NETHERLANDS GEODETIC COMMISSION

PUBLICATIONS ON GEODESY NEW SERIES

VOLUME 5 NUMBER 2

ON A GEODETIC APPLICATION OF MULTIPLE-STATION VERY LONG BASELINE INTERFEROMETRY

by

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1972

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PRINTED BY W. D. MEINEMA B.V., DELFT, NETHERLANDS

ISBN 90 6132 219 7

CONTENTS

SUMMARY

Multiple-station very long baseline interferometry (VLBI) is considered from a geodetic point of view. An approach is outlined to deal with relative phase delay data as obtained from measurements on both artificial radio sources at finite range and natural sources at infinity, observed in the simultaneous mode. The approach is basically a range-differences method and allows a treatment in theoretically convenient steps to which the several groups of parameters are primarily confined. This paper deals mainly with the interstation geometry of the VLBI-net. Attention is drawn to the possible occurrence of singular geometric situations of stations and sources for which no unique solution for the interstation geometry is possible in spite of the presence of an in general sufficient amount of measurement data. For a family of idealized model four-station VLBI-nets a numerical estimate is made of the accuracy likely attainable for the interstation distances, using two hypothetical observation programmes with sources at infinity in supposedly unknown directions. It was found that there is no unique solution if only one natural source is used. In the other programme two natural sources are used and variance-covariance matrices have been estimated with conventional least squares techniques. A variety of declination differences, VLBI-net orientations and maximum zenith distances is considered. It was found that the ratio between estimated standard deviations of the interstation distances and the range-difference measurements lies between 5 and 25 if 9 range-difference measurements are extracted from one pass of each of the two sources, provided cases of marginal observability of one of the sources are avoided and if it is assumed that the directions to the sources are not restricted to zenith distances below 55°. Station clocks are supposed to be perfectly synchronized.

ON A GEODETIC APPLICATION OF MULTIPLE-STATION VERY LONG BASELINE INTERFEROMETRY

1 Introduction

The feasibility of high angular resolution with interferometric systems operating in the centimetre and decimetre wavelength region resulted in recent years from the development of stable atomic frequency standards.

A subsequent cross-correlation of signals tape-recorded at the terminals of an interferometer base releases the baseline length limitation, the limit for earth-based baselines now being set by the dimensions of the earth only. This brings the angular resolution of very long baseline interferometry (VLBI) down to 0.001 of a second of arc and even better, a value not precedented by optical means. Ultimately however the precision limit will be set by atmospheric conditions again.

The sources usually considered for observation are natural, but artificial sources carried by geostationary or other artificial satellites or placed on the lunar surface have been considered as well. With such artificial sources the ground equipment could be simplified, due to better signal-to-noise ratios. The most desirable natural sources are strong, broad-band emitters of negligible angular size and sufficiently distant to have negligible proper motions. Some quasars seem to satisfy these requirements (Shapiro and Knight, 1969).

There are many potential scientific applications of VLBI, mainly in the fields of astronomy, geophysics and geodesy.

The quantity basically measured is the relative phase delay between terminals and this can be parameterized in terms of relative terminal locations, the earth's rotation and wobble, the positions of the radio sources and relative clock-offsets between terminals. The relative terminal locations are made up of baseline length and orientation and these need not to be invariable due to earth tides, global tectonics and other phenomena. In turn the natural sources may have proper motions. Parameters could also be included to account for unknown refractive effects and the bending of radio waves in the gravitational field of the sun. The latter would possibly provide a more definitive test of the theory of general relativity (Cohen et al., 1968).

Considering only those parameters which are obviously of geodetic interest, i.e. the relative antenna station locations and the earth's rotation rate and axis, much has been achieved recently or will be achieved in a near future by other techniques. Among these, laser range measurements to artificial satellites and to reflector packages on the moon should be mentioned.

Laser range measurements when made in the simultaneous or near-simultaneous mode are known not to provide a defined orientation, nor the location of a geodetic net with respect to the earth's mass centre. This is because in such an approach the positions of the objects ranged at are supposedly unknown and variable in a possibly complex way.

With VLBI in conjunction with distant natural radio sources an orientation of the net of antenna stations is obtained implicitly if the radio sources involved uniquely define a frame of reference, which in turn can be related to a conventional astronomical frame. Location of the net relative to the mass centre remains however unresolved.

The relation of the system defined by selected radio sources to a conventional system defined by the positions of selected optical sources, e.g. FK 4, could prove to be a difficult problem.

Therefore VLBI and laser ranging are able to supplement each other in a solution for a high precision global geodetic net, required for the monitoring of some geodynamic phenomena.

Also laser ranging and VLBI are closely related, if VLBI is looked upon as a method of range-difference measurement, like BROWN (1970) does. Indeed the VLBI-approach using distant radio sources in a simultaneous observation mode must be geometrically equivalent to simultaneous laser ranging to targets at infinity.

But pure simultaneous ground-to-satellite ranging is geodetically only meaningful if at least 4 stations participate in each observation event. Likewise simultaneous VLBI with distant sources would be geodetically only meaningful with at least 4 stations participating.

BROWN (1970) advocates such application of the VLBI approach in a coordinated net of receiving stations, rather than with isolated baselines. If (distant) natural sources are used with a net consisting of at least 4 stations, observing simultaneously, some geodetically significant parameters may be conveniently separated and other parameters, e.g. source positions, eventually subsequently be solved.

A similar approach is possible with artificial radio sources at finite distance, but then the net should exist of at least 5 simultaneously operating receiving stations, as will be pointed out in the following section.

The purpose of the present paper is to outline a model for the geodetic handling of multiple-station VLBI-data, collected in the simultaneous observation mode.

It should be remarked that valuable geodetic information may also be extracted from single-baseline VLBI, as pointed out by SHAPIRO and KNIGHT (1969). Such individual baselines may be combined as to form a net of three or more stations. If so, the solutions for the individual baselines should be constrained by a number of relations, although the measurements are not necessarily made simultaneously on the same sources.

2 Interstation geometry and relative clock-offsets

Each station $P_i (i = 1 \dots n)$ in an n-station ($n \geqslant 2$) VLBI-net is equipped with a radio receiving and recording system together with a very stable frequency standard which also locally controls time. Within this net there are $\frac{1}{2}n(n-1)$ baselines $P_iP_j (i, j = 1 \dots n; i < j)$. The tape-recorded signal voltages are brought together for a cross-correlation procedure which turns out an optimum value for the relative phase delay $\tau_{ij}(t)$ and a fringe pattern extending over the correlation interval chosen. Such a cross-correlation is analyzed in some detail by ROGERS (1970).

The measured relative phase delay can in general be written:

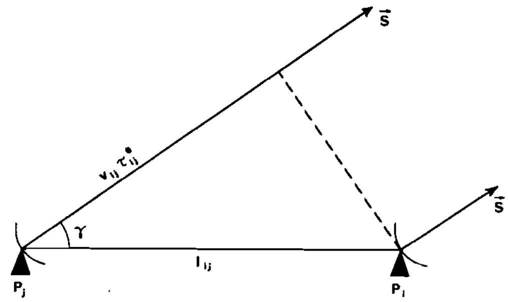

Fig. 1. Basic geometric relationship in VLBI with sources at infinity.

$$\tau_{ij}(t)+\varepsilon_{ij}(t)=\tau_{ij}^{g}(t)+k_{ij} \qquad (1)$$

where

$$\tau_{ij}^{g}(t)=\frac{1}{v_{ij}}[\vec{p}_j(t)-\vec{p}_i(t)]\cdot\vec{s} \qquad (2)$$

and k_{ij} stands for the supposedly constant, but unknown, amount by which the clock at P_j leads that of P_i; $\varepsilon_{ij}(t)$ is a correction to $\tau_{ij}(t)$ which allows for stochastic variation of the measurements.

$\vec{p}_i(t)$ and $\vec{p}_j(t)$ are vectors defining the locations of P_i and P_j, in a supposedly earth-fixed reference, and $\vec{s}$ is the direction to the source in inertial space; v_{ij} is an effective value for the velocity of the radio waves involved. In (2) it is tacitly assumed that the source is located at infinity.

If the distance between terminals P_i and P_j is assumed to be invariable (2) may conveniently be written (see Fig. 1):

$$\tau_{ij}^{g}(t)=\frac{l_{ij}}{v_{ij}}\cos\gamma \qquad (3)$$

where l_{ij} is the constant interstation distance P_iP_j.

The vacuum speed of light c requires a correction to obtain v_{ij}. This correction essentially only depends on the difference between the atmospheric propagation delays between the source and the respective terminals. This atmospheric effect has been studied by PEARLMAN and GROSSI (1969) and MATHUR et al. (1970). Denoting the propagation delays for stations P_i and P_j by d_i and d_j, we have:

$$v_{ij}=\left[1-\frac{d_j-d_i}{\tau_{ij}^{g}}\right]\cdot c$$

There is a total of $n-1$ independent relative clock-offsets k_{ij}. Unless these can be eliminated by an independent clock-comparison precise to at least $(\Delta\nu)^{-1}$, where $\Delta\nu$ is the effective bandwidth, these must be accommodated as unknown parameters.

There are practically two ways to extract information from VLBI-records: delay mapping and fringe-rate mapping. The present study is confined to delay mapping, and then the VLBI-method becomes indeed a range-difference method.

Denoting ranges from stations $P_i (i = 1 \ldots n)$ to sources $P_s (s = 1 \ldots q)$ by l_{is}, the range-differences are:

$$\delta_{ij,s} = l_{js} - l_{is} \quad \ldots \ldots \ldots \quad (4)$$

where sub- and superscripts are written in convenient places.

With (1):

$$\delta_{ij,s} = v_{ij,s} \tau^g_{ij,s} = v_{ij,s} \tau_{ij,s} + v_{ij,s} \varepsilon_{ij,s} - v_{ij,s} k_{ij} \quad \ldots \ldots \ldots \quad (5)$$

wherein $\tau_{ij,s}$ has been measured and $v_{ij,s}$ is known by assumption.

In an n-station observation event there are $\frac{1}{2}n(n-1)$ quantities $\tau_{ij,s}$ resulting from an equal number of supposedly independently performed cross-correlation processes. On the other hand there are only $n-1$ independent $\delta_{ij,s}$ and consequently quantities $v_{ij,s}(\tau_{ij,s} - k_{ij})$ need to be constrained by $\frac{1}{2}(n-1)(n-2)$ relations, which however in general involve $n-1$ free parameters k_{ij}. If $n = 4$ these relations read e.g. (omitting subscript s):

$$\delta_{23} = \delta_{13} - \delta_{12}$$

$$\delta_{24} = \delta_{14} - \delta_{12}$$

$$\delta_{34} = \delta_{14} - \delta_{13}$$

These are obviously equivalent to:

$$\left.\begin{aligned} (\tau_{12} + \varepsilon'_{12}) - (\tau_{13} + \varepsilon'_{13}) + (\tau_{23} + \varepsilon'_{23}) &= 0 \\ (\tau_{12} + \varepsilon'_{12}) - (\tau_{14} + \varepsilon'_{14}) + (\tau_{24} + \varepsilon'_{24}) &= 0 \\ (\tau_{13} + \varepsilon'_{13}) - (\tau_{14} + \varepsilon'_{14}) + (\tau_{34} + \varepsilon'_{34}) &= 0 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (6)$$

It should be noticed that relations (6) do neither involve the relative clock-offsets nor the atmospheric propagation delays. This enables a direct check on the results of the 6 pair-wise cross-correlations. In fact these relations should be used as to constrain the results of the cross-correlations, or rather be incorporated in a generalized cross-correlation procedure which turns out an optimum set of relative phase delays.

In section 3 relations (6) are used in the rather rudimentary way of simply least-squares adjusting the supposedly independently obtained relative phase delays.

The situation is quite similar if $n > 4$; in general there will be $\frac{1}{2}(n-1)(n-2)$ relations of type (6) involving $\frac{1}{2}n(n-1)$ relative phase delays.

It was remarked that in a restricted sense the VLBI-approach may be looked upon as a range-difference method. The geometric conditions imposed on the interstation geometry by simultaneous measurement of range differences can be easily derived from those imposed by such measurements of ranges.

For the latter a closed formula description developed previously (AARDOOM, 1970) will be adopted. Confining to the four-station case for a moment the condition resulting from a simultaneous ranging event can be written e.g.:

$$g^1 \equiv |G^1| = \begin{vmatrix} 1 & \cos\varphi_{23}^1 & \cos\varphi_{24}^1 & \cos\varphi_{2s}^1 \\ \cos\varphi_{32}^1 & 1 & \cos\varphi_{34}^1 & \cos\varphi_{3s}^1 \\ \cos\varphi_{42}^1 & \cos\varphi_{43}^1 & 1 & \cos\varphi_{4s}^1 \\ \cos\varphi_{s2}^1 & \cos\varphi_{s3}^1 & \cos\varphi_{s4}^1 & 1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (7)$$

where $\cos\varphi_{ab}^1$ stands for

$$\frac{l_{a1}^2 + l_{b1}^2 - l_{ab}^2}{2 l_{a1} l_{b1}}$$

and φ_{ab}^1 is the unknown angle subtended at P_1 by the directions to P_a and P_b, which supposedly have not been measured.

P_a and P_b can alternatively indicate the positions of ground stations P_2, P_3 and P_4 or source P_s. l_{a1} is the distance between P_a and P_1 etc.

Equivalent expressions would be:

$$g^2 = 0; \qquad g^3 = 0; \qquad g^4 = 0$$

Introducing range differences as defined by (4) and linearizing in view of subsequent application of linear procedures, like conventional least squares adjustment, (7) becomes after some manipulation:

$$\begin{aligned}
&\left[D_{22}^1 \frac{1}{l_{12}} + D_{23}^1 \frac{1}{l_{13}} + D_{24}^1 \frac{1}{l_{14}} + D_{2s}^1 \frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{12} + \\
&+\left[D_{32}^1 \frac{1}{l_{12}} + D_{33}^1 \frac{1}{l_{13}} + D_{34}^1 \frac{1}{l_{14}} + D_{3s}^1 \frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{13} + \\
&+\left[D_{42}^1 \frac{1}{l_{12}} + D_{43}^1 \frac{1}{l_{13}} + D_{44}^1 \frac{1}{l_{14}} + D_{4s}^1 \frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{14} - \\
&-\left(D_{23}^1 \frac{l_{23}}{l_{12} l_{13}}\right)_0 \mathrm{d}l_{23} - \left(D_{24}^1 \frac{l_{24}}{l_{12} l_{14}}\right)_0 \mathrm{d}l_{24} - \left(D_{34}^1 \frac{l_{34}}{l_{13} l_{14}}\right)_0 \mathrm{d}l_{34} - \\
&-\left[D_{2s}^1 \frac{\delta_{12,s}}{l_{12} l_{1s}} + D_{3s}^1 \frac{\delta_{13,s}}{l_{13} l_{1s}} + D_{4s}^1 \frac{\delta_{14,s}}{l_{14} l_{1s}} - D_{ss}^1 \frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{1s} = \\
&= \left[D_{2s}^1 \left(1 + \frac{\delta_{12,s}}{l_{1s}}\right) \frac{1}{l_{12}}\right]_0 \mathrm{d}\delta_{12,s} + \left[D_{3s}^1 \left(1 + \frac{\delta_{13,s}}{l_{1s}}\right) \frac{1}{l_{13}}\right]_0 \mathrm{d}\delta_{13,s} + \\
&+\left[D_{4s}^1 \left(1 + \frac{\delta_{14,s}}{l_{1s}}\right) \frac{1}{l_{14}}\right]_0 \mathrm{d}\delta_{14,s} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)
\end{aligned}$$

in which D_{ab}^1 stands for the cofactor of the corresponding element in G^1; subscripts "0" indicate approximate initial values.

Noting (5):

$$\mathrm{d}\delta_{12,s} = c(\mathrm{d}\tau_{12,s} - k_{12}) + c\varepsilon''_{12,s}$$

$$\mathrm{d}\delta_{13,s} = c(\mathrm{d}\tau_{13,s} - k_{13}) + c\varepsilon''_{13,s}$$

$$\mathrm{d}\delta_{14,s} = c(\mathrm{d}\tau_{14,s} - k_{14}) + c\varepsilon''_{14,s}$$

where $v_{ij,s}$ has been approximated by c.

Inserting this, omitting in view of present purposes corrections ε''_{ij}, (8) becomes:

$$\begin{aligned}
&\left[D^1_{22}\frac{1}{l_{12}} + D^1_{23}\frac{1}{l_{13}} + D^1_{24}\frac{1}{l_{14}} + D^1_{2s}\frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{12} + \\
&+\left[D^1_{32}\frac{1}{l_{12}} + D^1_{33}\frac{1}{l_{13}} + D^1_{34}\frac{1}{l_{14}} + D^1_{3s}\frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{13} + \\
&+\left[D^1_{42}\frac{1}{l_{12}} + D^1_{43}\frac{1}{l_{13}} + D^1_{44}\frac{1}{l_{14}} + D^1_{4s}\frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{14} - \\
&-\left(D^1_{23}\frac{l_{23}}{l_{12}l_{13}}\right)_0 \mathrm{d}l_{23} - \left(D^1_{24}\frac{l_{24}}{l_{12}l_{14}}\right)_0 \mathrm{d}l_{24} - \left(D^1_{34}\frac{l_{34}}{l_{13}l_{14}}\right)_0 \mathrm{d}l_{34} - \\
&-\left[D^1_{2s}\frac{\delta_{12,s}}{l_{12}l_{1s}} + D^1_{3s}\frac{\delta_{13,s}}{l_{13}l_{1s}} + D^1_{4s}\frac{\delta_{14,s}}{l_{14}l_{1s}} - D^1_{ss}\frac{1}{l_{1s}}\right]_0 \mathrm{d}l_{1s} + \\
&+\left[D^1_{2s}\left(1+\frac{\delta_{12,s}}{l_{1s}}\right)\frac{1}{l_{12}}\right]_0 ck_{12} + \left[D^1_{3s}\left(1+\frac{\delta_{13,s}}{l_{1s}}\right)\frac{1}{l_{13}}\right]_0 ck_{13} + \\
&+\left[D^1_{4s}\left(1+\frac{\delta_{14,s}}{l_{1s}}\right)\frac{1}{l_{14}}\right]_0 ck_{14} = \\
&= \left[D^1_{2s}\left(1+\frac{\delta_{12,s}}{l_{1s}}\right)\frac{1}{l_{12}}\right]_0 c\,\mathrm{d}\tau_{12,s} + \left[D^1_{3s}\left(1+\frac{\delta_{13,s}}{l_{1s}}\right)\frac{1}{l_{13}}\right]_0 c\,\mathrm{d}\tau_{13,s} + \\
&+\left[D^1_{4s}\left(1+\frac{\delta_{14,s}}{l_{1s}}\right)\frac{1}{l_{14}}\right]_0 c\,\mathrm{d}\tau_{14,s} \quad \ldots\ldots\ldots\ldots \quad (9)
\end{aligned}$$

Equation (9) contains 6 unknown interstation distances $l_{ij}(i, j = 1 \ldots 4; i < j)$, the unknown reference-range l_{1s} from station P_1 to source P_s and 3 unknown relative clock-offsets; in all 10 unknowns, conditioned by the relative phase delay measurements. Quantities $\mathrm{d}\tau_{ij,s}$ $(j = 2 \ldots 4)$ should be interpreted as:

$$\mathrm{d}\tau_{1j,s} = (\tau_{1j,s} + \varepsilon'_{1j,s}) - (\tau_{1j,s})_0$$

the difference between the measured relative phase delay $\tau_{1j,s}$, corrected by $\varepsilon'_{1j,s}$ according to (6), and the initial phase delay $(\tau_{1j,s})_0$ as calculated from approximated values for the interstation distances and the source position P_s relative to the network. Reduced observations $\mathrm{d}\tau_{1j,s}$ have the statistical properties of $\tau_{1j,s} + \varepsilon'_{1j,s}$ and finally:

$$\varepsilon_{1j,s} = \varepsilon'_{1j,s} + \varepsilon''_{1j,s}$$

The interstation distances l_{ij} are supposed to be constant throughout an observation campaign although this supposition is rather artificial considering e.g. the presence of earth tides and the precision aimed at. The relative clock-offsets k_{ij} are also taken as constants. The reference range l_{1s} however differs in general from event to event.

Since each additional event introduces a new unknown l_{1s} and since only one condition (9) results from each event, there is no solution for the interstation geometry l_{ij} from four-station VLBI, unless $l_{1s}\to\infty$, like for natural sources. Then the unknown l_{1s} simply disappears from (9), leaving 9 constant unknowns, which are likely to be determined from 9 four-station range difference measurements.

If $l_{1s}\to\infty$, (9) reduces to:

$$\begin{aligned}
&\left[D^1_{22}\frac{1}{l_{12}}+D^1_{23}\frac{1}{l_{13}}+D^1_{24}\frac{1}{l_{14}}\right]_0 \mathrm{d}l_{12}+\\
&+\left[D^1_{32}\frac{1}{l_{12}}+D^1_{33}\frac{1}{l_{13}}+D^1_{34}\frac{1}{l_{14}}\right]_0 \mathrm{d}l_{13}+\\
&+\left[D^1_{42}\frac{1}{l_{12}}+D^1_{43}\frac{1}{l_{13}}+D^1_{44}\frac{1}{l_{14}}\right]_0 \mathrm{d}l_{14}-\\
&-\left(D^1_{23}\frac{l_{23}}{l_{12}l_{13}}\right)_0 \mathrm{d}l_{23}-\left(D^1_{24}\frac{l_{24}}{l_{12}l_{14}}\right)_0 \mathrm{d}l_{24}-\left(D^1_{34}\frac{l_{34}}{l_{13}l_{14}}\right)_0 \mathrm{d}l_{34}+\\
&+\left(D^1_{2s}\frac{1}{l_{12}}\right)_0 ck_{12}+\left(D^1_{3s}\frac{1}{l_{13}}\right)_0 ck_{13}+\left(D^1_{4s}\frac{1}{l_{14}}\right)_0 ck_{14}=\\
&=\left(D^1_{2s}\frac{1}{l_{12}}\right)_0 c\mathrm{d}\tau_{12,s}+\left(D^1_{3s}\frac{1}{l_{13}}\right)_0 c\mathrm{d}\tau_{13,s}+\left(D^1_{4s}\frac{1}{l_{14}}\right)_0 c\mathrm{d}\tau_{14,s} \quad \ldots\ldots (10)
\end{aligned}$$

In both (9) and (10) the number of unknowns is reduced by 3, if the k_{ij} are discarded as unknowns. This simplifies the situation and only 6 range difference measurements between 4 stations are required for a unique solution of the interstation geometry, provided the measurements are made on natural sources.

If artificial sources at finite range l_{1s} are used, at least 5 stations should join in VLBI in order to achieve unique interstation geometry. This can be seen as follows, distinguishing cases of known from cases with unknown clock-offsets k_{ij}.

n-station simultaneous ranging yields $n-3$ independent relations of type (7) to be satisfied by the $3(n-2)$ independent interstation distances (see AARDOOM, 1970). With range difference measurements this is also true, but each range difference measurement event adds one unknown reference range. Hence, to obtain possibly a unique solution for the interstation distances (and incidentally the reference range) from q n-station range difference measurements,

$$q(n-3) \geqslant 3(n-2)+q \quad \ldots\ldots\ldots\ldots (11)$$

should be satisfied, or:

$$q \geqslant 3\frac{n-2}{n-4},$$

if the k_{ij} are known.

This inequality has no possitive integer solution q if $n < 5$.

If however the clock-offsets are introduced as unknown parameters, the inequality becomes:

$$q(n-3) \geqslant 3(n-2)+(n-1)+q$$

or:

$$q \geqslant \frac{4n-7}{n-4} \quad \text{(12)}$$

To obtain feasible combinations n, q for the general infinite-source-case with *a priori* known and unknown clock-offsets respectively, the q-term in the right hand members of (11) and (12) should be suppressed, leading to:

$$q \geqslant 3\frac{n-2}{n-3} \quad \text{(13)}$$

and:

$$q \geqslant \frac{4n-7}{n-3} \quad \text{(14)}$$

respectively.

Table 1. Minimum number q of n-station VLBI-measurements required for unique determination of interstation distances.

artificial sources				natural sources			
clock-offsets known (11)		clock-offsets unknown (12)		clock-offsets known (13)		clock-offsets unknown (14)	
n	$q\geqslant$	n	$q\geqslant$	n	$q\geqslant$	n	$q\geqslant$
				4	6	4	9
5	9	5	13	5	5	5	7
6	6	6	9	$\geqslant$6	4	6	6
7	5	7	7			7	6
8	5	8	7			$\geqslant$8	5
8	5	9	6				
$\geqslant$10	4	10	6				
		11	6				
		12	6				
		$\geqslant$13	5				

Table 1 summarizes solutions of (11) through (14). The minimum value q of n-station measurements decreases with increasing n towards a definite lower bound. q is larger with (artificial) sources at finite range than with (natural) sources at infinity. Obviously more observations q are required if the clock-offsets are initially unknown.

The conditions stated in table 1 are necessary but not sufficient. With some exceptional arrangements of station and/or source positions there might be no unique geometric solution although the necessary conditions of table 1 are amply fulfilled. As regards simultaneous station-to-satellite ranging such singular cases have been studied by KILLIAN and MEISSL (1969), and in more detail by BLAHA (1971). Of special practical interest is that a coplanar four-station configuration is indeterminate with simultaneous station-to-satellite ranging. Such singular cases never occur exactly, but due to intervisibility conditions related to the curvature of the earth, they will be sometimes approached by practical station configurations and are then termed critical.

No careful study of critical configurations has been made here in connection with VLBI, but reiterating that in a sense VLBI can be understood as a range difference approach, it is conjectured that in VLBI- and range networks similar critical situations occur. In fact VLBI with sources at infinite range must be geometrically equivalent to simultaneous ranging to a target at infinity. VLBI involves at least as many unknown parameters as ranging; therefore a critical configuration with ranging is expected to be also critical with VLBI.

If this is correct, in particular all coplanar, or rather, nearly coplanar, four-station VLBI-net configurations are ill-determined and so are all VLBI-net configurations if any of the stations is coplanar with all its observed sources.

Applying the method presently outlined one will usually not measure on a great number of different sources, but rather on a relatively small number which appear in a sequence of related positions with respect to the rotating earth.

Considering sources at infinite range for a moment, then sufficiently frequent observation of one source at an arbitrary declination will usually determine the interstation geometry, although the use of several sources with different declinations would be preferable. If however only one source is observed with zero-declination, and even if all observed sources have zero-declination, the problem to determine the interstation geometry would be singular. This is only one example, which nevertheless warrants some care in setting up VLBI-networks to be used partly for geodetic purposes. The simulation-calculations of section 3 strongly indicate that there occurs a singularity even in the general case of observation of one source at infinity, irrespective of the declination of that source.

If however the entire spectrum of the rotation of the earth is presupposed, the relative source positions are so, and this eliminates the singularity.

A characteristic of the present approach to VLBI is that, apart from eventual optimization considerations, and the wish to avoid critical situations, the positions of neither natural nor artificial sources do appear explicitly in the mathematics leading to the interstation geometry. This feature could be seen as an advantage of this approach, because it enables a tentative separation of the unknown source position parameters from the interstation distances and eventually the relative clock-offsets. This enables a treatment of the integral and involved VLBI-problem in functional steps, the first being an initial determination of the interstation geometry and the clock-offsets, the second yielding source positions. Possibly these positions are considered geodetically irrelevant; if so, the second step could be omitted from a purely geodetic point of view. Positions of an individual source however will be related; for artificial sources through a known or parameterized orbital theory; for all sources through the rotation of the earth-fixed VLBI-net with respect to the inertial frame to which the source positions are related. These relationships in principle feed back into the interstation geometry as initially determined in the first step. If this feed-back proves to be sensible the purely geodetic point of view is contestable, even if e.g. the rotation of earth, involved through the consecutive positions of the sources with respect to the earth-fixed interstation net, is considered geodetically irrelevant.

In this paper the second step of computations is disregarded. This, although it simplifies the argument, implies the neglect of information on the rotation axis and rotation rate of earth (if it is supposed that the VLBI-net is rigidly tied to it), and in case of artificial sources, on a number of force field parameters, in particular the location of the centre of mass relative to the VLBI-net.

3 Some simulation calculations

This section serves two purposes: firstly, to illustrate the approach outlined in section 2; and secondly, to obtain some numerical estimates of the accuracy likely attainable for the interstation distances l_{ij} in an idealized four-station VLBI-net with simulated data on sources at infinity.

In these calculations the station clocks are taken as perfectly synchronized, i.e. $k_{ij} = 0$, so that model (10) is applicable.

(10) may briefly be written:

$$\vec{a}_s \cdot \vec{x} = \vec{b}_s \cdot \vec{p}_s \qquad (15)$$

with:

$$(\vec{x})^* = (\mathrm{d}l_{12}, \mathrm{d}l_{13}, \mathrm{d}l_{14}, \mathrm{d}l_{23}, \mathrm{d}l_{24}, \mathrm{d}l_{34})$$

and:

$$(\vec{p}_s)^* = c(\mathrm{d}\tau_{12,s}, \mathrm{d}\tau_{13,s}, \mathrm{d}\tau_{14,s})$$

$\vec{a}_s$ and $\vec{b}_s$ are coefficient-vectors the components of which may be read from (10).

If s ranges from 1 to q, relations (15) are combined into:

$$A \cdot \vec{x} = B \cdot \vec{p} \qquad (16)$$

wherein matrices A and B are:

$$A = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_s \end{pmatrix}; \qquad B = \begin{pmatrix} \vec{b}_1 & & & \\ & \vec{b}_2 & & \\ & & \ddots & \\ & & & \vec{b}_s \end{pmatrix}$$

B is composed of q 3×3 diagonal sub-matrices and zeros elsewhere.

Further:

$$\vec{p} = \begin{pmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \vdots \\ \vec{p}_s \end{pmatrix}$$

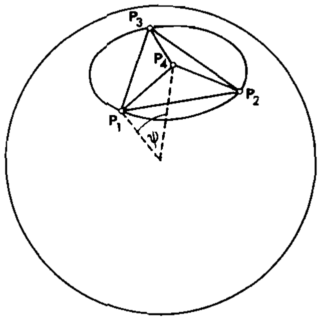

Fig. 2. Model four-station geometry.

As idealized model station configuration, the configuration illustrated by Fig. 2 was chosen: three stations (P_1, P_2 and P_3) arranged as to form an equilateral spherical triangle around a central station (P_4), all lying on a sphere with radius 6370 km, a figure close enough to the mean radius of the earth. The mutual separation of the stations, and thus the size of the configuration, is specified by geocentric angle ψ. The orientation of the configuration relative to the earth's rotation axis is defined by the latitude L of station P_4 and the azimuth Φ of line P_4P_1, counted from the north-meridian of P_4, eastward positive.

All four stations are supposed to be able to track radio sources up to a maximum zenith-angle z. z is the semi vertex angle of the cone of observability for each station, having the geocentric direction to that station as axis. Thinking in terms of radio sources at infinity, the visibility domain for a station is simply the intersection of the appropriate cone with the geocentric unit sphere, or rather the interior of a circular region on this unit sphere. The domain of simultaneous observations by all four stations is specified by the spherical region interior to all individual spherical domains as defined above. This domain of common observations is approximated by the spherical interior of a circle, inscribed as in Fig. 3, with radius:

$$\beta = z - \psi$$

As was mentioned already, in the subsequent calculations the sources will be taken at infinity and will have declinations $\delta_s (s = 1 \ldots q)$.

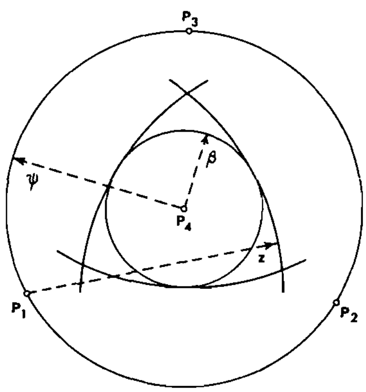

Fig. 3. Definition of domain of common observability between stations P_1, P_2, P_3 and P_4.

As regards data handling much depends on the statistical pattern adopted for the relative phase delay measurements τ_{ij}. If this is taken as multidimensional Gaussian, then the standard least squares apparatus is the proper tool. To do something specific, a Gaussian distribution is indeed presupposed here for the τ_{ij}, though it is recognized that doing so, possibly a rather severe violation to the rules of optimum data handling is committed. Moreover the variance-covariance matrix $V\{\tau\}$, which apart from its mean, completely defines such a Gaussian distribution, is taken as diagonal, neglecting all covariances.

More specific, and also more restrictive,

$$V\{\tau\} = \sigma^2 I,$$

where I is the $(6q)^{\text{th}}$ order unit matrix. σ will be on the order of the correlation half-width $(2\Delta\nu)^{-1}$, where $\Delta\nu$ is the effective bandwidth.

Under these assumptions the variance-covariance matrix $V\{\text{p}\}$ of $\vec{p}$ becomes after an initial least squares adjustment on (6):

$$V\{p\} = (c\sigma)^2 \begin{pmatrix} E & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{pmatrix} \qquad (17)$$

a matrix composed of q 3×3 sub-matrices E on the diagonal and zeros elsewhere. Sub-matrices are explicitly:

$$E = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad (18)$$

Then the variance-covariance matrix of the solution for $\vec{x}$ will be:

$$V\{x\} = (A^*WA)^{-1} \qquad (19)$$

with

$$W = [BV\{p\}B^*]^{-1}$$

More explicitly:

$$V\{x\} = (c\sigma)^2 \begin{pmatrix} (12,12) & (12,13) & (12,14) & (12,23) & (12,24) & (12,34) \\ (13,12) & (13,13) & (13,14) & (13,23) & (13,24) & (13,34) \\ (14,12) & (14,13) & (14,14) & (14,23) & (14,24) & (14,34) \\ (23,12) & (23,13) & (23,14) & (23,23) & (23,24) & (23,34) \\ (24,12) & (24,13) & (24,14) & (24,23) & (24,24) & (24,34) \\ (34,12) & (34,13) & (34,14) & (34,23) & (34,24) & (34,34) \end{pmatrix} = (c\sigma)^2 V'\{x\} \qquad (20)$$

$V'\{x\}$ has been evaluated for two classes of simulated observation programmes:

a. one source at declination δ

b. two sources at declinations δ^{I} and δ^{II}

In each programme the source(s) is (are) tracked by all four stations throughout their domain of common observability. Depending on L and δ a source attains, because of the rotation of the earth, hour angles from $-\alpha$ to $+\alpha$ referred to the meridian of P_4. A source is tracked during one pass only and from this pass $2n+1$ supposedly statistically independent cross-correlations are extracted, with mutual intervals α/n evenly devided over the hour angle range $-\alpha$ to $+\alpha$. n is put 4 throughout, yielding 9 four-station relative phase delay measurements for each source if $\alpha > 0$.

If only one source is used (programme (a)), matrix A^*WA turned out to be singular, irrespective of L, Φ and δ. Hence there is computational evidence for the occurrence of a critical situation as regards the determination of the selected model interstation geometry from VLBI phase delay measurements using one natural source only. This singularity could conceivably be understood as an analog to a singularity occurring in simultaneous ranging with all points (stations and targets) lying on one second order surface, which was demonstrated by KILLIAN and MEISSL (1969) and more general by BLAHA (1971). If this is correct, then no multiple-station VLBI-net could be determined with one natural source only. The singularity noticed here may also be looked upon as one of which that anticipated in section 2 is a special case. Like there, the singularity will be removed if the entire spectrum of the earth's rotation is known *a priori*.

Continuing with programme (b), $V'\{x\}$ was first evaluated with $z = 75°$, $\psi = 30°$ and $\Phi = 0°$ for values L between 10° and 80° and several combinations δ^{I}, δ^{II} as follows: 40°, 50°; 35°, 55°; 30°, 60°; 25°, 65°; 20°, 70°; 15°, 75°; 10°, 80° in so far as both sources appeared, not necessarily simultaneously, in the common observation domain as defined by L and $\beta = 45°$.

Restricting the discussion to the variances, although this might be rather misleading occasionally, there will be, due to the special station configuration, four numerically distinct quantities:

$$(12,12) = (13,13);\ (14,14);\ (23,23);\ (24,24) = (34,34),$$

(ij, ij) denoting the reduced variances of l_{ij}, as appearing in $V'\{x\}$.

The results obtained for the various combinations δ^{I}, δ^{II} are rather similar. Those for 35°, 55° have been selected as typical and are presented in Fig. 4 in terms of reduced standard deviations $m_{ij} = [(ij, ij)]^{\frac{1}{2}}$. These are related to standard deviations σ_{ij} by $\sigma_{ij} = c\sigma \cdot m_{ij}$ according to (20).

Noticing the singularity encountered with programme (a), it is obvious that there is no solution for $L < 10°$ and $L > 80°$ and that the m_{ij} will rapidly increase when these latitude limits are approached. Between these limits the curves of Fig. 4 have some absolute minimum, but these minima are not attained for the same value of L. In general $m_{12} = m_{13}$ and m_{14} are remarkably larger than m_{23} and $m_{24} = m_{34}$. This is interpreted as a feature typical for the selected station configuration and its orientation relative to the rotation axis of the earth as enforced by the positive declinations δ^{I} and δ^{II}. It should be recalled that the m_{ij} as plotted are valid for a total of 18 relative delay measurements, 9 on each source, as determined by $n = 4$. The m_{ij} would decrease with increasing n, probably roughly proportional to $n^{-\frac{1}{2}}$.

When reading Fig. 4 it should not be overlooked that the l_{ij} to which the m_{ij} refer, are of different magnitude, their ratio being either 1 or $\frac{1}{2}\sqrt{3} \cdot \sin\psi / \sin\frac{1}{2}\psi = \sqrt{3} \cdot \cos\frac{1}{2}\psi$.

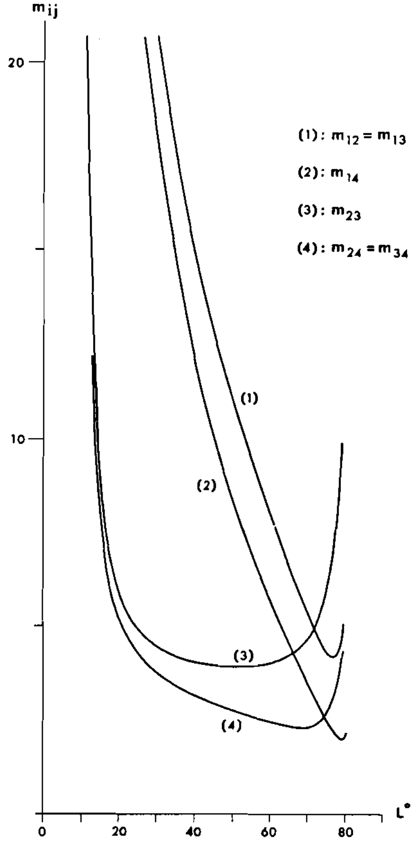

Fig. 4. Reduced standard deviations of interstation distances in model station configuration with $\delta^{I} = 35°$; $\delta^{II} = 55°$. $z = 75°$. $\psi = 30°$. $\Phi = 0°$.

It appears from Fig. 4 that all m_{ij} are smaller than say 25, and as regards l_{23}, l_{24} and l_{34} smaller than about 5, provided the critical regions $10° < L < 20°$ and $70° < L < 80°$ are avoided. These figures for m_{ij} indicate, in a sense, the loss of precision from measured range differences, through the station-sources-geometry to interstation distances.

Within the practical observation limits set by declinations δ^{I} and δ^{II} these figures are of the same order of magnitude for all combinations δ^{I}, δ^{II} selected. The usable range $2\beta - |\delta^{II} - \delta^{I}|$ of L decreases however with increasing difference $|\delta^{II} - \delta^{I}|$.

From now on only the combination of declinations $\delta^{I} = 35°$, $\delta^{II} = 55°$ is considered and successively a study is made of the effect on interstation distance accuracy of a variation in:

azimuth Φ,
maximum zenith distance z and
geocentric angle ψ

In these calculations latitude L is varied within the usable range, but apart from that, only the selected parameter is varied, the parameters being otherwise fixed at $\Phi = 0°$, $\psi = 30°$ and $z = 75°$. The results are presented in graphs similar to Fig. 4.

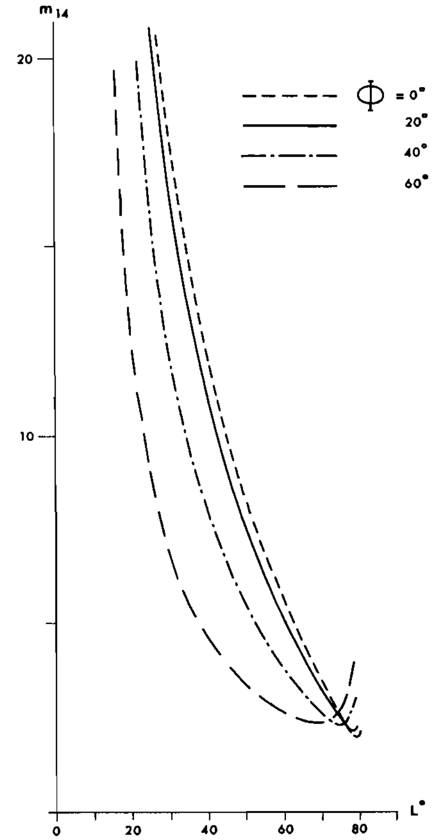

Fig. 5. Reduced standard deviation m_{14} for some values of azimuth Φ. $\delta^{I} = 35°$; $\delta^{II} = 55°$. $z = 75°$. $\psi = 30°$.

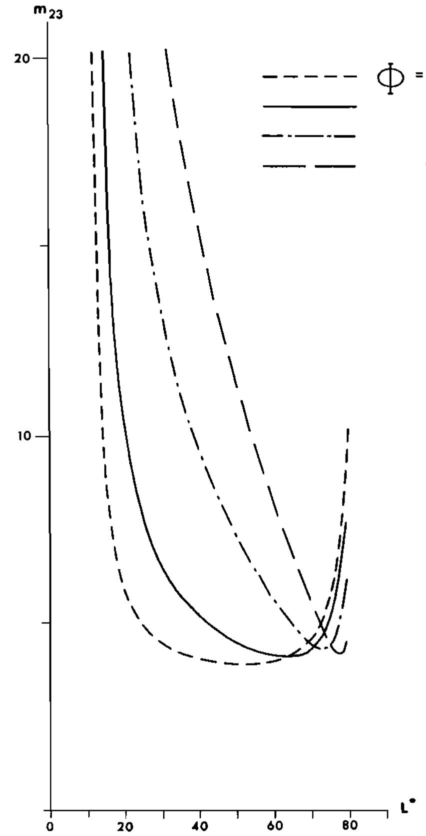

Fig. 6. Reduced standard deviation m_{23} for some values of azimuth Φ. $\delta^{I} = 35°$; $\delta^{II} = 55°$. $z = 75°$. $\psi = 30°$.

Because of geometric symmetry, all information concerning the variation of Φ is contained in the range $0° \leqslant \Phi \leqslant 60°$. As examples in Figs. 5 and 6 computed m_{14}- and m_{23}-curves have been plotted for some discrete values of Φ in this range, taking the selected curves for $\Phi = 0°$ from Fig. 4. As before, the independent variable is the latitude L of station P_4. Moreover the arithmetic means of m_{12}, m_{13} and m_{23}, respectively m_{14}, m_{24} and m_{34} have been averaged over all selected values of Φ as to form M and m respectively. These averages M and m have been plotted against L in Fig. 7. It is recalled that Figs. 5, 6 and 7 are valid for $\psi = 30°$ and $z = 75°$. These latter values are rather arbitrary and the choice $z = 75°$ appears even as somewhat artificial because of the expected high levels of noise and atmospheric disturbances at such large zenith distances.

Fig. 5 demonstrates that if $L < 70°$, m_{14} decreases with increasing Φ; Fig. 6 shows that m_{23} however increases with Φ. There remains a strong tendency for both m_{14} and m_{23} to increase with decreasing L. It does not make sense to consider values $\Phi > 60°$, the results being immediately predictable from those already obtained by an argument of symmetry and periodicity. Both M and m increase steeply if L decreases from 70°. When noticing that

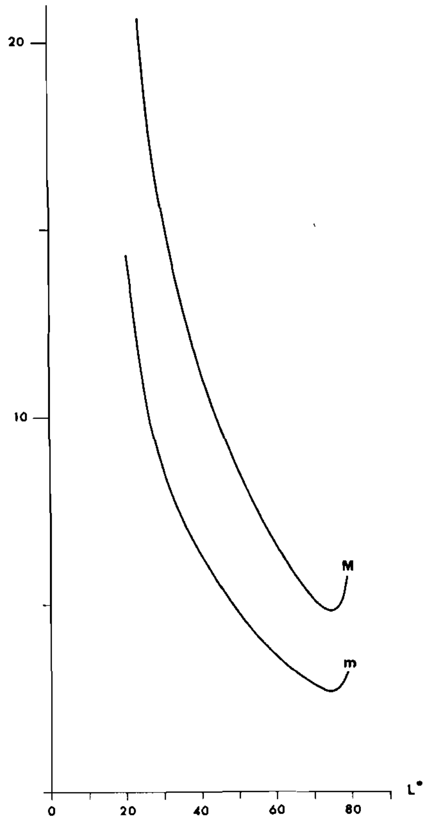

Fig. 7. Averages of means of m_{12}, m_{13}, m_{23} (M) and m_{14}, m_{24}, m_{34} (m) taken over $\Phi = 0°, 20°, 40°$ and $60°$. $\delta^{I} = 35°$; $\delta^{II} = 55°$. $z = 75°$. $\psi = 30°$.

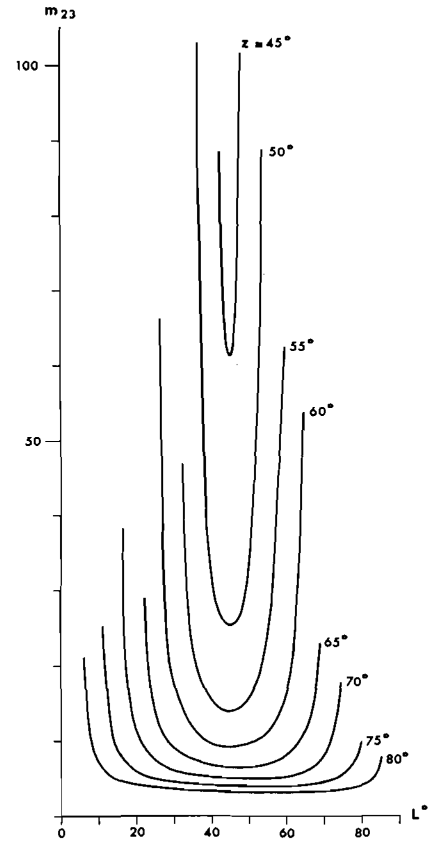

Fig. 8. Reduced standard deviation m_{23} for a variety of maximum zenith distances z. $\delta^{I} = 35°$; $\delta^{II} = 55°$. $\Phi = 0°$. $\psi = 30°$.

$M > m$, it should not be overlooked that M and m are associated with interstation distances of different magnitude in the ratio $\sqrt{3} \cdot \cos 15° \approx 1.7$, which agrees rather well with the ratio between M and m. This means that the averaged proportional standard deviations are nearly the same for both groups of interstation distances. It is worth while to note that the means of m_{12}, m_{13} and m_{23} for any L turned out to be almost independent of Φ; the same holds for the means of m_{14}, m_{24} and m_{34}.

Adopting fixed values $\psi = 30°$, $\Phi = 0°$, the maximum zenith distance z was made to vary in steps of 5° from 80° down to 45°. The larger of these values, say 70° and above are, as mentioned, somewhat artificial, the smaller may be too pessimistic. The computed standard deviation estimates m_{23} were plotted in Fig. 8 for the selected values of z. Interstation distance l_{23} is neither the best, nor the poorest determined one in the model net. Of course the usable interval of L decreases with decreasing z. Within these intervals m_{23} increases rapidly when z is reduced to below 55°. Combining this property with the simultaneous reduction of the usable interval for L, leads to the tentative conclusion that the

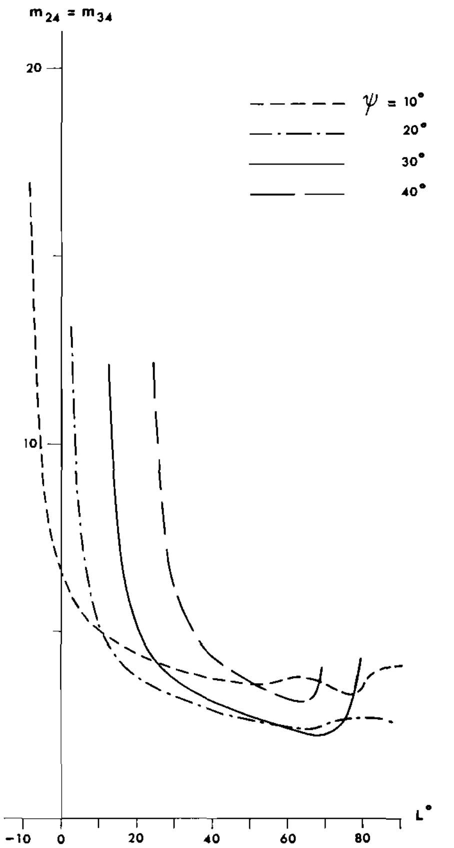

Fig. 9. Reduced standard deviation $m_{12} = m_{13}$ for some values of geocentric angle ψ. $\delta^{I} = 35°$; $\delta^{II} = 55°$. $z = 75°$. $\Phi = 0°$.

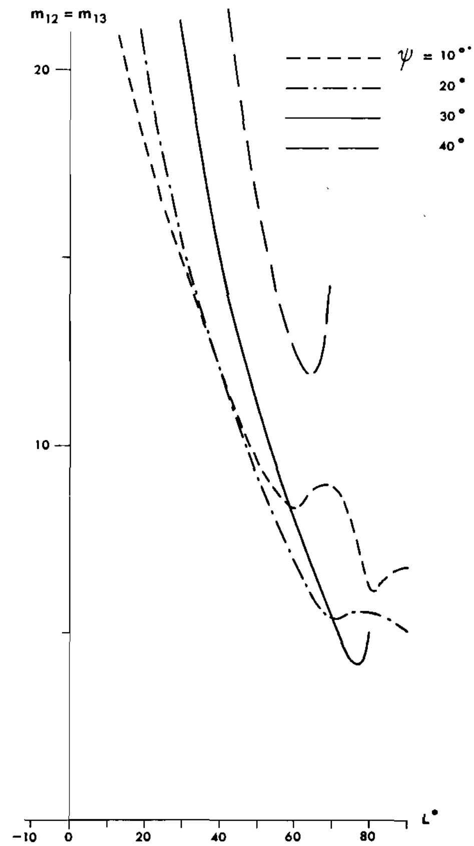

Fig. 10. Reduced standard deviation $m_{24} = m_{34}$ for some values of geocentric angle ψ. $\delta^{I} = 35°$; $\delta^{II} = 55°$. $z = 75°$. $\Phi = 0°$.

feasibility of the discussed approach to multiple-station VLBI for geodesy is doubtful if no zenith distances larger than say 55° are tolerated because of noise and atmospheric conditions.

Finally the geocentric angle ψ, which controls the size of the model configuration, is varied. In addition to $\psi = 30°$, which was used throughout before, values $\psi = 10°$, 20° and 40° were taken successively, adopting $\Phi = 0°$, $z = 75°$. The numerical results are similar to those of Fig. 4, valid for $\psi = 30°$ and the same combination of source declinations $\delta^{I} = 35°$, $\delta^{II} = 55°$. Of course ψ effects β and this in turn the range of L, but within these ranges curves corresponding to the same interstation line stay on the same general level. $m_{12} = m_{13}$ and $m_{24} = m_{34}$ have been plotted for $\psi = 10°$, 20°, 30° and 40° in Figs. 9 and 10 respectively. At first glance the oscillations in the curves for $\psi = 10°$ and 20° are quite remarkable. After all, this feature has a simple qualitative explanation. For small values of L the angle α, which defines the observable hour angle range, is less than 180° for both stellar sources. With increasing L, α increases for both sources, reaching 180° for the sources at declination 55° and 35° successively. The distribution in hour angles of sources has an important effect on the estimated interstation distance accuracies. These accuracies improve if α increases and if α reaches 180° no further improvement is possible in so far this is concerned. The observed departures of the curves for $\psi = 10°$ and 20° occur at values L for which α reaches 180° for either one of the sources. Due to the choice of source declinations $\delta^{I} = 35°$, $\delta^{II} = 55°$ and $z = 75°$ this situation does not occur if $\psi > 20°$, because then α does not reach 180° for $\delta^{II} = 55°$ before α reaches 0° for $\delta^{I} = 35°$ and the latter source becomes unobservable.

Acknowledgement

For the numerical work in section 3 the author is indebted to Mr. K. H. OTTEN of the Sub-Department of Geodesy of the Delft University of Technology.

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