NETHERLANDS GEODETIC COMMISSION

PUBLICATIONS ON GEODESY NEW SERIES

VOLUME 2 NUMBER 3

SIMULTANEOUS DETERMINATION OF LATITUDE, LONGITUDE AND AZIMUTH BY HORIZONTAL DIRECTIONS AT THE SUN

by IR. G. J. HUSTI

1966

RIJKSCOMMISSIE VOOR GEODESIE, KANAALWEG 4, DELFT, NETHERLANDS

PRINTED IN THE NETHERLANDS BY W. D. MEINEMA N.V., DELFT

CONTENTS

SIMULTANEOUS DETERMINATION OF LATITUDE, LONGITUDE AND AZIMUTH BY HORIZONTAL DIRECTIONS AT THE SUN

Summary

In this paper the investigation of the method of simultaneous determination of latitude, longitude and azimuth by horizontal directions at the sun, as described in [2], is continued and amended in some respects. The sun's altitude at observations is restricted between 15° and 60°, a numerical adjustment applying different weights is assumed and in addition to the accuracy of the quantities mentioned, that of the Laplace quantity has been studied as well. The results obtained are represented in a number of diagrams.

Design of the measurements

When stars are used (GOUGENHEIM-BLACK) the method has practically unlimited possibilities with regard to the selection of stars. A satisfactory solution may be obtained by measuring a number of stars distributed regularly in azimuth and at approximately equal altitudes. In the case of *sun* observations however the choice is limited as the sun can obviously only be observed in a limited range of azimuth and at different altitudes.

It will be assumed that the sun will be observed at altitudes not smaller than 15° (to avoid lateral refraction) and not larger than 60° (in connection with the influence of errors in levelling the theodolite). Different cases can be distinguished, depending on the latitude of the station and the declination of the sun:

1. $30° \leqslant |\varphi - \delta| < 75°$ i.e. the meridian altitude of the sun is smaller than or equal to 60° and larger than 15°. The first observation is made in the morning when the sun reaches approximately 15° and the last when the sun again reaches approximately 15° altitude in the afternoon. An odd number of observations are regularly distributed in azimuth within this sector. In this case the number of observations (one of the observations being close to the meridian) may be: $s = 3, 5, 7, 9$ etc.
2. $|\varphi - \delta| < 30°$ i.e. the meridian altitude of the sun is larger than 60°. The required measurements will be distributed regularly in two sectors: in the morning with sun altitudes from 15° to 60°, and in the afternoon from altitudes 60° to 15°. In this case the number of measurements may be $s = 4, 6, 8, 10$ etc.
3. If $|\varphi - \delta| \geqslant 75°$ i.e. the meridian altitude of the sun is smaller than or equal to 15° and in the special case where $\varphi = \delta = 0°$, the method cannot be applied.

In the former two cases the determination of the three unknowns is theoretically always possible. The adjustment is carried out according to the rules of Standard Problem II.

Computation and adjustment

Applying the cotangent rule in the position triangle gives:

$$\cot a_i = \cot (A+\psi_i) = \frac{\sin\varphi \cos (t_i^G - \lambda) - \cos\varphi \tan\delta_i}{\sin (t_i^G - \lambda)} \quad \ldots\ldots \quad (1)$$

where:

A = unknown azimuth of the terrestrial line
ψ_i = the measured horizontal angle between the terrestrial line and the direction to the sun.
t_i^G = the sun's Greenwich hour angle computed from the chronometer reading T_i, the chronometer correction ΔT_i to Universal Time, and the equation of time e_i:

$$t_i^G = T_i + \Delta T_i + e_i + 12^h \quad \ldots\ldots \quad (2)$$

After introduction of φ_0, λ_0 and A_0 as approximate values for the unknowns, and differentiating (1), we obtain the correction equations:

$$\sin a_i' \tan h_i' \Delta\varphi + (\cos a_i' \tan h_i' - \tan\varphi_0)\, \Delta\lambda \cos\varphi_0 - \Delta A = -(\Delta a_i + v_i) \quad \ldots \quad (3)$$

where:

$$\Delta a_i = a_i' - (A_0 + \psi_i) \quad \text{(quasi-observation)} \quad \ldots\ldots \quad (4)$$

The value of a_i' is computed from the approximate values of the unknowns and t_i^G (2):

$$\cot a_i' = \frac{\sin\varphi_0 \cos (t_i^G - \lambda_0) - \cos\varphi_0 \tan\delta_i}{\sin (t_i^G - \lambda_0)} \quad \ldots\ldots \quad (5)$$

while h_i' is computed from:

$$\sin h_i' = \sin\varphi_0 \sin\delta_i + \cos\varphi_0 \cos\delta_i \cos (t_i^G - \lambda_0) \quad \ldots\ldots \quad (6)$$

The weights are inversely proportional to the variances (square of the standard deviations) of the quasi-observations:

$$g_i = \frac{m^2}{m_{\Delta a_i}^2} \quad \ldots\ldots \quad (7)$$

Applying the law of propagation of errors to (4) gives:

$$m_{\Delta a_i}^2 = m_{a_i'}^2 + m_{\psi_i}^2 \quad \ldots\ldots \quad (8)$$

The variances of the azimuth a_i' can be derived from (5):

$$m_{a_i'}^2 = \cos^2\varphi_0 (\tan\varphi_0 - \cos a_i' \tan h_i')^2 m_{t^G}^2 \quad \ldots\ldots \quad (9)$$

where, according to (2), $m_{t^G} = m_T$, the standard deviation of the recorded time. The variance of the measured horizontal angle between the terrestrial line and the direction to the sun is according to [1] (page 109–110):

$$m_{\psi_i}^2 = m_1^2 \sec^2 h_i + m_2^2 + m_3^2 \tan^2 h_i \quad \ldots\ldots \quad (10)$$

where:

m_1 = standard deviation of pointing at the sun,
m_2 = the resultant of the standard deviation of the two circle readings and the standard deviation of pointing at the terrestrial mark.
m_3 = standard deviation of levelling the theodolite or reading the striding level.

The substitution of formulas (8), (9) and (10) into (7) gives the following formula for the weights:

$$g_i = \frac{nm^2}{m_1^2 \sec^2 h_i + m_2^2 + m_3^2 \tan^2 h_i + \cos^2 \varphi_0 (\tan \varphi_0 - \cos a_i \tan h_i)^2 m_T^2} \quad . \; . \; (11)$$

where:

n = number of measurements of the horizontal angle between the terrestrial point and the sun and the corresponding chronometer times; the mean value of these being considered as a single observation.
m^2 = variance of unit weight.

From (3) and (11) we obtain the normal equations:

$$\mathbf{N} \begin{pmatrix} \Delta\varphi \\ \Delta\lambda \cos \varphi_0 \\ \Delta A \end{pmatrix} = \begin{pmatrix} -[g_i \sin a_i' \tan h_i' \, \Delta a_i] \\ -[g_i (\cos a_i' \tan h_i' - \tan \varphi_0) \, \Delta a_i] \\ [g_i \, \Delta a_i] \end{pmatrix} \quad . \; . \; . \; (12)$$

where $\mathbf{N}$ is a symmetrical matrix:

$$\mathbf{N} = \begin{pmatrix} [g_i \sin^2 a_i' \tan^2 h_i'] & [g_i (\sin a_i' \tan h_i')(\cos a_i' \tan h_i' - \tan \varphi_0)] & -[g_i \sin a_i' \tan h_i'] \\ & [g_i (\cos a_i' \tan h_i' - \tan \varphi_0)^2] & -[g_i (\cos a_i' \tan h_i' - \tan \varphi_0)] \\ & & [g_i] \end{pmatrix} \quad (13a)$$

or taking the directions symmetrical to the meridian:

$$\mathbf{N} = \begin{pmatrix} [g_i \sin a_i' \tan^2 h_i'] & 0 & 0 \\ 0 & [g_i (\cos a_i' \tan h_i' - \tan \varphi_0)^2] & [g_i \cos a_i' \tan h_i' - \tan \varphi_0] \\ 0 & [g_i \cos a_i' \tan h_i' - \tan \varphi_0] & [g_i] \end{pmatrix} \quad (13b)$$

The solution of (12) gives the unknowns $\Delta\varphi$, $\Delta\lambda \cos \varphi_0$ and ΔA. Finally,

$$\left.\begin{aligned} \varphi &= \varphi_0 + \Delta\varphi \\ \lambda &= \lambda_0 + \Delta\lambda \\ A &= A_0 + \Delta A \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (14)$$

Accuracy obtained

By substitution of the unknowns into equation (3) the corrections (v_i) can be determined and the estimation of the variance of unit weight is then:

$$m_0^2 = \frac{[g_i v_i^2]}{s-3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (15)$$

and the standard deviations of the unknowns:

$$\begin{pmatrix} m_\varphi \\ m_\lambda \cos\varphi_0 \\ m_A \end{pmatrix} = m_0 \begin{pmatrix} \sqrt{Q_{\varphi\varphi}} \\ \sqrt{Q_{\lambda'\lambda'}} \\ \sqrt{Q_{AA}} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (16)$$

where the weight coefficients are obtained by inverting the **N**-matrix:

$$\mathbf{N}^{-1} = \begin{pmatrix} Q_{\varphi\varphi} & Q_{\varphi\lambda'} & Q_{\varphi A} \\ Q_{\lambda'\varphi} & Q_{\lambda'\lambda'} & Q_{\lambda' A} \\ Q_{A\varphi} & Q_{A\lambda'} & Q_{AA} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (17)$$

Accuracy expected

Since the standard deviation depends strongly on the latitude of the station and the declination of the sun, it is proposed to make a survey of the expected standard deviations at different latitudes and varying declinations. The weight coefficients of matrix (17) have been computed for $s = 5$ or 6 and $s = 9$ or 10 directions for every 5° in latitude and every 3° in declination, using the TR 4 computer of the Technological University at Delft.

The following table contains the standard deviations assumed in the computation of the weights from formula (11):

Standard deviation	Type A	Remark	Type B	Remark
m_1	2.5″	with solar prism attachment (v=28) *)	1.8″	with solar prism attachment (v=40) *)
m_2	2.5″		0.8″	
m_3	1.5″		0.9″	
m_T	1.0″ = 0.07ˢ	with chronograph record	1.0″ = 0.07ˢ	with chronograph record

*) v = magnifying power of the telescope.

The instrument of type A has an accuracy equivalent to the theodolite Wild T2, while the instrument of type B is comparable with the Wild T3. It is supposed that one observation is composed of the mean value of four measurements of the angle between the terrestrial point and the sun, with the corresponding chronometer times ($n = 4$). The expected standard deviations are then computed from (16).

It is interesting to determine from the above data the standard deviation of the Laplace quantity as well. It is well known that the Laplace equation is a relation between astronomic (a) and geodetic (g) quantities (see [3]):

$$A^a - A^g + (\lambda^a - \lambda^g) \sin \varphi_0 = 0$$

or in a different form:

$$A^a+\lambda^a \sin \varphi_0 = A^g+\lambda^g \sin \varphi_0$$

The left hand side of this equation is called the astronomic Laplace quantity. The standard deviation of this according to the rule of TIENSTRA is:

$$m^2_{A+\lambda \sin\varphi_0} = m^2(Q_A+\tan \varphi_0 Q_{\lambda'})^2 = m^2(Q_{AA}+\tan^2\varphi_0 Q_{\lambda'\lambda'}+2 \tan \varphi_0 Q_{A\lambda'}) \quad . \; . \; . \; (18)$$

where the Q-coefficients are obtained from (17). The numerical evaluation of the equations (16) and (18) is graphically represented in the appended figures.

Conclusion

The conclusion is that the method of simultaneous determination of latitude, longitude and azimuth from horizontal directions to the sun can in many cases give satisfactory results. The field in the diagrams where the latitude and declinations are approximately equal (the surrounding of the dotted line) must be avoided as far as possible, because the standard deviations of the computed longitude and azimuth are not acceptable here. This is where $|\varphi-\delta| < 5°$ to $10°$. Another unsuitable field for the application of the method, $|\varphi-\delta| > 60°$ to $65°$, is of no practical importance.

On the basis of these investigations it is concluded that a reasonable result may be obtained if the absolute value of the difference between latitude and declination fulfills the condition:

$$10° < |\varphi-\delta| < 60°$$

Numerical example

A practical example, which was solved graphically in [2], will be worked out numerically here using different weights computed from equation (11). The measurements where made by Mr. DE VRIES, technical assistant, at the Observation Tower of the Geodetic Institute in Delft with a Wild T2 and ROELOFS' solar prism attachment.

Some important data used in the computations are given in the following table:

Appr. declination of the sun: $+11°.2$					Date: April 19, 1961		
i	a_i'	h_i'	g_i	$-\Delta a_i$	v_i	$g_i v_i$	$g_i v_i^2$
1	92°.7	16°.6	1.9	−18″.2	+0″.58	+1.10	0.639
2	134°.0	42°.2	1.3	−21″.3	−1″.20	−1.56	1.872
3	180°.0	50°.8	1.0	−36″.6	+1″.38	+1.38	1.904
4	228°.6	41°.0	1.3	−31″.4	−0″.95	−1.24	1.173
5	268°.7	16°.0	1.9	−22″.5	+0″.19	+0.36	0.069
Σ						+0.04	5.657
					$m_0 = \sqrt{\frac{5.657}{2}} = 1.68$		

According to (12) we obtain the normal equations:

$$\begin{pmatrix} +1.4338 & -0.0805 & -0.0219 \\ -0.0805 & +21.8097 & +12.2999 \\ -0.0219 & +12.2999 & +7.4000 \end{pmatrix} \begin{pmatrix} \Delta\varphi \\ \Delta\lambda\cos\varphi_0 \\ \Delta A \end{pmatrix} = \begin{pmatrix} 10''.40 \\ 319''.96 \\ 182''.42 \end{pmatrix}$$

from which the unknowns are solved:

$$\begin{pmatrix} \Delta\varphi \\ \Delta\lambda\cos\varphi_0 \\ \Delta A \end{pmatrix} = \begin{pmatrix} +0.6982 & +0.0225 & -0.0354 \\ +0.0225 & +0.7331 & -1.2185 \\ -0.0354 & -1.2185 & +2.1603 \end{pmatrix} \begin{pmatrix} 10''.40 \\ 319''.96 \\ 182''.42 \end{pmatrix}$$

From this follows:

	obtained	*expected*
$\Delta\varphi = +8''.0$	$m_\varphi = 1.68\sqrt{0.70} = 1''.4$	$m_\varphi = 2''.3$
$\Delta\lambda\cos\varphi_0 = +12''.5$	$m_{\lambda\cos\varphi_0} = 1.68\sqrt{0.73} = 1''.4$	$m_{\lambda\cos\varphi_0} = 2''.3$
$\Delta A = +3''.8$	$m_A = 1.68\sqrt{2.16} = 2''.5$	$m_A = 3''.8$

Then:

$\varphi_0 = +52°00'30''.0$	$\lambda_0 = -0^h17^m30^s.00$	$A_0 = 81^c.3400$ *)
$\Delta\varphi = +\quad 8''.0$	$\Delta\lambda = +\quad 1^s.36$	$\Delta A = +\quad 12$
$\varphi = +52°00'38''.0$	$\lambda = -0^h17^m28^s.64$	$A = 81.3412$
$32''.9$ **)	$28^s.51$ **)	3402 **)
$d\varphi = +5''.1$	$d\lambda = -0^s.13$	$dA = +10^{cc} = 3''.2$

Furthermore the standard deviation of the Laplace quantity is calculated to be:

$$m_{A+\lambda\sin\varphi_0} = 1.68\sqrt{2.16+1.28^2\times 0.73+2\times 1.28(-1.22)} = 0''.82$$

while the expected error from the corresponding diagram is 1″.3.

It can be concluded that the internal accuracy of this measurement is quite satisfactory. The external accuracy in latitude ($d\varphi = +5''.1$) could be suspected of systematic error. Applying the student *t*-test one finds that a deviation of at least $\pm 5''.1$ in the determined latitude could arise in 7% of the measurements, consequently it can be accepted.

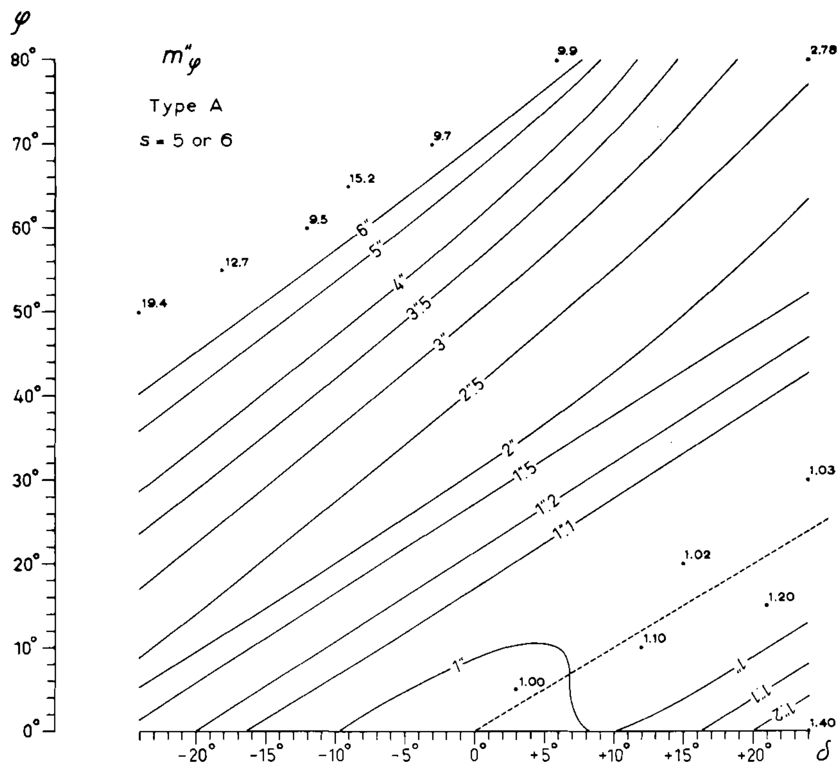
From an examination of the very small differences between the results of the graphical method with equal weights [2] and the numerical solution given here, it would appear that the effect of using different weights is not significant.

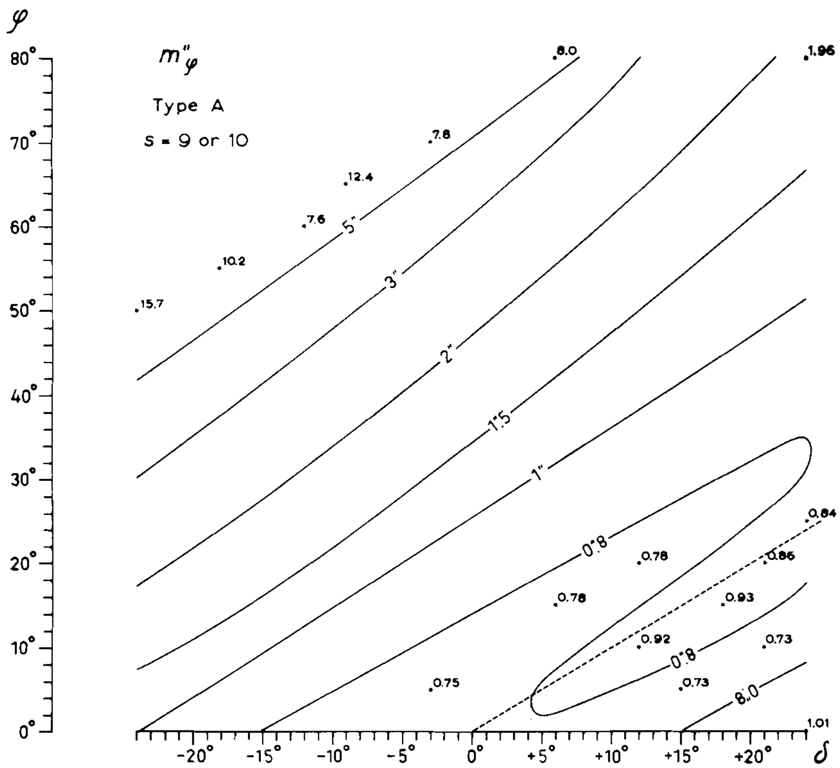
---

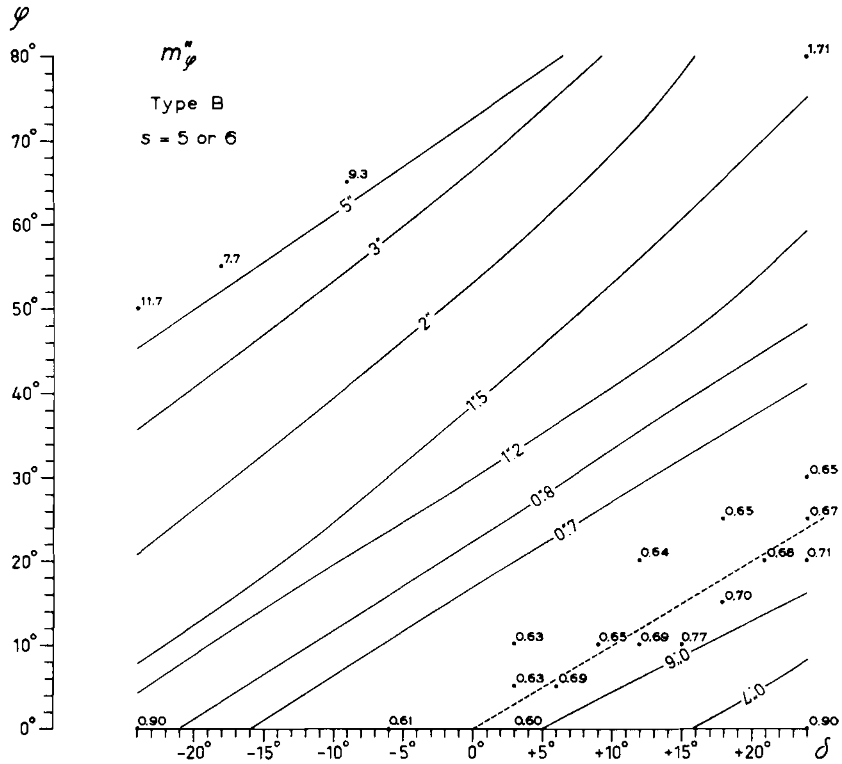
*) Church of Pijnacker.
**) "Known values" (mean value of a number of observations with different methods).

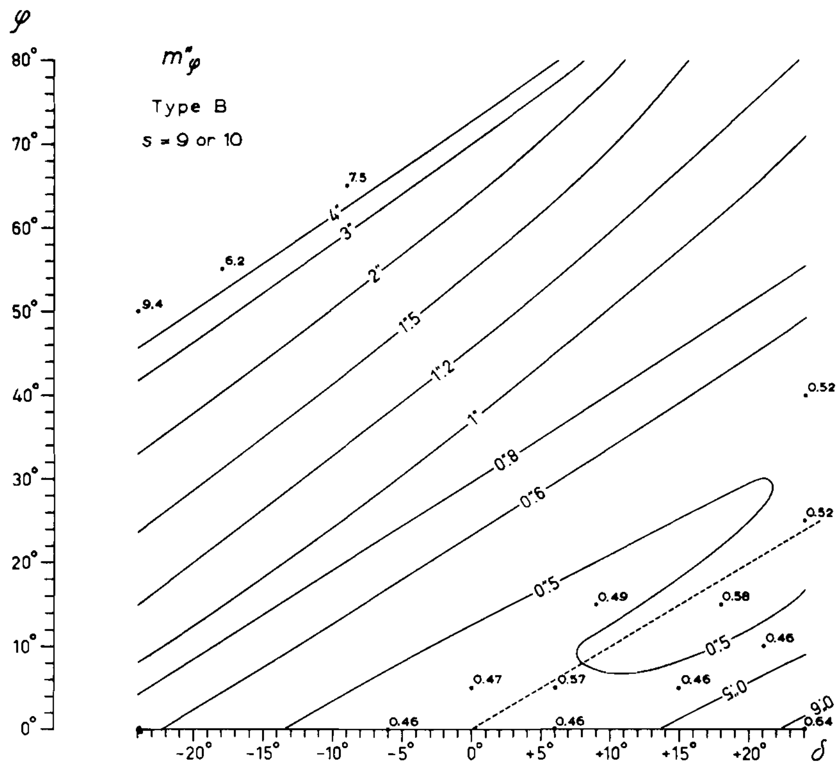
REFERENCES

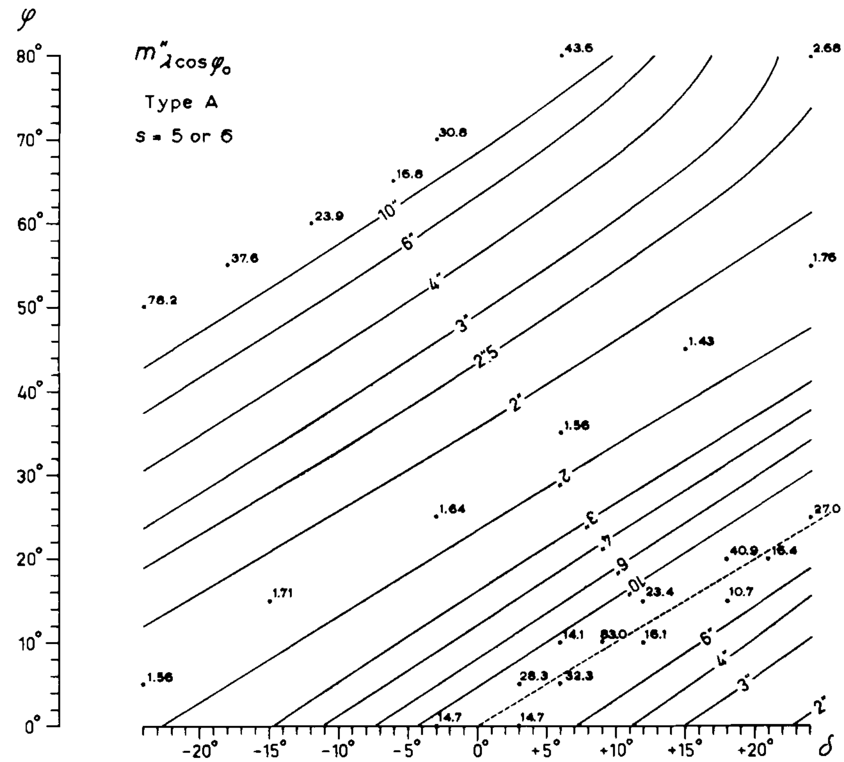
[1] R. Roelofs, Astronomy Applied to Land Surveying. Argus, Amsterdam, 1950.
[2] R. Roelofs, Simultaneous Determination of Latitude, Longitude and Azimuth by Horizontal Directions at the Sun. Annales Academiae Scientiarum Fennicae. Series A, III. Geologica-Geographica, 61.
[3] Th. Niethammer, Die Bestimmung der in der Laplaceschen Gleichung auftretenden Grössen astronomischer Natur. Annexe au Procès-Verbal de la 87e séance de la Commission Géodésique Suisse, 2 mai 1942.

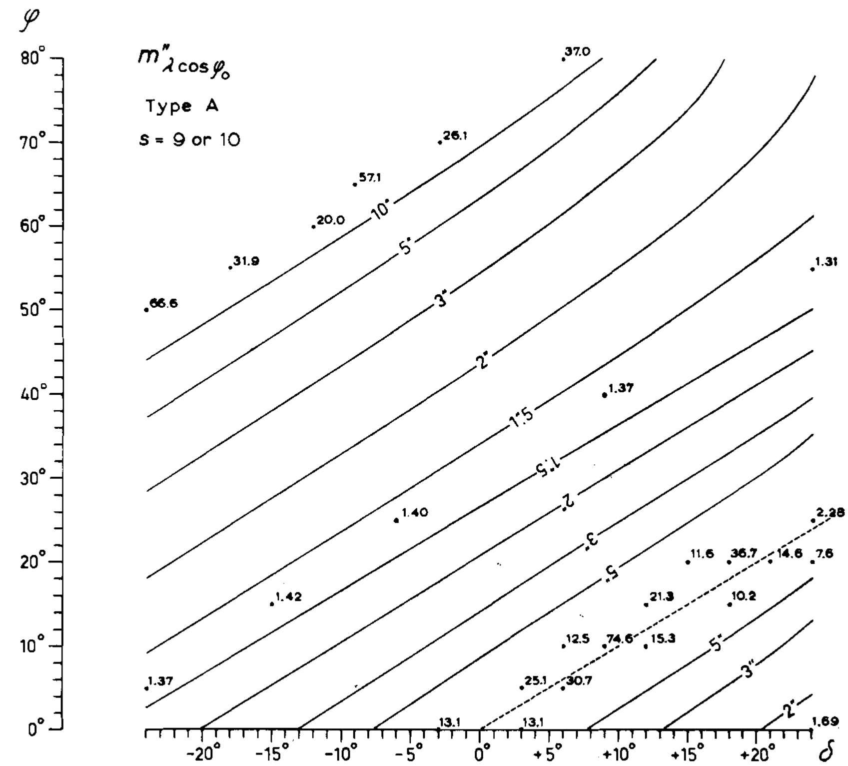

φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
m''φ
Type A
s = 5 or 6
9.9
2.78
9.7
15.2
9.5
12.7
19.4
6"
5"
4"
3"5
3"
2"5
2"
1"5
1"2
1"1
1.03
1.02
1.20
1.10
1"
1.00
1"
1"1
1"2
1.40
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ

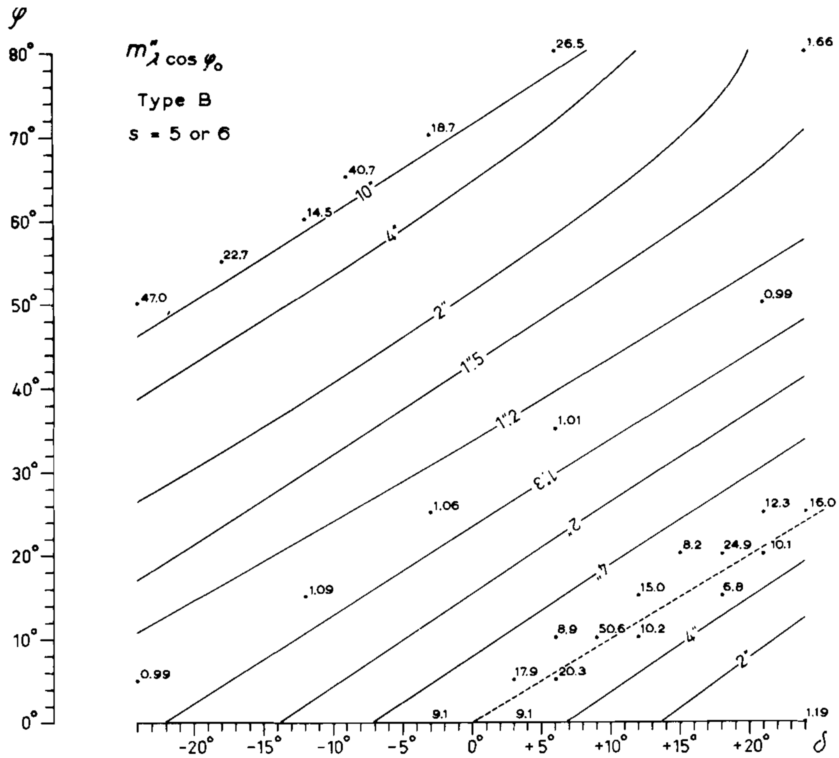

φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
m''φ
Type A
s = 9 or 10
8.0
1.96
7.8
12.4
7.6
10.2
15.7
5"
3"
2"
1"5
1"
0.84
0"8
0.78
0.86
0.78
0.93
0.92
0"8
0.73
0.75
0.73
0"8
1.01
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ


φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
m"φ
Type B
s = 5 or 6
9.3
7.7
11.7
5"
3"
2"
1"5
1"2
0"8
0"7
0"6
0"7
1.71
0.65
0.65
0.67
0.64
0.68
0.71
0.70
0.63
0.65
0.69
0.77
0.63
0.69
0.90
0.61
0.60
0.90
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ

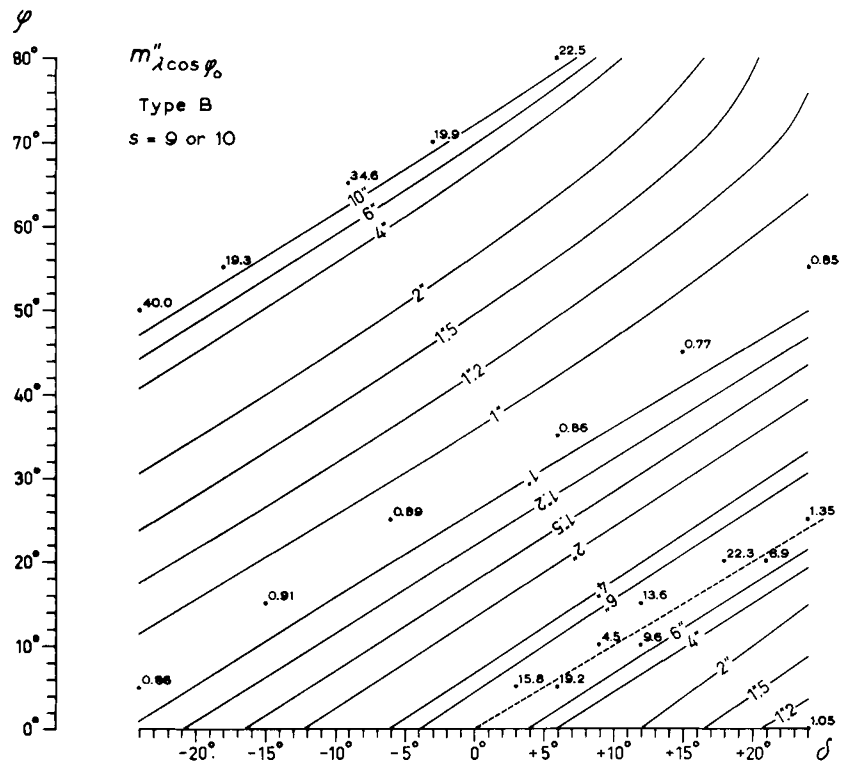

φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
m"φ
Type B
s = 9 or 10
7.5
6.2
9.4
4"
3"
2"
1"5
1"2
1"
0"8
0"6
0"5
0"5
0"5
0"6
0.52
0.52
0.49
0.58
0.46
0.47
0.57
0.46
0.46
0.46
0.64
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ


φ
$m''_\lambda \cos\varphi_0$
Type A
s = 5 or 6
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
10"
6"
4"
3"
2"5
2"
43.6
2.68
30.8
16.8
23.9
37.6
76.2
1.76
1.43
1.56
1.64
27.0
40.9
16.4
1.71
23.4
10.7
14.1
83.0
16.1
1.56
28.3
32.3
14.7
14.7

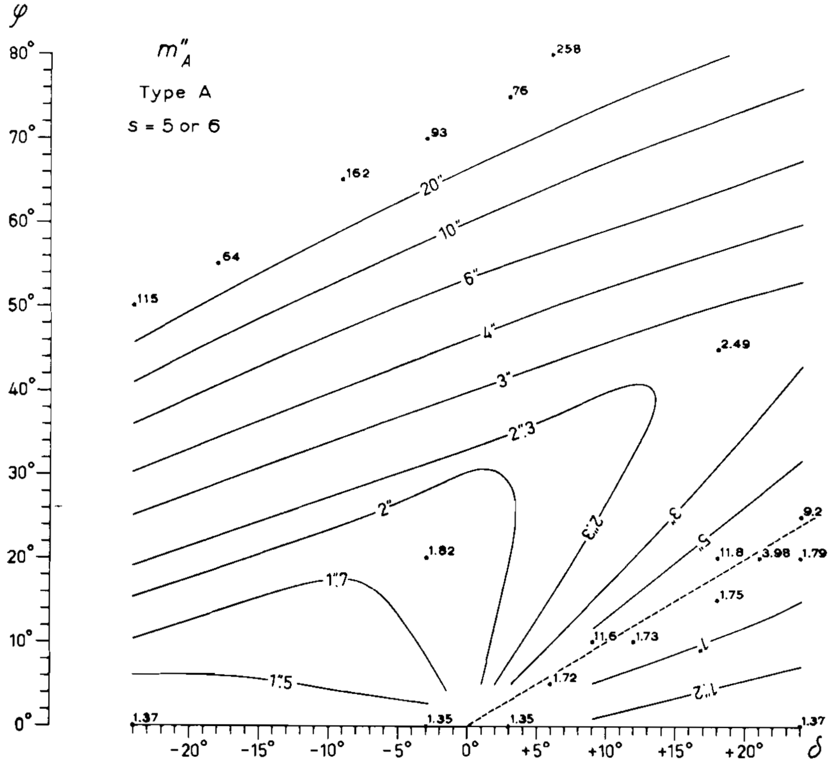

φ
$m''_\lambda \cos\varphi_0$
Type A
s = 9 or 10
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
10"
5"
3"
2"
1"5
37.0
26.1
57.1
20.0
31.9
66.6
1.31
1.37
1.40
2.28
11.6
36.7
14.6
7.6
1.42
21.3
10.2
12.5
74.6
15.3
1.37
25.1
30.7
13.1
13.1
1.69

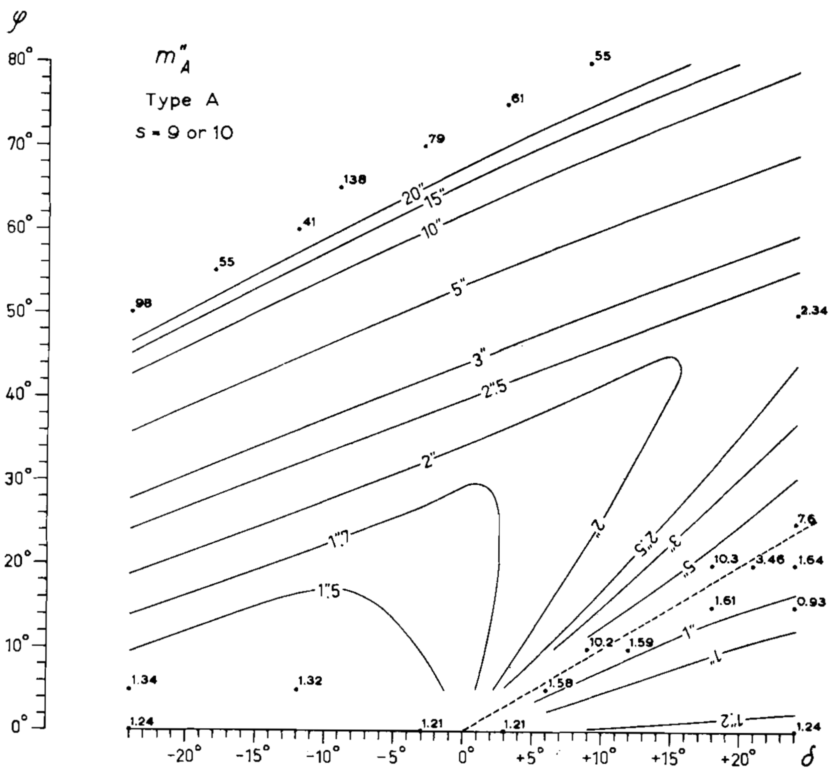


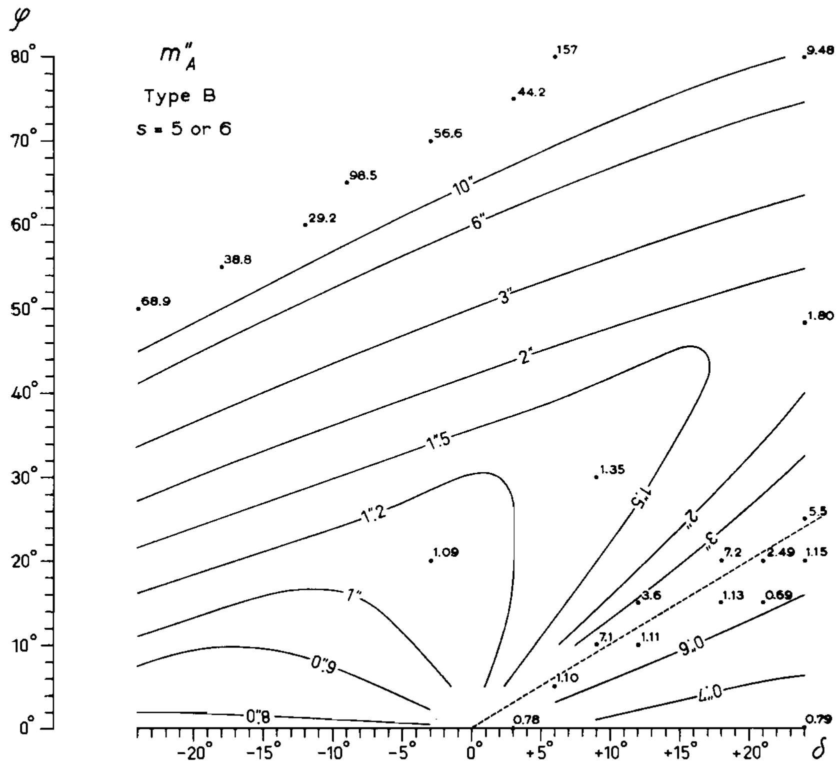
$m''_\lambda \cos \varphi_0$
Type B
s = 5 or 6
φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
26.5
1.66
18.7
40.7
10"
14.5
4"
22.7
47.0
2"
0.99
1"5
1"2
1.01
1"3
1.06
2"
12.3
16.0
8.2
24.9
10.1
4"
1.09
15.0
6.8
8.9
50.6
10.2
4"
2"
0.99
17.9
20.3
9.1
9.1
1.19



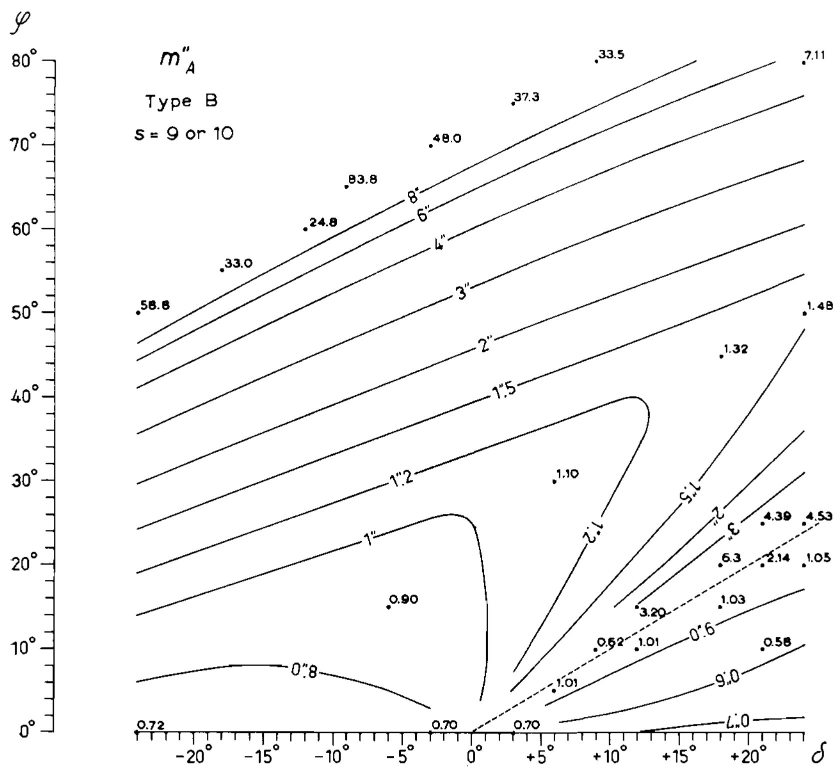
$m''_\lambda \cos \varphi_0$
Type B
s = 9 or 10
φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
22.5
19.9
34.6
10"
6"
4"
19.3
40.0
2"
0.85
1"5
1"2
0.77
1"
0.86
1"
1"2
1"5
2"
0.89
1.35
22.3
6.9
0.91
4"
6"
13.6
4.5
9.6
6"
4"
2"
0.86
15.8
19.2
1"5
1"2
1.05


m''_A
Type A
s = 5 or 6
φ
δ
20"
10"
6"
4"
3"
2"3
2"
1"7
1"5
1.82
2.49
9.2
11.8
3.98
1.79
1.75
11.6
1.73
1.72
1.37
1.35
1.35
1.37
258
76
93
162
64
115

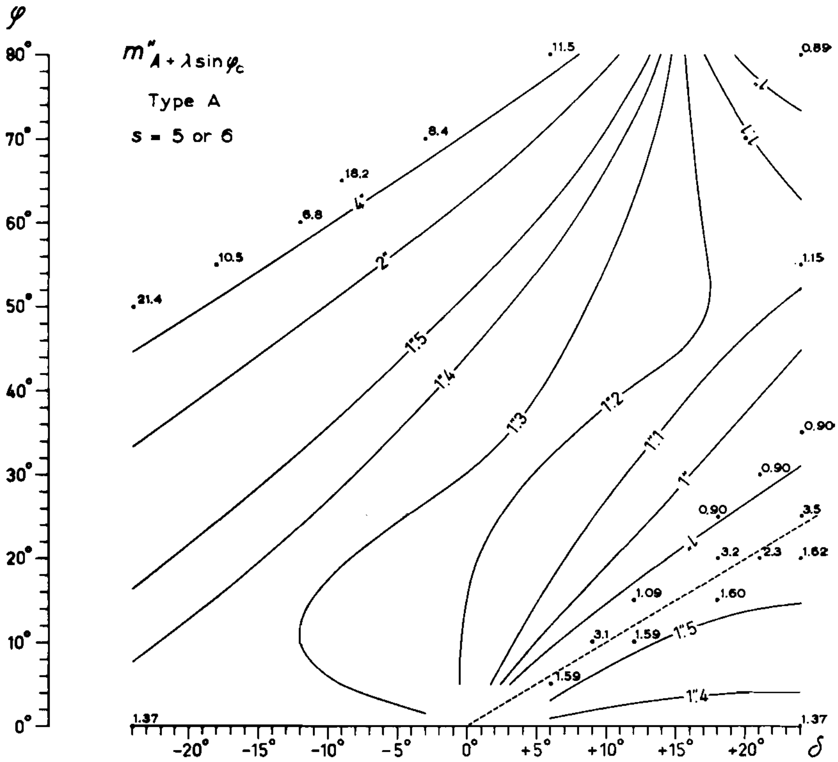

m''_A
Type A
s = 9 or 10
φ
δ
20"
15"
10"
5"
3"
2"5
2"
1"7
1"5
2.34
7.6
10.3
3.46
1.64
1.61
0.93
10.2
1.59
1.58
1.34
1.32
1.24
1.21
1.21
1.24
55
61
79
138
41
55
98

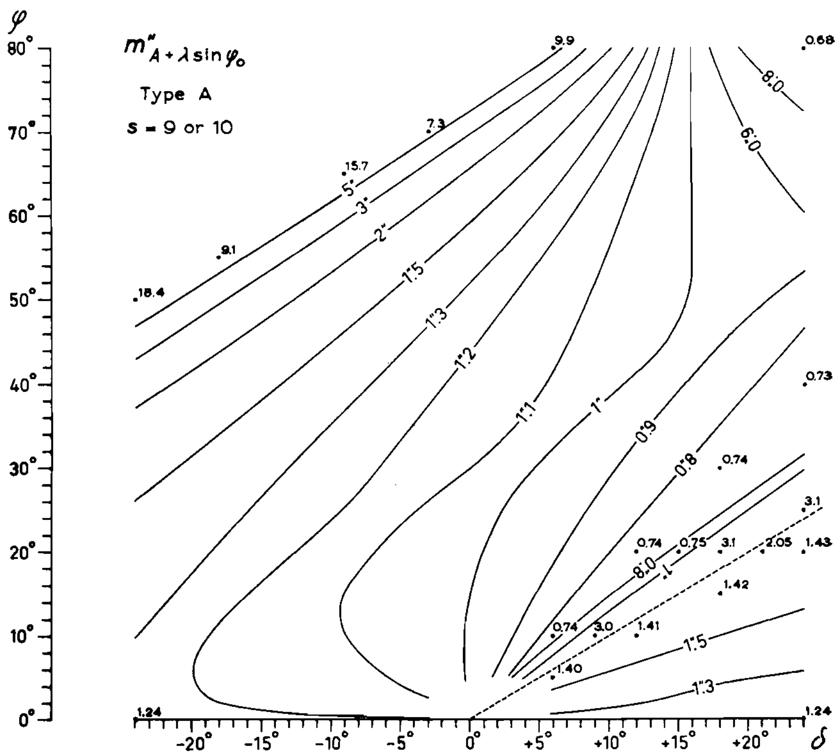


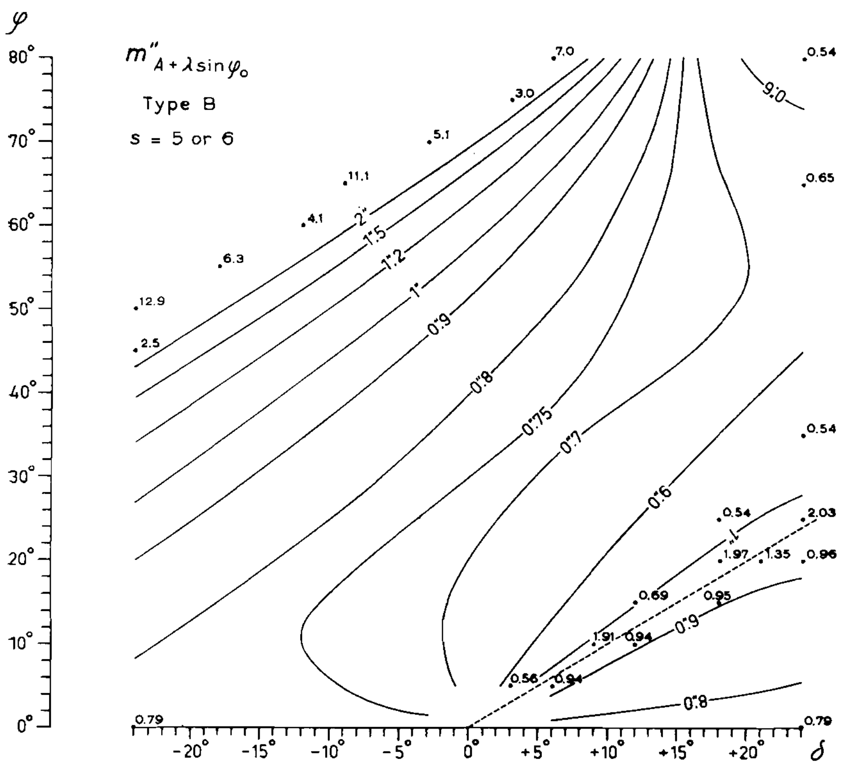
m''_A
Type B
s = 5 or 6

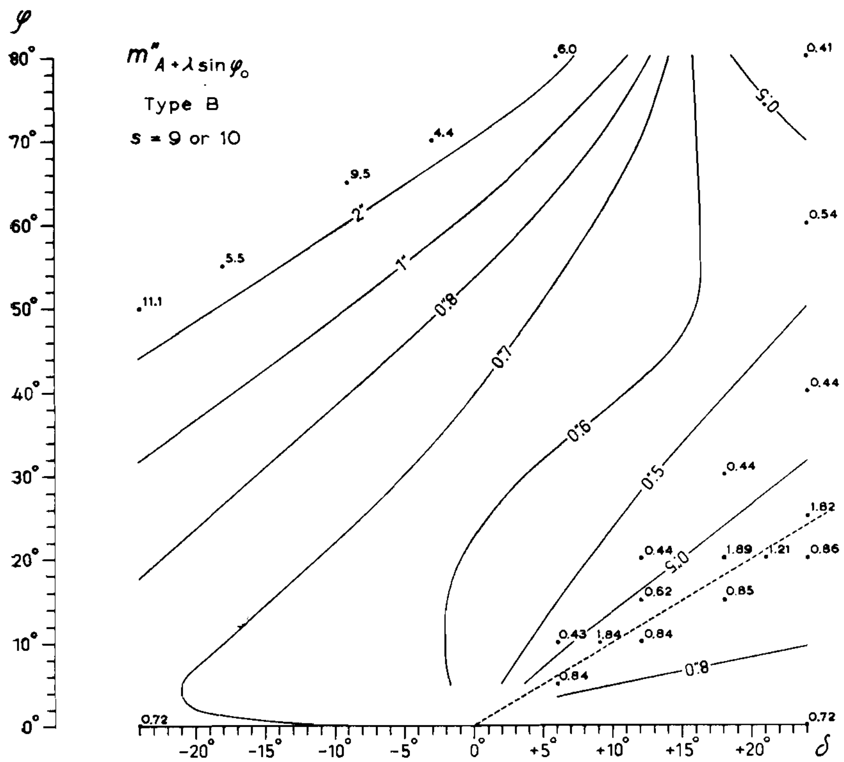


m''_A
Type B
s = 9 or 10


$m''_{A+\lambda \sin \varphi_c}$
Type A
s = 5 or 6
φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
11.5
0.89
8.4
18.2
6.8
10.5
21.4
1.15
0.90
0.90
0.90
3.5
3.2
2.3
1.62
1.09
1.60
3.1
1.59
1.59
1.37
1.37
4"
2"
1".5
1".4
1".3
1".2
1".1
1"
1".5
1".4


$m''_{A+\lambda \sin \varphi_o}$
Type A
s = 9 or 10
φ
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
9.9
0.68
7.3
15.7
9.1
18.4
0.73
0.74
3.1
0.74
0.75
3.1
2.05
1.43
1.42
0.74
3.0
1.41
1.40
1.24
1.24
5"
3"
2"
1".5
1".3
1".2
1".1
1"
0".9
0".8
0".8
0".9
1".5
1".3


φ
m″ A + λ sin φ₀
Type B
s = 5 or 6
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
2″
1″5
1″2
1″
0″9
0″8
0″75
0″7
0″6
0″5
0″9
0″8
7.0
3.0
5.1
11.1
4.1
6.3
12.9
2.5
0.54
0.65
0.54
0.54
2.03
1.97
1.35
0.96
0.69
0.95
1.91
0.94
0.56
0.94
0.79
0.79


φ
m″ A + λ sin φ₀
Type B
s = 9 or 10
80°
70°
60°
50°
40°
30°
20°
10°
0°
-20°
-15°
-10°
-5°
0°
+5°
+10°
+15°
+20°
δ
2″
1″
0″8
0″7
0″6
0″5
0″5
0″5
0″8
6.0
4.4
9.5
5.5
11.1
0.41
0.54
0.44
0.44
1.82
0.44
1.89
1.21
0.86
0.62
0.85
0.43
1.84
0.84
0.84
0.72
0.72