin zpr[s,n,1])^2 +$$
$$-\frac{1}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}}(\sin zpr[s,n,i] - \sin zpr[s,n,1])\left(\sum_{i=2}^{\mathbf{m}}\sin zpr[s,n,i] - \sin zpr[s,n,1]\right)$$

Let us focus the attention to the first expression. This expression can successively be written as:

$$\frac{\mathbf{m}-1}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}}(\cos zpr[s,n,i] - \cos zpr[s,n,1])^2 +$$
$$-\frac{2}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}(\cos zpr[s,n,i] - \cos zpr[s,n,1])(\cos zpr[s,n,j] - \cos zpr[s,n,1])$$

Carrying out first the summation over **n** using (7) and (9):

$$\frac{\mathbf{m}-1}{\mathbf{m}}2\mathbf{n}\sum\sum_{i=2}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[s,1,i] - r[s,1,1]) +$$
$$-\frac{2}{\mathbf{m}}\mathbf{n}\sum\sum_{i=2}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\Big(\sin^2\tfrac{1}{2}zp(r[s,1,i] - r[s,1,1]) + \sin^2\tfrac{1}{2}zp(r[s,1,j] - r[s,1,1]) +$$
$$-\sin^2\tfrac{1}{2}zp(r[s,1,j] - r[s,1,i])\Big)$$

which equals:

$$\mathbf{s}\left(\frac{\mathbf{m}-1}{\mathbf{m}}2\mathbf{n} - \frac{\mathbf{m}-2}{\mathbf{m}}2\mathbf{n}\right)\sum_{i=2}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[i]-r[1]) + \frac{2\mathbf{sn}}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[j]-r[i]) =$$

$$\frac{2\mathbf{sn}}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[i]-r[1]) + \frac{2\mathbf{sn}}{\mathbf{m}}\sum_{i=2}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[j]-r[i]) =$$

$$\frac{2\mathbf{sn}}{\mathbf{m}}\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[j]-r[i])$$

The elaboration of the second expression gives the same results so that the $(2p,2p)$ sub-matrix of $(g_{\beta\alpha\cdot\text{III}})$ consists of non-diagonal elements zero and diagonal elements:

$$\boxed{\frac{2\mathbf{s}\cdot\mathbf{n}\cdot\mathbf{n1}\cdot\mathbf{n2}}{\mathbf{m}}\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[j]-r[i])} \tag{10}$$

This general analytic expression for the inverse of the weight coefficients of the parameters of the periodic part of the deviation in the graduation lines proved to be the key to simple computational programmes for the Bessel and Schreiber method of direction measurement.

As to the vector $(\underline{y}_{\beta\cdot\text{III}})$ it must be stressed that the first $\mathbf{s}(\mathbf{m}-1)$ elements of $(\underline{y}_{\beta\cdot\text{III}})$ need not be computed separately because from the general relation:

$$(\underline{y}^{\alpha}) = (\bar{g}^{\alpha\beta})(\underline{y}_{\beta})$$

immediately follows that the first $s(m-1)$ elements of $(\underline{y}^{\alpha\cdot\text{III}})$ are equal to:

$$\underline{l1}[s,i,1] = \frac{1}{\mathbf{n}}\sum_{n=1}^{n}\underline{l2}[s,n,i,1]$$

which means a simple averaging of the observations of each angle in the $\mathbf{n}$ circle positions in each set s.

The general expression given in the diagram for the last $2\mathbf{p}$ elements of the vector $(\underline{y}_{\beta\cdot\text{III}})$, denoted by $\underline{y}_{\beta\cdot\text{III}}[1,zp]$ and $\underline{y}_{\beta\cdot\text{III}}[2,zp]$, indicating that they refer to the unknown variates $\underline{a}[1,zp]$ and $\underline{a}[2,zp]$, are transformed in the following general expressions which are more suitable for the elaboration with a computer.

With $c[s,n] \equiv r[s,n,1]$, the following relations, given in matrix form, hold:

$$\begin{pmatrix}\cos zpr[s,n,i]-\cos zpr[s,n,1]\\ \sin zpr[s,n,i]-\sin zpr[s,n,1]\end{pmatrix} = -2\begin{pmatrix} ss[s,n,p,i,1] & sc[s,n,p,i,1]\\ -sc[s,n,p,i,1] & ss[s,n,p,i,1]\end{pmatrix}\begin{pmatrix}\cos zpc[s,n]\\ \sin zpc[s,n]\end{pmatrix}$$

with:

$$ss[s,n,p,i,1] = \sin\tfrac{1}{2}zp(r[s,n,i]-r[s,n,1])\sin\tfrac{1}{2}zp(r[s,n,i]-r[s,n,1])$$

$$sc[s,n,p,i,1] = \sin\tfrac{1}{2}zp(r[s,n,i]-r[s,n,1])\cos\tfrac{1}{2}zp(r[s,n,i]-\mathrm{r}[s,n,1])$$

In these last two expressions the directions need only to be given in tenth of a degree, hence:

$$r[s,n,i]-r[s,n,1] \equiv r[i]-r[1]$$

and:

$$ss[s,n,p,i,1] \equiv ss[p,i,1]$$

$$sc[s,n,p,i,1] \equiv sc[p,i,1]$$

From the diagram for $(\underline{y}_{\beta\cdot\text{III}})$ it follows that:

$$\left.\begin{aligned}
\frac{1}{\mathbf{n1}\cdot\mathbf{n2}}\underline{y}_{\beta\cdot\text{III}}[1,zp] = & -2\sum\ldots\sum_{i=2}^{\mathbf{m}} ss[p,i,1]\cdot\cos zpc[s,n]\cdot\underline{l2}[s,n,i,1] \\
& -2\sum\ldots\sum_{i=2}^{\mathbf{m}} sc[p,i,1]\cdot\sin zpc[s,n]\cdot\underline{l2}[s,n,i,1] \\
& +\frac{2}{\mathbf{m}}\left(\sum_{i=2}^{\mathbf{m}} ss[p,i,1]\right)\sum\ldots\sum_{i=2}^{\mathbf{m}}\cos zpc[s,n]\cdot\underline{l2}[s,n,i,1] \\
& +\frac{2}{\mathbf{m}}\left(\sum_{i=2}^{\mathbf{m}} sc[p,i,1]\right)\sum\ldots\sum_{i=2}^{\mathbf{m}}\sin zpc[s,n]\cdot\underline{l2}[s,n,i,1] \\
\text{and:} & \\
\frac{1}{\mathbf{n1}\cdot\mathbf{n2}}\underline{y}_{\beta\cdot\text{III}}[2,zp] = & +2\sum\ldots\sum_{i=2}^{\mathbf{m}} sc[p,i,1]\cdot\cos zpc[s,n]\cdot\underline{l2}[s,n,i,1] \\
& -2\sum\ldots\sum_{i=2}^{\mathbf{m}} ss[p,i,1]\cdot\sin zpc[s,n]\cdot\underline{l2}[s,n,i,1] \\
& -\frac{2}{\mathbf{m}}\left(\sum_{i=2}^{\mathbf{m}} sc[p,i,1]\right)\sum\ldots\sum_{i=2}^{\mathbf{m}}\cos zpc[s,n]\cdot\underline{l2}[s,n,i,1] \\
& +\frac{2}{\mathbf{m}}\left(\sum_{i=2}^{\mathbf{m}} ss[p,i,1]\right)\sum\ldots\sum_{i=2}^{\mathbf{m}}\sin zpc[s,n]\cdot\underline{l2}[s,n,i,1]
\end{aligned}\right\}\quad(11)$$

The shifting variate becomes:

$$\frac{1}{\mathbf{n1}\cdot\mathbf{n2}}\underline{E}^{\text{III}} = \frac{\mathbf{m}-1}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}}(\underline{l1}[s,i,1]-\underline{l2}[s,n,i,1])^2+$$

$$-\frac{2}{\mathbf{m}}\sum\ldots\sum_{i=2}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}(\underline{l1}[s,i,1]-\underline{l2}[s,n,i,1])(\underline{l1}[s,j,1]-\underline{l2}[s,n,j,1])$$

$$-\frac{2\mathbf{ns}}{\mathbf{m}}\sum_{p=1}^{\mathbf{p}}\left(\{\underline{a}[1,zp]^2+\underline{a}[2,zp]^2\}\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[j]-r[i])\right)$$

and the number of redundant observations:

$$b^{\text{III}} = \mathbf{sn}(\mathbf{m}-1)-\mathbf{s}(\mathbf{m}-1)-2\mathbf{p}$$

The estimate of the variance factor in the third phase includes, besides the effect of pointing and reading, the random effect of the deviation in the graduation lines and thus depends on the degree of approximation **p** of the Fourier series. Consequently it should be borne in mind that this estimate cannot simply be compared with the estimation in the first two phases by computing a value for the statistic $\underline{F}_{b^{\text{I}},b^{\text{III}}}$ or $\underline{F}_{b^{\text{II}},b^{\text{III}}}$. The question as to the total number of periods **p** that should be determined is touched upon in section 4.

Finally the following remark can be made. When taking the mean of all sets:

$$\underline{l}[i,1] = \frac{1}{\mathbf{s}}\sum_{s=1}^{\mathbf{s}}\underline{l1}[s,i,1]$$

it follows from the inverse of $(g_{\beta\alpha\cdot\mathrm{III}})$ (see page 19), that the variates $\underline{l}[i,1]$ correlate:

$$\overline{l[i,1],\, l[j,1]} = \frac{2}{\mathbf{n}\cdot\mathbf{n1}\cdot\mathbf{n2}\cdot\mathbf{s}} \quad \text{for} \quad i=j \quad \text{and} \quad \frac{1}{\mathbf{n}\cdot\mathbf{n1}\cdot\mathbf{n2}\cdot\mathbf{s}} \quad \text{for} \quad i \neq j$$

Although a horizontal direction variate cannot be obtained by means of an experiment [10, p. 45, 46] the probability distribution of the angles $\underline{l}[i,1]$ may be described by the probability distribution of theoretically defined direction variates $\underline{r}[i]$ which appear to be non-correlating. With $\underline{l}[i,1] = \underline{r}[i] - \underline{r}[1]$, the $(\mathbf{m}-1)$ variates $\underline{l}[i,1]$ are substituted by $\mathbf{m}$ variates $\underline{r}[i]$. The probability distribution of one of these latter variates is arbitrary.

When taking:

$$\overline{r[1],\, r[1]} = \frac{1}{\mathbf{n}\cdot\mathbf{n1}\cdot\mathbf{n2}\cdot\mathbf{s}}$$

$$\overline{r[1],\, r[i]} = 0 \quad \text{for} \quad i = 2,\ldots,\mathbf{m}$$

and the sample value $r[1] = 0$, the following information is obtained:

$$r[1] = 0$$

$$r[i] = l[i,1] \quad \text{for} \quad i = 2,\ldots,\mathbf{m}$$

and:

$$\overline{r[i],\, r[j]} = \delta^i_j \cdot \frac{1}{\mathbf{n}\cdot\mathbf{n1}\cdot\mathbf{n2}\cdot\mathbf{s}} \quad \text{with} \quad \delta^i_j = \begin{cases} 1 & \text{for} \quad i=j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

3 THE ADJUSTMENT OF A SCHREIBER PROGRAMME

According to the Schreiber method all possible combinations of two directions which can not be suppplemented to a full round, are observed.

Let **m** be the number of directions, then $\binom{\mathbf{m}}{2}$ combinations or angles are possible. Let these angles be arranged in the usual sequence (1,2), (1,3), …, (1,**m**), (2,3), (2,4), …, (2,**m**), …, (i,j), … (**m** − 1, **m**), then the sequence number of the angle (i,j) is

$$k = \sum_{l=1}^{i} (l-1)(\mathbf{m}+1-i)+(j-i)$$

Let each angle be observed in s positions of the telescope, in order to eliminate the collimation error, and let each telescope position include **n** circle positions. The **sn** circle positions for each angle (i,j) are equally distributed over $2\pi/z$ where z takes the values 1 or 2 depending on whether a single point or diametrical reading device has been fitted into the instrument.

For the angle (i,j) with sequence number k these circle positions, denoted by $c[s,n,k]$, are usually taken as follows:

$$c[s,n,k] = c[1,1,k] + \left(\frac{n-1}{\mathbf{n}} + s - 1\right)\frac{2\pi}{\mathbf{s}z}$$

$$c[1,1,k] = c[1,1,1] + \frac{k-1}{\mathbf{ns}\binom{\mathbf{m}}{2}}\cdot\frac{2\pi}{z}$$

thus with:

$$c[1,1,1] = c$$

$$c[s,n,k] = c + \left(\frac{k-1}{\mathbf{n}\binom{\mathbf{m}}{2}} + \frac{n-1}{\mathbf{n}} + s - 1\right)\frac{2\pi}{\mathbf{s}z}$$

For the adjustment the following matrix of weight coefficients is introduced:

$$\overline{r[s,n,i],\, r[s,n,j]} = \delta^i_j$$

with the unknown variance factor σ^2.

The transfer from directions to angles can be represented by:

$$\underline{l2}[s,n,i,j] = \underline{r}[s,n,j] - \underline{r}[s,n,i]$$

with:

$$\overline{l2[s,n,i,j],\, l2[s,n,k,l]} = \delta^i_k \delta^j_l \cdot 2$$

and:

$$\begin{aligned} s &= 1, \ldots, \mathbf{s} \\ n &= 1, \ldots, \mathbf{n} \\ i &= 1, \ldots, \mathbf{m}-1 \\ j &= i+1, \ldots, \mathbf{m} \end{aligned}$$

In a first phase $\binom{\mathbf{m}}{2}$ **sn** correction equations, according to the 2nd standard problem, can be formed with $\binom{\mathbf{m}}{2}$ **s** unknown variates $\underline{l1}[s,i,j]$ and **2p** unknown variates $\underline{a}[1,zp]$, $\underline{a}[2,zp]$:

$$\underline{l2}[s,n,i,j] + \underline{\varepsilon 2}[s,n,i,j] = \underline{l1}[s,i,j] + \sum_{p=1}^{\mathbf{p}} \underline{a}[1,zp](\cos zpr[s,n,j] - \cos zpr[s,n,i]) + \\ + \sum_{p=1}^{\mathbf{p}} \underline{a}[2,zp](\sin zpr[s,n,j] - \sin zpr[s,n,i])$$

As to the unknown variates $\underline{l1}[s,i,j]$, it can easily be seen that the adjustment is reduced to a simple averaging of the observations of each angle viz.:

$$\underline{l1}[s,i,j] = \frac{1}{\mathbf{n}} \sum \underline{l2}[s,n,i,j]$$

with:

$$\overline{l1[s,i,j], l1[s,k,l]} = \delta_k^i \delta_l^j \cdot \frac{2}{\mathbf{n}}$$

As to the unknown variates $\underline{a}[1,zp]$, $\underline{a}[2,zp]$, expressions for $\underline{y}_\beta[1,zp]$, $\underline{y}_\beta[2,zp]$ and $g_{\beta\alpha}[1,zp]$ $g_{\beta\alpha}[2,zp]$, denoting the elements of $(\underline{y}_\beta)$ and the diagonal elements of $(g_{\beta\alpha})$, can easily be found from the corresponding expressions obtained for the Bessel method.

The contribution of the observations of the angle (i,j) to the above mentioned elements of $(\underline{y}_\beta)$ is found by substituting $1 = i$, $i = j$, $\mathbf{m} = 2$ and $\mathbf{n1} = \mathbf{n2} = 1$ in the expressions for $(\underline{y}_\beta)$ of the Bessel method, see (11).

The elements of $\underline{y}_\beta$ are found after summation over i and j:

$$\left.\begin{aligned} \underline{y}_\beta[1,zp] &= \sum \ldots \sum_{i=1}^{\mathbf{m}-1} \sum_{j=i+1}^{\mathbf{m}} - ss[zp,i,j] \cos zpc[s,n,k] \cdot \underline{l2}[s,n,i,j] \\ &\qquad - sc[zp,i,j] \sin zpc[s,n,k] \cdot \underline{l2}[s,n,i,j] \\ \underline{y}_\beta[2,zp] &= \sum \ldots \sum_{i=1}^{\mathbf{m}-1} \sum_{j=i+1}^{\mathbf{m}} + sc[zp,i,j] \cos zpc[s,n,k] \cdot \underline{l2}[s,n,i,j] \\ &\qquad - ss[zp,i,j] \sin zpc[s,n,k] \cdot \underline{l2}[s,n,i,j] \end{aligned}\right\} \tag{12}$$

Substituting in the expression for $(g_{\beta\alpha})$ of the Bessel method in (10): $\mathbf{m} = 2$, $\mathbf{n1} = \mathbf{n2} = 1$ and summing over i and j gives for the diagonal elements of $(g_{\beta\alpha})$:

$$g_{\beta\alpha}[1,zp] = g_{\beta\alpha}[2,zp] = \mathbf{sn} \sum_{i=2}^{\mathbf{m}-1} \sum_{j=i+1}^{\mathbf{m}} \sin^2 \tfrac{1}{2} zp(r[j] - r[i]) \tag{13}$$

Least square estimates of the unknown variates $\underline{a}[1,zp]$, $\underline{a}[2,zp]$ are found from:

$$\underline{y}^{\alpha}[1,zp] \equiv \underline{a}[1,zp] = \bar{g}^{\alpha\beta}[1,zp]\cdot\underline{y}_{\beta}[1,zp]$$

$$\underline{y}^{\alpha}[2,zp] \equiv \underline{a}[2,zp] = \bar{g}^{\alpha\beta}[2,zp]\cdot\underline{y}_{\beta}[2,zp]$$

If the adjustment in the first phase has been carried out without taking into consideration a periodical deviation in the circle graduation, which means $\mathbf{p} = 0$, the shifting variate becomes:

$$\underline{E}^{\mathrm{I}}(0) = \tfrac{1}{2}\sum \ldots \sum_{i=1}^{\mathbf{m}-1} \sum_{j=i+1}^{\mathbf{m}} \{\underline{l1}[s,i,j] - \underline{l2}[s,n,i,j]\}^2$$

with:

$$b^{\mathrm{I}}(0) = (\mathbf{n}-1)\mathbf{s}\binom{\mathbf{m}}{2}$$

If $\mathbf{p}$ periods have been taken into account this value should be decreased by $(\underline{y}_{\alpha})^{*}(\bar{g}^{\alpha\beta})(\underline{y}_{\beta})$, thus (see [10], p. 9):

$$\underline{E}^{\mathrm{I}}(\mathbf{p}) = \underline{E}^{\mathrm{I}}(0) - (\underline{y}_{\alpha})^{*}(\bar{g}^{\alpha\beta})(\underline{y}_{\beta})$$

and:

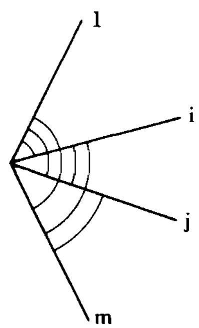
$$b^{\mathrm{I}}(\mathbf{p}) = b^{\mathrm{I}}(0) - 2\mathbf{p}$$

Before carrying out the second phase of the adjustment, means are taken of the angles (i,j) as obtained in the $\mathbf{s}$ face positions

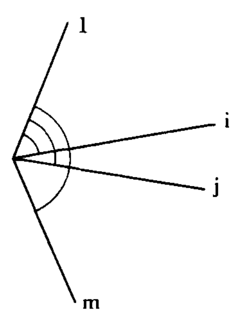
$$\underline{l}[i,j] = \frac{1}{\mathbf{s}}\sum \underline{l1}[s,i,j] \quad \text{with:}$$

$$\overline{l[i,j],\, l[k,l]} = \delta^{i}_{k}\delta^{j}_{l}\,\frac{2}{\mathbf{ns}}$$

In a second phase $\binom{\mathbf{m}}{2}$ condition equations according to the 2nd standard problem can be formed with $(\mathbf{m}-1)$ unknown variates $\underline{ll}[1,j]$ with $j = 2, \ldots, \mathbf{m}$



observation variates $\underline{l}[i,j]$



unknown variates $\underline{ll}[1,j]$

In order to give the algorithm in a generalized form, fictitious observations $ll[1,1] = 0$, $l[j,i] = -l[i,j]$ and $l[i,i] = 0$ are introduced. These „observations" may be interpreted as sample values of observational variates which need no further definition.

A general expression can now be given for the correction equations:

$$l[i,j]+\varepsilon[i,j] = ll[1,j]-ll[1,i]$$

with:

$$i = 1, \ldots, \mathbf{m}-1$$

$$j = i+1, \ldots, \mathbf{m}$$

Applying the algorithm of the 2nd standard problem the following expressions are found:

$$\underline{y}_\beta[j-1] = \frac{\mathbf{ns}}{2}\sum_{i=1}^{\mathbf{m}} \underline{l}[i,j], \quad \text{for} \quad j = 2, \ldots, \mathbf{m}$$

$$g_{\beta\alpha} = \begin{cases} \dfrac{(\mathbf{m}-1)\,\mathbf{ns}}{2} & \text{for} \quad \alpha = \beta \\[2ex] -\dfrac{\mathbf{ns}}{2} & \text{for} \quad \alpha \neq \beta \end{cases}$$

$$\bar{g}^{\alpha\beta} = \begin{cases} \dfrac{2\cdot 2}{\mathbf{mns}} & \text{for} \quad \alpha = \beta \\[2ex] \dfrac{2}{\mathbf{mns}} & \text{for} \quad \alpha \neq \beta \end{cases}$$

and:

$$\underline{y}^\alpha[j-1] \equiv \underline{ll}[1,j] = \frac{1}{\mathbf{m}}\sum_{i=1}^{\mathbf{m}} (\underline{l}[1,i] - \underline{l}[j,i]), \quad \text{for} \quad j = 2, \ldots, \mathbf{m}$$

The above expressions can be verified for $\mathbf{m} = 5$ with the example in [1, p. 302].

The shifting variate in the 2nd phase becomes:

$$\underline{E}^{\text{II}} = \frac{\mathbf{ns}}{2}\sum_{i=1}^{\mathbf{m}-1} \sum_{j=i+1}^{\mathbf{m}} \{\underline{ll}[1,j] - \underline{ll}[1,i] - \underline{l}[i,j]\}^2$$

with:

$$b^{\text{II}} = \binom{\mathbf{m}-1}{2}$$

From $(\bar{g}^{\alpha\beta})$, which elements are $2.2/\mathbf{nsm}$ for $\alpha = \beta$ and $2/\mathbf{nsm}$ for $\alpha \neq \beta$, it follows that the quantities $\underline{ll}[1,j]$, with $j = 2, \ldots, \mathbf{m}$, are correlating.

When introducing (see the reasoning in section 2):

$$\underline{ll}[1,j] = \underline{r}[j]-\underline{r}[1]$$

with:

$$\overline{r[1], r[1]} = \frac{1}{\mathbf{nsm}} \quad \text{and} \quad \overline{r[1], r[j]} = 0$$

and taking the sample value $r[1] = 0$, it follows that:

$$\overline{r[i], r[j]} = \delta^i_j \frac{1}{\mathbf{nsm}}$$

which means that the probability distribution of $ll[1,j]$ may be substituted by the probability distribution of the non-correlating direction variates $\underset{\sim}{r}[i]$.

In order to obtain equal precision for the directions in different stations, **nsm** is made constant for the whole network.

4 SOME SPECIAL REFERENCES TO CIRCLE TESTING

a. As to the relation between the weight coefficients of $\underset{\sim}{a}[1,zp]$, $\underset{\sim}{a}[2,zp]$ and the function $\underline{Rr}$ on the one hand and the directions of the bundle of rays on the other hand, the following remark can be made.

The formulas for the weight coefficients:

$$\overline{a[1,zp],a[1,zp]} = \overline{a[2,zp],a[2,zp]} = \frac{\mathbf{m}}{2\mathbf{ns}}\left(\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[j]-r[i])\right)^{-1}$$

and:

$$\overline{Rr,Rr} = \frac{\mathbf{m}}{2\mathbf{ns}}\sum_{p=1}^{\mathbf{p}}\left(\sum_{i=1}^{\mathbf{m}-1}\sum_{j=i+1}^{\mathbf{m}}\sin^2\tfrac{1}{2}zp(r[j]-r[i])\right)^{-1}$$

are evaluated for three bundles of 4 rays for a theodolite with diametrical reading on the same circle graduation, thus $z = 2$.

The total number of circle positions is $ns = 50$

Table of weight coefficients $\overline{a[1,zp],a[1,zp]}$

p \ r	0, 20, 40, 80 gr.	0, 40, 80, 100 gr.	0, 15, 37, 90 gr. (cf. ROELOFS [8])
1	0.0164	0.0127	0.0141
2	0.0106	0.0117	0.0132
3	0.0113	0.0102	0.0116
4	0.0106	0.0140	0.0113
5	0.0133	0.0133	0.0123
6	0.0106	0.0117	0.0102
7	0.0113	0.0103	0.0120
Rr	0.0841	0.0849	0.0847

From the table it appears that the weight coefficients of the parameters $\underset{\sim}{a}$, and especially of $\underline{Rr}$, hardly depend on the choice of the bundle of rays in a circle testing. The conclusion is that any sensible combination of test directions is acceptable, cf. ROELOFS [8].

b. An important question is: how many members of the Fourier series are needed for an adequate representation of the regular deviation of the circle graduation? An upper limit for p is set by the ALGOL programme as this programme makes use of the diagonal form of the $[\mathbf{2p},\mathbf{2p}]$ submatrix of $(g_{\beta\alpha})$, which form is only adapted for $\mathbf{p} < \mathbf{n}/2$ as is demonstrated in section 1. Consequently the ALGOL programme permits only the determination of the amplitudes of the harmonics with p less than $\mathbf{n}/2$. The choice of $\mathbf{p}$, however, should not be dictated by numerical considerations but should be based on the experience and intuition of the observer.

In [11] a possible occurrence of short periodical components has been made plausible. These components, with a period of about 2,5 gr, are directly due to the manufacturing process – e.g. run of screws – and manufacturers will certainly do their utmost to keep these deviations within acceptable limits.

The methods of circle testing under discussion are primarily concerned with the determination of regular deviations of a larger extent. The shorter periods are only signalized by an off-balance ratio of $\hat{\sigma}^2_{\text{II}}$ and $\hat{\sigma}^2_{\text{I}}$.

The larger periods might be brought about by stress in the glass introduced by temperature - or other effects. It is also possible that repeated checks, carried out during the manufacturing process have led to a more or less continual change of the interval of graduation. In view of the very nature of these deviations it is not very likely, that, when developing the deviation in a Fourier series, some of the harmonics will have significant large amplitudes with respect to the amplitudes of the other harmonics. It is only the combination of all harmonics of subsequent order up to a certain limit, that may give a fair picture of the regular deviation in the circle graduation. Theoretically it is not possible to indicate *a priori* the order of the harmonic from which the deviations of the graduation should be considered of stochastic nature, and therefore it has been suggested ([8], [10]) to consider the normal use of a theodolite in the field.

In primary field measurement directions are observed in at least 8 sets. That implies that in the means the effect of all harmonics of *Rr* is eliminated except those of order 8, 16, and so on. Consequently it should be convenient to determine in a circle testing programme the first seven harmonics of *Rr*, as this means that the estimate of the variance factor in the corresponding adjustment will give a fairly good insight into the combined effect of pointing, circle reading, and that part of the circle deviation which actually contributes to the variance of primary field measurements.

It should, however, be remembered that pointings under laboratory and field conditions are not comparable. Therefore it may be useful to have the effect of pointing and reading separated from the random circle deviation. This can be realised as most circle testing programmes have the possibility of determining separately the precision of pointing and reading. A measure for this precision is obtained from the estimate of the variance factor in the first two phases.

A few experiments [7], [8] have shown that the determination of seven harmonics means a considerable improvement with respect to the original Heuvelink programme [5], where three harmonics were determined.

c. The determination of $\underline{Rr}$ is not of interest when the theodolite is exclusively used for normal field measurements where observations can be repeated in different circle positions. For such instruments a general testing programme under field conditions is preferred to a special circle testing programme in a laboratory. Such a testing programme in the field need not be designed specially. The usual primary field measurements, carried out in a specified period during which the theodolite is subjected to trial, may be used.

All these programmes now should be adjusted to a mathematical model, as given in sections 2 and 3, that takes full account of the presence of the regular part of the deviation in the circle graduation, as represented by a number of lower order harmonics in the Fourier series. The estimate of the thus obtained variance factor, multiplied by the weight coefficient which equals unity, represents the precision – including the effects of pointing, reading, and random circle deviation – in a single direction and this value actually characterizes the quality of a theodolite as a whole.

A special circle testing programme in a laboratory is only necessary when the utmost precision is wanted. Then it might be useful to separate the effects and have an idea of the quality of the circle graduation itself. A special circle test is also needed when the theodolite is used for special purposes – e.g. for technical measurements or astronomical work – where observations in different positions of the circle are not always possible.

5 THE ALGOL PROGRAMMES

Explanation to the programmes

The programmes are written in an ALGOL 60 version, especially adapted to the TR 4 computer. The instructions for use are given below. The notation before the brackets is used in the ALGOL programme whereas in the preceding sections the same quantity is denoted by the bold type between the brackets.

Programme for the Bessel method

The following data must be introduced in the given order:

observer, date, instrument,

em(**m**), the number of directions that has been observed.

es(**s**), for the field measurements: the number of telescope positions. **s** is two when the telescope is reversed halfway the observations.
for circle testing measurements: the number of partial programmes; each partial programme is designed in such a way that it can be considered as a complete circle testing programme itself.

en(**n**), the number of circle positions in each partial programme or telescope position. These circle positions are regularly distributed over $2\pi/z$.

en1(**n1**), the number of sets observed in the same circle position, immediately after each other; for normal field measurements this quantity is always equal to 1.

en2(**n2**), the number of subprogrammes in each partial programme.
The first subprogramme is observed in ascending progression of the circle positions, the second subprogramme in the reversed order, and so on.
Consequently the total number of observations of each direction in the same circle position is **n1**·**n2**.
For normal field measurements this quantity is always equal to 1.

pe(**p**), the number of periods that is to be determined in the Fourier series for the regular part $\underline{Rr}$ of the circle deviation..

z, a variable that takes the value 1 or 2; $z = 1$ when the fundamental period of $\underline{Rr}$ is 2π; $z = 2$ when this period is π.

c, the very first circle reading in degrees given with one decimal; c is usually 0.0.

q, a variable that takes the value 1 or 2;
$q = 1$ if the observations are fully introduced in degrees and with all digits; this will be the case for the field measurements.
$q = 2$ if the observations only consist of the micrometer readings in cc; this will always be the case for a circle testing programme.

tel[1:**m**], this array must be introduced only if $q = 2$; it consists of the approximate values for the directions in degrees with two decimals.

r[1:**s**, 1:**n**, 1:**n2**, 1:**n1**, 1:**m**], the array of observations;
if $q = 1$ the observations are fully introduced in degrees and with all digits;
if $q = 2$ the micrometer readings are introduced in cc.

Programme for the Schreiber method

The same quantities in the same order must be introduced with the exception that the quantities **n1**, **n2** and the array tel [1 : **m**] are omitted; **s**, **n** and q are now respectively the number of telescope positions, the number of circle positions in these positions and the number of decimals of the circle reading. The observations are arranged in the array: r[1 : **k**, 1 : **s**, 1 : **n**, 1 : 2], and are always introduced with all digits in degrees. The quantity k(k = 1, ..., **k**) is the sequence number of the angle, see section 3.

Programme for the Bessel method

```
'BEGIN'  'INTEGER' EM,ES,EN,EN1,EN2,PE,Z,C,Q;
  'REAL' RHO,PI;
  'PROCEDURE' STREEP;
  WRITE('':'');
  'PROCEDURE' DASH(I);'INTEGER'I;
  'BEGIN' 'INTEGER' K;
  'FOR' K:=1'STEP' 1'UNTIL' I'DO' WRITE(''-'')
  'END';
  'ARRAY' NAAM1,NAAM2,NAAM3[1:1o];
  READ(NAAM1,NAAM2,NAAM3);
  WRITE(''THE ADJUSTMENT OF A BESSEL PROGRAM'');NLCR(4);
  WRITE(''INSTRUMENT: '',NAAM1,''
                                        DATE:        '',NAAM2,''
                                                              OBSERVER:    '',NAAM3); NLCR(2);
  READ(EM,ES,EN,EN1,EN2,PE,Z,C,Q);
  RHO:=63.66197724;PI:=3.141592654;
  'BEGIN' 'INTEGER' I,J,S,N,N1,N2,P;

    'REAL' E1,E2,F1,F2,F3,G1,G2,G3,WCO,B1,B2,CL1,CL2,CL3,SL1,SL2,SL3,SS,SC,SI,CO;
    'REAL' 'ARRAY' R,L4[1:ES,1:EN,1:EN2,1:EN1,1:EM],TEL,TELL[1:EM],
    L3[1:ES,1:EN,1:EN2,1:EM],
    L2[1:ES,1:EN,1:EM],
    L1[1:ES,1:EM],
    L[1:EM],CL,SL[2:EM],E3,B3[o:PE],
    GALFBE[1:PE],YBETA[1:2,1:PE];
    'IF' Q'EQUAL' 2 'THEN' READ(TEL)'ELSE' 'FOR'
    I:=1'STEP' 1'UNTIL' EM'DO' TEL[I]:=o;
    F1:=F2:=F3:=G1:=G2:=G3:=o;
    WCO:=1/(EN×EN1×EN2);
    READ(R);
    'FOR' S:=1'STEP' 1'UNTIL'  ES'DO'
    'FOR' N:=1'STEP' 1'UNTIL' EN'DO'
    'FOR' N2:=1'STEP' 1'UNTIL' EN2'DO'
    'FOR' N1:=1'STEP' 1'UNTIL' EN1'DO'
    'FOR' I:=1'STEP' 1'UNTIL' EM'DO'
```

```
'BEGIN'L4[S,N,N2,N1,I]:=R[S,N,N2,N1,I]-R[S,N,N2,N1,1];
'END';
WRITE(''ADJUSTED DIRECTIONS;THE NUMBERS IN THE FIRST ROW REFER TO
THE DIRECTION I,THE NUMBERS IN THE FIRST COLUMN
TO THE PARTIAL PROGRAMS'');NLCR(2);SPACE(4);STREEP;
'IF' Q'EQUAL' 1'AND' L4[S,N,N2,N1,I]
'LESS' o'THEN' L4[S,N,N2,N1,I]:=L4[S,N,N2,N1,I]+4oo'ELSE' 'BEGIN' TELL[I]:=TEL[I]-TEL[1];
L4[S,N,N2,N1,I]:=L4[S,N,N2,N1,I]/1oooo+TELL[I];'END'
'FOR' I:=1'STEP' 1'UNTIL' EM 'DO'
'BEGIN' SPACE(3); VASKO(1,o,I);SPACE(5);STREEP;
'END';
WRITE('' WCOEF  '');STREEP; NLCR(1);
'FOR' S:=1'STEP' 1'UNTIL' ES 'DO'
'BEGIN' VASKO(1,o,S);STREEP;
  'FOR' I:=1'STEP' 1'UNTIL' EM'DO'
  'BEGIN' L1[S,I]:=o;
    'FOR' N:=1'STEP' 1'UNTIL' EN'DO'
    'BEGIN' L2[S,N,I]:=o;
      'FOR' N2:=1'STEP' 1'UNTIL' EN2'DO'
      'BEGIN' L3[S,N,N2,I]:=o;
        'FOR' N1:=1'STEP' 1'UNTIL' EN1'DO'
        L3[S,N,N2,I]:=L3[S,N,N2,I]+L4[S,N,N2,N1,I]/EN1;
        L2[S,N,I]:=L2[S,N,I]+L3[S,N,N2,I]/EN2
      'END';
      L1[S,I]:=L1[S,I]+L2[S,N,I]/EN
    'END';
    VASKO(3,6,L1[S,I]);STREEP
  'END';
  VASKO(1,4,WCO);STREEP;NLCR(1);
'END';
WRITE(''MEAN'');STREEP;
'FOR' I:=1'STEP' 1'UNTIL' EM'DO'
'BEGIN' L[I]:=o; 'FOR' S:= 1'STEP' 1'UNTIL' ES'DO'
       L[I]:= L[I]+L1[S,I]/ES;
       VASKO(3,6,L[I]);
  STREEP
'END';
VASKO(1,4,WCO/ES);STREEP;
'FOR' S:=1'STEP' 1'UNTIL' ES'DO'
'BEGIN' 'FOR' I:=2'STEP' 1'UNTIL' EM'DO'
'FOR' N:=1'STEP' 1'UNTIL' EN 'DO'
'BEGIN' 'FOR' N2:=1'STEP' 1'UNTIL' EN2'DO'
  'BEGIN' 'FOR' N1:=1'STEP' 1'UNTIL' EN1'DO'
    F1:=F1+((EM-1)/EM)×(L3[S,N,N2,I]-L4[S,N,N2,N1,I])×(L3[S,N,N2,I]-L4[S,N,N2,N1,I]);
    F2:=F2+((EM-1)/EM)×(L2[S,N,I]-L3[S,N,N2,I])×(L2[S,N,I]-L3[S,N,N2,I])
  'END';
  F3:=F3+((EM-1)/EM)×(L1[S,I]-L2[S,N,I])×(L1[S,I]-L2[S,N,I])
'END';
'FOR' I:=2'STEP' 1'UNTIL' EM-1'DO'
```

```
 'FOR' J:=I+1'STEP' 1'UNTIL' EM'DO'
'FOR' N:=1'STEP' 1'UNTIL' EN'DO'
 'BEGIN' 'FOR' N2:=1'STEP' 1'UNTIL' EN2'DO'
    'BEGIN' 'FOR' N1:=1'STEP' 1'UNTIL' EN1'DO'
       G1:=G1-(2/EM)×(L3[S,N,N2,I]-L4[S,N,N2,N1,I])×(L3[S,N,N2,J]-L4[S,N,N2,N1,J]);
       G2:=G2-(2/EM)×(L2[S,N,I]-L3[S,N,N2,I])×(L2[S,N,J]-L3[S,N,N2,J])
    'END';
    G3:=G3-(2/EM)×(L1[S,I]-L2[S,N,I])×(L1[S,J]-L2[S,N,J])
 'END';
 'END';
E1:=(F1+G1)×100000000;E2:=EN1×(F2+G2)×100000000;E3[o]:=EN1×EN2×(F3+G3)×100000000;
B1:=ES×EN×EN2×(EN1-1)×(EM-1);
B2:=ES×EN×(EN2-1)×(EM-1);
B3[o]:=ES×(EN-1)×(EM-1);
NLCR(3);
WRITE(''PHASE  : SIGKWA  :    B  '');STREEP;NLCR(1);DASH(7);
STREEP;DASH(9);STREEP;DASH(6);STREEP;NLCR(1);
WRITE(''FIRST  '');STREEP;'IF' EN1'EQUAL' 1'THEN' WRITE(''UNDEFINED'')
'ELSE' VASKO(4,2,E1/B1);STREEP;VASKO(3,o,B1);STREEP;
NLCR(1);WRITE(''SECOND '');STREEP;'IF' EN2'EQUAL' 1'THEN' WRITE(''UNDEFINED'')
'ELSE' VASKO(4,2,E2/B2);STREEP;VASKO(3,o,B2);STREEP;NLCR(3);
WRITE(''THIRD PHASE (EXPRESSED IN CC)'');NLCR(2);
WRITE('' P   A[1]   A[2]    WCOEFF   SIGKWA   B'');
NLCR(1);WRITE(''  o'');SPACE(24);
VASKO(4,2,E3[o]/B3[o]);
VASKO(3,o,B3[o]);
NLCR(1);
'FOR' P:=1'STEP' 1'UNTIL' PE 'DO'
'BEGIN' GALFBE[P]:=o;
   SS:=SC:=CL1:=CL2:=CL3:=SL1:=SL2:=SL3:=o;
   'FOR' I:=1'STEP' 1'UNTIL' EM-1'DO'
   'FOR' J:=I+1'STEP' 1'UNTIL' EM'DO'
   GALFBE[P]:=GALFBE[P]+(ES×EN×EN2×EN1×2×(SIN(P×Z×(L[J]-L[I])/(2×RHO)))
   ×(SIN(P×Z×(L[J]-L[I])/(2×RHO))))/EM;GALFBE[P]:=1/GALFBE[P];
   'IF' GALFBE[P]'GREATER' 1oo'THEN'
   'BEGIN' VASKO(2,o,P×Z);WRITE(''UNDEFINED''); NLCR(1);
      E3[P]:=E3[P-1];B3[P]:=B3[P-1];
      'GOTO' EINDE
   'END';
   'FOR' I:=2'STEP' 1'UNTIL' EM'DO'
   'BEGIN' CL[I]:=SL[I]:=o;
      'FOR' S:=1'STEP' 1'UNTIL' ES'DO'
      'FOR' N:=1'STEP' 1'UNTIL' EN'DO'
         'BEGIN' CL[I]:=CL[I]+(COS(P×(Z×C/RHO+((S-1)/ES+N-1)×2×PI/EN)))×L2[S,N,I]×10000;
            SL[I]:=SL[I]+(SIN(P×(Z×C/RHO+((S-1)/ES+N-1)×2×PI/EN)))×L2[S,N,I]×10000;
         'END';
      SI:=SIN(P×Z×L[I]/(2×RHO));
      CO:=COS(P×Z×L[I]/(2×RHO));
      CL1:=CL1+CL[I]×SI×SI;
```

```
            CL2:=CL2+CL[I]×SI×CO;
            CL3:=CL3+CL[I];
            SL1:=SL1+SL[I]×SI×SI;
            SL2:=SL2+SL[I]×SI×CO;
            SL3:=SL3+SL[I];
            SS:=SS+SI×SI;
            SC:=SC+SI×CO;
         'END';
         YBETA[1,P]:=2×EN2×EN1×(-CL1-SL2+(SS×CL3+SC×SL3)/EM);
         YBETA[2,P]:=2×EN2×EN1×(+CL2-SL1+(-SC×CL3+SS×SL3)/EM);
         E3[P]:=E3[P-1]-(YBETA[1,P]×YBETA[1,P]+YBETA[2,P]×YBETA[2,P])×GALFBE[P];
         B3[P]:=B3[P-1]-2;
         VASKO(2,o,P×Z);
         VASKO(2,2,GALFBE[P]×YBETA[1,P]);
         VASKO(2,2,GALFBE[P]×YBETA[2,P]);
         VASKO(1,4,GALFBE[P]);
         VASKO(4,2,E3[P]/B3[P]);
         VASKO(3,o,B3[P]);
         NLCR(1);
      EINDE:'END';
   NPAG;
   'END'
'END'
```

Example applying the Bessel programme

```
INSTRUMENT: T3 NR18651
DATE:       15 OKT 1968
OBSERVER:   BUURMAN

ADJUSTED DIRECTIONS;THE NUMBERS IN THE FIRST ROW REFER TO
     THE DIRECTION I,THE NUMBERS IN THE FIRST COLUMN
     TO THE PARTIAL PROGRAMS
```

	1	2	3	4	WCOEF
1	0.000000	15.001665	37.001158	90.000821	0.0500
2	0.000000	15.001761	37.001144	90.000557	0.0500
3	0.000000	15.001424	37.000922	90.001089	0.0500
4	0.000000	15.001145	37.000666	90.000493	0.0500
MEAN	0.000000	15.001499	37.000973	90.000740	0.0125

PHASE	SIGKWA	B
FIRST	UNDEFINED	0
SECOND	35.05	120

THIRD PHASE (EXPRESSED IN CC)

P	A[1]	A[2]	WCOEFF	SIGKWA	B
0				49.31	108
2	2.59	- 0.36	0.0088	42.87	106
4	0.98	0.31	0.0082	42.46	104
6	- 0.59	- 2.01	0.0073	37.40	102
8	- 0.84	- 0.87	0.0071	36.07	100
10	0.68	- 0.64	0.0077	35.65	98

Note: The above observational programme was carried out by an inexperienced student-group which is revealed by the large value of the estimate of the variance factor. This should be taken into consideration when a comparison is made between this example and the one shown on page 41.

Programme for the Schreiber method

```
'BEGIN' 'INTEGER' EM,ES,EN,PE,Z,C,Q,KA;
   'REAL' RHO,PI;
   'PROCEDURE' STREEP;
   WRITE(''|'');
   'PROCEDURE' DASH(I);'INTEGER' I;
   'BEGIN' 'INTEGER' K;
      'FOR' K:=1'STEP' 1'UNTIL' I'DO' WRITE(''-'')
   'END';
   'ARRAY'NAAM1,NAAM2,NAAM3[1:10];
   READ(NAAM1,NAAM2,NAAM3);
    WRITE(''THE ADJUSTMENT OF A SCHREIBER PROGRAM'');NLCR(4);
    WRITE(''INSTRUMENT: '',NAAM1,''
                                          DATE:        '',NAAM2,''
                                                             OBSERVER:    '',NAAM3);NLCR(2);
   READ(EM,ES,EN,PE,Z,C,Q);RHO:=63.66197724; PI:=3.141592654;
   KA:=EM×(EM-1)/2;
   'BEGIN' 'INTEGER' I,J,S,N,P,K,B2;
      'REAL' CL,SL,SI,CO;
      'REAL' E2,WC;
      'REAL' 'ARRAY' R[1:KA,1:ES,1:EN,1:2],
      L2[1:EM,1:EM,1:ES,1:EN],L1[1:EM,1:EM,1:ES],
      L[1:EM,1:EM],LL[1:EM],B1,E1[0:PE],
      GALFBE[1:PE],A,YBETA[1:2,1:PE];
      READ(R);K:=0;WC:=1/(ES×EN×EM);
      'FOR' I:=1'STEP' 1'UNTIL' EM-1'DO'
      'FOR' J:=I+1'STEP' 1'UNTIL' EM'DO'
      'BEGIN' L[I,J]:=0;K:=K+1;
         'FOR' S:=1'STEP' 1'UNTIL' ES'DO'
         'BEGIN' L1[I,J,S]:=0;
            'FOR' N:=1'STEP' 1'UNTIL' EN'DO'
            'BEGIN' L2[I,J,S,N]:=R[K,S,N,2]-R[K,S,N,1];
               'IF' L2[I,J,S,N] 'LESS' 0 'THEN' L2[I,J,S,N]:=L2[I,J,S,N]+400;
               L1[I,J,S]:=L1[I,J,S]+L2[I,J,S,N]/EN
            'END';
            L[I,J]:=L[I,J]+L1[I,J,S]/ES
         'END';
         L[J,I]:=-L[I,J]
      'END';
      'FOR' I:=1 'STEP' 1 'UNTIL' EM 'DO' L[I,I]:=0;
      'FOR' J:=1'STEP' 1'UNTIL' EM'DO'
      'BEGIN' LL[J]:=0;
        'FOR' I:=1'STEP' 1'UNTIL' EM'DO'
        LL[J]:=LL[J]+(L[1,I]-L[J,I])/EM;
      'END';
```

```
E1[o]:=o;E2:=o;B1[o]:=ES×(EN-1)×KA;
B2:=(EM-1)×(EM-2)/2;
'FOR' I:=1'STEP' 1'UNTIL' EM-1'DO'
'FOR' J:=I+1'STEP' 1'UNTIL' EM'DO'
'BEGIN' 'FOR' S:=1'STEP' 1'UNTIL' ES'DO'
   'FOR' N:=1'STEP' 1'UNTIL' EN'DO'
   E1[o]:=E1[o]+((L1[I,J,S]-L2[I,J,S,N])×(L1[I,J,S]-L2[I,J,S,N])/2)×100000000;
   E2:=E2+EN×ES×((LL[J]-LL[I]-L[I,J])×(LL[J]-LL[I]-L[I,J])/2)×100000000;
'END';
WRITE(''FIRST PHASE (EXPRESSED IN CC)'');NLCR(2);
WRITE(''  P  |  A[1] |  A[2] | WCOEFF |  SIGKWA |   B  |'');NLCR(1);
DASH(5);STREEP;DASH(7);STREEP;DASH(7);STREEP;
DASH(8);STREEP;DASH(9);STREEP;DASH(6);STREEP;
NLCR(1); WRITE(''  o  ''); STREEP;SPACE(7);STREEP;SPACE(7);STREEP;
SPACE(8);STREEP;VASKO(4,2,E1[o]/B1[o]);STREEP;
VASKO(3,o,B1[o]);STREEP;NLCR(1);
'FOR' P:=1'STEP' 1'UNTIL' PE'DO'
'BEGIN' K:=o;YBETA[1,P]:=YBETA[2,P]:=GALFBE[P]:=o;
   'FOR' I:=1'STEP' 1'UNTIL' EM-1'DO'
   'FOR' J:=I+1'STEP' 1'UNTIL' EM'DO'
   'BEGIN' CL:=SL:=o; K:=K+1;
      'FOR' S:=1'STEP' 1'UNTIL' ES'DO'
      'FOR' N:=1'STEP' 1'UNTIL' EN'DO'
      'BEGIN' CL:=CL+((COS(P×(Z×C/RHO+((N-1)/
EN+S-1)×2×PI/ES+(K-1)×2×PI/(KA×EN×ES))))×L2[I,J,S,N])×10000;
         SL:=SL+((SIN(P×(Z×C/RHO+((N-1)/EN+S-1)×2×PI/ES+(K-1)×2×PI/(KA×EN×ES))))×
L2[I,J,S,N])×10000;
      'END';
      SI:=SIN(P×Z×(LL[J]-LL[I])/(2×RHO));
      CO:=COS(P×Z×(LL[J]-LL[I])/(2×RHO));
      YBETA[1,P]:=YBETA[1,P]-SI×SI×CL-SI×CO×SL;
      YBETA[2,P]:=YBETA[2,P]+SI×CO×CL-SI×SI×SL;
     GALFBE[P]:=GALFBE[P]+EN×ES×SI×SI;
   'END';
   GALFBE[P]:=1/GALFBE[P];
   A[1,P]:=GALFBE[P]×YBETA[1,P];
   A[2,P]:=GALFBE[P]×YBETA[2,P];
   E1[P]:=E1[P-1]-(A[1,P]×A[1,P]+A[2,P]×A[2,P])/GALFBE[P];
   B1[P]:=B1[P-1]-2;
'END';
'FOR' P:=1'STEP' 1'UNTIL' PE'DO'
'BEGIN' VASKO(2,o,P×Z);STREEP;
   VASKO(2,2,A[1,P]);STREEP;
   VASKO(2,2,A[2,P]);STREEP;
   VASKO(1,4,GALFBE[P]);STREEP;
   VASKO(4,2,E1[P]/B1[P]);STREEP;
   VASKO(3,o,B1[P]);STREEP;
   NLCR(1);
'END';
```

```
        NLCR(3); WRITE(''SECOND PHASE'');NLCR(3);
        WRITE(''THE ESTIMATE OF THE VARIANCEFACTOR IS SIGKWA=''); 
        VASKO(4,2,E2/B2);NLCR(1);
        WRITE(''THE NUMBER OF SUPERNUMEROUS OBSERVATIONS IS B='');
        VASKO(2,o,B2);
        NLCR(1);
        WRITE(''THE WEIGHTCOEFFICIENT OF THE ADJUSTED DIRECTIONS IS'');
        VASKO(1,4,WC);NLCR(1);WRITE(''THE ADJUSTED DIRECTIONS ARE'');NLCR(1);
        'FOR' I:=1'STEP' 1'UNTIL' EM'DO'
        'BEGIN' VASKO(3,Q+1,LL[I]);NLCR(1);
        'END';
        NPAG;
    'END'
'END'
```

Example applying the Schreiber programme

```
INSTRUMENT: WILD T3 NO 18651
DATE:       14 OKT 68
OBSERVER:   VERHOEF

FIRST PHASE (EXPRESSED IN CC)
```

P	A[1]	A[2]	WCOEFF	SIGKWA	B
o				1o.51	42
2	2.71	o.88	o.o439	6.41	4o
4	2.12	o.49	o.o411	3.7o	38
6	- o.12	- 1.57	o.o364	2.o2	36
8	o.o2	- o.42	o.o354	1.99	34
1o	o.57	- o.56	o.o384	1.59	32

```
SECOND PHASE


THE ESTIMATE OF THE VARIANCEFACTOR IS SIGKWA=    2.43
THE NUMBER OF SUPERNUMEROUS OBSERVATIONS IS B=  3
THE WEIGHTCOEFFICIENT OF THE ADJUSTED DIRECTIONS IS o.o313
THE ADJUSTED DIRECTIONS ARE
   o.oooooo
  15.ooo995
  37.ooo644
  9o.ooo543
```

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