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LINKING UP SPATIAL MODELS IN GEODESY
EXTENDED S-TRANSFORMATIONS

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1. Introduction

In the nineteen-sixties I posed myself some questions when searching for a geodetic model based on the three division algebras, viz. the algebras of real numbers, complex numbers and quaternions. In the present more or less sketchy treatise the answers, found in recent years, are formulated. The division algebras had proved necessary in order to define dimensionless quantities which were instrumental in the coupling between dimensioned measurable quantities and fictitious mathematical quantities. The term "dimensionless" is here to be interpreted in a narrower sense than in everyday language, namely such that dimensional units which are eliminated are defined with a comparable sharpness by the process of measurement or computation; an example is the unit of length in the quotient of two distances measured under similar circumstances.

Division algebras are associative and commutative, except quaternion algebra, which is non-commutative with respect to multiplication. The latter fact is no objection, in fact it is even an advantage because one is forced to be more careful when establishing relationships. It is more awkward that there is no Analysis applicable to quaternions, contrary to the case for real numbers and complex numbers (the theory of functions).

In three-dimensional geodesy one therefore usually prefers to use vectors and tensors with their vector- and tensoranalysis respectively. But vector- and tensoralgebra are no division algebras, which makes the formation of dimensionless quantities difficult, if not impossible.

The objection to quaternions can, however, be circumvented by linearizing non-linear relations, using difference-quantities or -variates, such as $Q - Q^0$, the difference of a quaternion-variate Q and its approximate value Q^0. Then one can artificially define e.g. the logarithm of a quaternion via a difference quantity, in analogy with the difference quantity of the logarithm of a vector in the complex plane. In the complex plane one finds for the relation between rectangular and polar coordinates (i, j, k are point numbers):

$$z_{ij} = x_{ij} + \bar{e}y_{ij} = s_{ij}\left(\cos\varphi_{ij} + \bar{e}\sin\varphi_{ij}\right) = e^{\ln s_{ij} + \bar{e}\varphi_{ij}} \quad , \bar{e}\bar{e} = -1$$

$$\Lambda_{ij} \underset{\text{def}}{=} \ln z_{ij} = \ln s_{ij} + \bar{e}\varphi_{ij} \quad , \Delta\Lambda_{ij} = \frac{\Delta z_{ij}}{z^0_{ij}}$$

$$\Pi_{jik} \underset{\text{def}}{=} \Lambda_{ik} - \Lambda_{ij} = \ln\frac{s_{ik}}{s_{ij}} + \bar{e}\left(\varphi_{ik} - \varphi_{ij}\right)$$

in which $\frac{s_{ik}}{s_{ij}}$ and $\left(\varphi_{ik} - \varphi_{ij}\right)$ are dimensionless (and estimable) quantities. In the complex number theory Π_{jik} is invariant with respect to a similarity transformation and can therefore be called a form element. In our quaternion theory the following is analogically introduced (Section 3.2.2.1 ff.):

$$\Delta\Lambda_{ik} \underset{\text{def}}{=} \left(q_{ik}^{0}\right)^{-1} \Delta q_{ik} \quad , \quad \Delta\Pi_{jik} = \Delta\Lambda_{ik} - \Delta\Lambda_{ij}$$

but here $\Delta\Pi_{jik}$ is not completely invariant with respect to a similarity transformation.

This invariance can, however, be attained by applying a so-called S-transformation to co-ordinate variates, whereby also the coordinate system is defined. But there appears a typical difference between the two- and the three-dimensional situations. Whereas for coordinate variates in the complex plane an S-transformation does not affect the property of circularity of point- and relative standard ellipses, the analogous property of sphericity in the three-dimensional situation is lost when an S-transformation is applied. This had already been shown by numerical computations, the theoretical proof has now been given in connection with the formulation of a Criterion Matrix for coordinate variates.

Because in the application of quaternion theory one is practically compelled to work with difference quantities, the choice of approximate values must be carefully considered. In this context I remember being puzzled when first working on the S-transformation in the complex plane in 1944. The approximate values chosen for coordinates were those resulting from an adjusted network. All Δ-values were zero and remained zero, and yet the covariance matrix of the coordinate variates was transformed. Of course the explanation was that the Δ-values were indeed zero, but the Δ-variates were not. I was an "isoparametric mapping" *avant la lettre*, later so well-known in physical geodesy, although there the distinction between values and variates was not always observed, to the detriment of conclusions drawn.

Following up [Baarda 1979] [1)], the present publication once more pays attention to the coupling and interaction between geometric and gravimetric or physical geodesy, with some further conclusions. Aspects of physical theory are elaborated on the basis of real numbers. A further development of "corrections" to the Stokes-type integral formulas in [Baarda 1979] has been taken over from [Baarda 1989]; the results deviate from the customary form. It is regretted that neither the line of thought nor the results have so far drawn the attention of the geodetic community.

In the present state of geodesy it is logical to devote attention to the coupling between on one hand terrestrial geometric and physical geodesy and on the other hand the much more spatially oriented satellite geodesy. Here also, the search is for dimensionless quantities and the application of quaternion theory, so that a similar testing theory for errors and a similar criterion theory for the precision of coordinates can be applied.

Since 1963, when I contributed my paper "Modeleffecten in de geodesy" to a discussion in the Netherlands Geodetic Commission, I am convinced that the origin P_M of a geodetic coordinate frame (as a part of the set of approximate values) will never exactly coincide with

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1) See "Notes and References"

the centre of mass P_C of the earth. In [Baarda 1979] this effect has therefore been accounted for in the entire system of formulas. In the present study this is also observed in satellite orbit computations and in the establishment of terrestrial control stations by satellite methods. The results indicate a small, but by no means negligible, deformation in station coordinates caused by the non-geocentricity of the coordinate frame. Owing to VLBI-methods the eccentricity appears to be estimable, but it is impossible to reduce it exactly to zero.

Furthermore there is the remarkable possibility of another small effect, viz. a difference in scale between the computation of (among others) the gravity potential from satellite data and the computation from terrestrial data.

The author feels obliged to apologize for the lack of homogeneity in this treatise. The cause is the serious calamity which has fallen upon his family in 1971 and has since seriously impeded the completion and publication of research. The aggravation of recent years allowed only short periods for rounding off research and writing down the results, while checking remained inadequate.

The plan of this essay is based on a contribution to the commemoration of F.A. Vening Meinesz by the Royal Netherlands Academy of Arts and Sciences in 1987. In this contribution, geodesy had to be explained to non-geodesists, traces of which can still be found in the present text.

A contribution to the "Festschrift to Torben Krarup", 1989, entitled "Tentative Remarks on Adjustment Models in Geodesy" can be seen as a second version. The third version, further completed, is now presented. In each version parts of a previous one have been included, after correction of errors and mistakes that inevitably had been made. There can be little hope that the present version contains no errors, but it is hoped that they do not invalidate the train of thought developed.
The basis for all three versions is still [Baarda 1979], which publication found its origin in an unpublished essay for the Festschrift in honour of A. Marussi's 70th birthday.

The text is restricted to the main lines; various small but necessary corrections have been omitted in order to avoid disturbance of the essential line of thought. Of course it is acknowledged that the figure of the earth depends on time, but this only enhances the value of a good momentary computing model.

In order to facilitate reference, short abstracts of the sections are now given:

1. Introduction.
2. A preliminary consideration of the three-dimensional S-coordinate frame preferred by the author.

3.1 An estimation of the position of the centre of mass of the earth in an S-coordinate frame according to Section 2, with possible consequences for the linking up of mathematical models in geodesy.

3.2 An intermezzo treating by means of quaternions the mathematical formulation of the S-coordinate frame chosen, and some consequences.

3.2.0 Introduction

1.

3.2.1 The similarity transformation including the gravity potential. A form element for this potential.

3.2.2.1 The formula for the three-dimensional S-coordinate. The S-transformation as a connection onto assumed fixed coordinates.

3.2.2.2 The relation between two- and three-dimensional S-coordinates. An elegant formula for the three-dimensional S-coordinate as a function of three intrinsic quantities.

3.2.3 Application of the law of propagation of variances by means of isomorphic matrices.

3.2.4 The construction of Criterion Matrices. The important theorem stating that sphericity of three-dimensional point- and relative standard ellipsoids is not conserved in an S-transformation, contrary to the corresponding property of circularity in the two-dimensional situation.

4.1 Again the connection between gravimetric and geometric theory, treated sketchily but with concentration on the fundamentals. The linking up of a mathematical model by dimensionless quantities. An objectionable interpretation of compound quantities as "free-air reduction to the geoid". A more appropriate definition of "relative sea-topography".

4.2 An analysis of the modified integral formulas of Stokes and Hotine, based on ideas of Rummel and Teunissen. A choice for the present satellite era. The lasting influence of the transition from sea to land.

4.3 The influence of $\overrightarrow{P_M P_C} \neq 0$, P_C being the centre of mass of the earth and P_M the origin of a quasi-centric S-coordinate frame for terrestrial data.

4.4 Once more the integral formula of Hotine. Effects of first degree spherical harmonics. A suggestion for application. An afterthought.

5. Supplementary remarks on the linking up of the gravimetric-geometric model. Correction terms in the modified integral formulas of Stokes, Hotine and Vening Meinesz, deviating from the terms found in the literature.

6. Possible consequences of the gravimetric-geometric S-system for (terrestrial) mechanics. The corresponding dimensionless time quantity.

7.1 A sketch of problems in point positioning on the earth by means of satellite observations.

7.2 The effect of $\overrightarrow{P_M P_C} \neq 0$ on launch data of a satellite.

7.3 Correction of $\Delta\Pi$-quantities (orbit data) for earth rotation.

8.1 An alternative way of writing the formulas of the Kepler ellipse for the computation of a satellite orbit. Difference formulas for dimensionless quantities, such as the dimensionless time interval.

8.2.1 The linking up of the mathematical model from section 8.1. The influence of $\overrightarrow{P_M P_C} \neq 0$.

8.2.2 Comparison of the S-system in satellite orbit computation with the S-system in physical geodesy; possible small differences in scale in mass, potential and time.

8.2.3 Questions arising when rewriting the higher-order terms of orbit computations by means of the dimensionless quantities introduced.

9.1 Establishment of control by satellite measurements. The bird's-tail construction. Difference formulas with an appraisal of the influence of $\overrightarrow{P_M P_C} \neq 0$.

9.2 An investigation into possibilities for the estimation of $\overrightarrow{P_M P_C}$.

9.3.1 A more realistic process of measurement by means of series of pseudo-distances (distance measurements from one station with the same but unknown length scale).

9.3.2 Synchronous measurements in several stations, with an estimation of the maximum distance between stations if a (practically acceptable) elimination of the influence of orbit errors is to be attained. Measurement of the distance differences.

10. A short after-consideration. The application of dimensionless quantities in satellite gradiometry.

11. Concluding word with remarks concerning relativity theory, a possible influence of the choice of terrestrial datum points on the precision of the determination of points of satellite orbits, doubts about the alleged precision of computed quantities in physical geodesy obtained by satellite gradiometry.

Notes and References.

2.

In order to get out of the tangle of systematic and pseudo-systematic errors in plane control networks, I developed around 1960 a system of measurement and computation which was based on angles and distance ratios, compounded into complex quantities. The aim was to eliminate uncertainties in instrument orientation and -scale. Consequently, only the form of a group of terrain points was determined, to be described in a coordinate frame attached to these points in a precisely defined way, making use of the well-known four degrees of freedom. A closer analysis shows that this coordinate frame is part of the set of approximate coordinate values.

In interaction with the theory of complex numbers, the spatial terrestrial method grew, using quaternions as quotients of vectors. An extra complication is that although a quaternion is invariant with respect to rotation and stretching in the plane perpendicular to its unit vector, the direction of this unit vector has to be fixed with respect to a coordinate frame. This proves to be essential for satellite problems to be considered later.

Here again it is necessary for the definition of coordinates to attach the coordinate frame to a group of terrain points, for example by considering as non-stochastic the coordinates of two terrain points and the coordinate component of a third terrain point perpendicular to the plane of the three points. This is a typical example of the definition of an S-system, in which to a certain extent the plane of the three datum points takes the place of the plane of the complex number theory. It is clear that the number of degrees of freedom in the spatial case is seven. The measurement of vertical angles, astronomical latitude, longitude and azimuth, determines the direction of the vertical in the terrain stations in the S-system chosen.

The vertical angles are influenced by refraction, which has an adverse effect on the coordinate component perpendicular to the earth's surface. The coordinate components along the earth's surface are hardly affected by this, as was already experimentally established by Hotine. No wonder that the classical ellipsoidal network computations could retain their value, and, be it in different variations, are still being used for terrestrial work.

3.

3.1

The last remark illustrates the fact that part of the terrestrial achievements in geodesy is always conserved. This will now be gratefully used, namely by the introduction of an earth model, to begin with having a simplicity adapted to transparent problems, later becoming more complicated as subsequent problems have a greater complexity.

Begin by considering the earth as a homogeneous sphere with radius R, rotating around a constant axis with a constant velocity. We have to consider the rotation because of astronomical measurements, but for simplicity we shall ignore centrifugal potential. In this model the centre of mass P_C of the earth is the centre of the sphere.

Now imagine this earth model described in a rectangular X', Y', Z' coordinate system, whose origin is in P_C, with the Z'-axis along the axis of rotation.

Suppose that a continental network has been measured on this earth model, so that angles and distance ratios can be computed between three points P_1, P_2, P_3 having mutual distances of 2000 km. The standard deviations of the angles and distance ratios are assumed to be $\sigma = 10^{-5}$. Now measure the astronomical latitude and longitude in P_1 and P_2, as well as the distance s_{12} (possibly in an indirect way).

Choose:

$$\varphi_1 = \varphi_2 \quad (\simeq 52^\circ)$$

P_3 south of P_1 and P_2

$$\sigma_{\varphi_1} = \sigma_{\varphi_2} = \cos\varphi_1 \cdot \sigma_{\lambda_{12}} = 0.5 \cdot 10^{-5}$$

$$\sigma_{\ln s_{12}} = 10^{-5}$$

One can then compute estimates for R, φ_3 and λ_{13}, with, among others, $\sigma_{\ln R} \simeq 2 \cdot 10^{-5}$ which is in reasonable agreement with earlier analyses of the dimensions of reference ellipsoids. One can also compute the set of X', Y', Z'-coordinates of P_1, P_2 and P_3 with their covariance matrix. In this system the coordinate variances of P_C are zero, as well as the variance of the direction of the axis of rotation.

At this stage P_C and the axis of rotation still are fictitious mathematical entities; only the terrain points P_1, P_2, P_3 are visible and accessible to man. Therefore the problem must be posed the other way around. In order to do so, execute a similarity transformation preserving the estimated values, but now putting equal to zero the variances of the coordinates of

P_1 and P_2 and the coordinate component of P_3 perpendicular to the plane of the three points; call this the X, Y, Z-system. The variances of the coordinates of P_C and the direction of the axis of rotation are not zero in this system; these quantities are now estimated with respect to the datum points P_1, P_2 and P_3 of an S-system as introduced in the quaternion theory. The X, Y, Z-frame is thus fixed by the terrain points P_1, P_2, P_3; now the origin P_M does not in general coincide with P_C, nor is the Z-axis parallel to the axis of rotation. For the example computed one finds roughly:

$$\frac{1}{R}\sigma_{\text{coord. } \hat{P}_C} \simeq \sigma_{\text{direction axis}} \simeq 10^{-5}$$

a result in reasonable agreement with the discrepancies found in the connection of classical continental networks by satellite methods. The only estimate of $\overline{P_M P_C}$ I know in classical geodesy is given by Ledersteger in Volume V of the 10th edition of the Jordan Handbuch der Vermessungskunde. On page 37 he arrives at an estimated $10^{-4}R$. The appraisal which was made above thus look reasonable, and its seems possible already to draw some consequences.

Physical geodesy and geometric geodesy are inseparably connected, as was also shown in my 1979 publication. Consequently, the statement in some textbooks, that one can choose a reference ellipsoid centred in P_C, having its minor axis parallel to the axis of rotation of the earth, cannot be put into practice.

Only form elements are determined, for if the unit of length, in which the distance s_{12} is expressed, changes, all distance quantities are proportionally reduced or enlarged. Such a change may be brought about by the stochastic effects of measurement, and/or by more or less doubtful reductions which are inevitable when a more complicated earth model is considered. But form alone does not determine volume; therefore a requirement concerning the volume of a reference ellipsoid, which is often found in the literature, cannot be fulfilled either.

In fact the unit in which distances are measured is determined by the value (estimated or without a measurement process) assigned to the distance between the terrain points P_1 and P_2. This means that the metre loses its role as a unit of measurement. If now a velocity has been expressed in metres per second, then in our model a function of this velocity must be introduced, which takes care of the difference between the model unit of measurement and the metre. Also the unit of time will then have to be subjected to a suitable transformation. The same applies to accelerations and consequently to gravity. We shall come back to this later.

For the computation of satellite orbits one **thinks** it is justified to choose the origin of the orthogonal coordinate system in P_C. But when a satellite is launched, the position vector and the velocity vector are essentially determined via measurement in points on earth whose coordinates are known in an S-system. This implies that, in computing the orbit, one has to introduce corrections for the eccentricity of the origin of this S-coordinate frame with respect to P_C; corrections which eventually should make it possible to estimate this eccentricity. In analogy with terrestrial situations, one meets here form problems that inevitable lead to suitable transformations of quantities. This also will be treated in the sequel.

3.2 Intermezzo

In this Section 3.2. a three-dimensional form element will be developed, which is invariant with respect to a similarity transformation close to identity. The formula developed may be called an S-transformation of three-dimensional coordinates. It has already been published in [M. Molenaar 1981] and [H. Quee 1983], but the present derivation is aimed at the construction of a Criterion Matrix for three-dimensional coordinates, and shows a fundamental difference with the two-dimensional situation. This is relevant for the problems in satellite geodesy to be treated in the following sections, because in our approach the terrestrial X, Y, Z-frame in principle also remains valid for satellite orbit computations. The determination of the coordinates of terrestrial stations from "known" satellite coordinates may imply the necessity to establish a Criterion Matrix - as a substitute for a "real" covariance matrix - for these satellite coordinates. Such a Criterion Matrix can also be used for judging the covariance matrix of the computed coordinates of a global network of terrestrial stations. In both cases the third dimension is conspicuously present.

3.2.1

Consider a point P_k having coordinates x_k, y_k, z_k. Unit vectors on the x, y and z-axes are e_x, e_y, e_z respectively [1)]. Then the vector q_k is:

$$q_k = x_k e_x + y_k e_y + z_k e_z$$

The similarity transformation from q to q' is then, written in quaternion notation:

$$q'_k = \lambda p q_k p^{-1} + q_0$$

in which λ = length scale factor
p = rotation quaternion with norm 1
q_0 = constant vector.

Now consider, for simplicity, a differential transformation with approximate values:

$$\lambda^{(\mathrm{appr})} = 1 \quad , \quad p^{(\mathrm{appr})} = 1 \quad , \quad q_0^{(\mathrm{appr})} = 0$$

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1) For a summary of quaterniontheory, see "Notes and Refernces, Section 3."

3.

hence $\quad q'^{(\text{appr})}_k = q_k^{(\text{appr})}$

and put (leaving out the "appr" notation in the sequel):

$$\lambda = 1 + \Delta\lambda \quad , \quad p = 1 + \Delta p \quad , \quad q_0 = 0 + \Delta q_0$$

$$q_k' = q_k^{(\text{appr})} + \Delta q_k' \quad , \quad q_k = q_k^{(\text{appr})} + \Delta q_k$$

with $\quad \Delta(p^{-1}) = -p \cdot \Delta p p^{-1} = -\Delta p$ one obtains:

$$\boxed{\Delta q_k' = q_k \cdot \Delta\lambda + \Delta p \cdot q_k - q_k \cdot \Delta p + \Delta q_k + \Delta q_0}$$

By subtracting from this formula the corresponding one for a "datum point" P_1, Δq_0 is eliminated. Or, with $q_k - q_1 = q_{1k}$:

$$\boxed{\Delta q_{1k}' = q_{1k} \cdot \Delta\lambda + \Delta p \cdot q_{1k} - q_{1k} \cdot \Delta p + \Delta q_{1k}}$$

It will be clear from the following sections that there is a continuous interaction between the gravity potential W and q as a compound of "geocentric" coordinates. The orders of magnitude and the dimensions are the same if one takes $\frac{W}{g_1}$ instead of W, g_1 being an (assumed) value of gravity in a datum point P_1.

Then it is plausible to add a scalar w to q, and we think of:

$$w_k = \frac{W_k}{g_1}$$

The three-dimensional case then becomes four-dimensional. Now let us see where the application of our similarity transformation leads us:

$$(w_k' + q_k') = \lambda p (w_k + q_k) p^{-1} + (w_0 + q_0)$$

Since $p w_k p^{-1} = w_k$, this equation splits into:

$$w_k' = \lambda w_k' + w_0$$

$$q_k' = \lambda p q_k p^{-1} + q_0 \qquad \text{(as before)}$$

Now introduce the additional approximate values:

$$w_k'^{(\mathrm{appr})} = w_k^{(\mathrm{appr})} \quad , \quad w_0 = 0$$

Then one obtains:

$$w_k \,\Delta\left(\ln w_k'\right) = w_k \left[\Delta\lambda + \Delta\left(\ln w_k\right) + \frac{\Delta w_0}{w_k}\right]$$

or:

$$\Delta\left(\ln w_k'\right) = \Delta\lambda + \Delta\left(\ln w_k\right) + \frac{\Delta w_0}{w_k}$$

In analogy for the "datum point" P_1:

$$\Delta\left(\ln w_1'\right) = \Delta\lambda + \Delta\left(\ln w_1\right) + \frac{\Delta w_0}{w_1}$$

Now, calling in mind the introduction of w, the often applied spherical approximation in coefficients of difference formulas results in:

$$w_k^{(\mathrm{appr})} \simeq w_1^{(\mathrm{appr})}$$

Using this, with:

$$\Delta\left(\ln w_k'\right) - \Delta\left(\ln w_1'\right) = \Delta\left(\ln \frac{w_k'}{w_1'}\right)$$

one obtains

$$\boxed{\Delta\left(\ln \frac{w_k'}{w_1'}\right) = \Delta\left(\ln \frac{w_k}{w_1}\right)}$$

so that for the scalar part of $(w + q)$ a separate form element has been found. It must be noted that the formulation is chosen with a view to the train of thought which will be developed later in Section 4.1.

3.2.2.1

We now continue with the three-dimensional case. From $\Delta q_{1k}'$ follows, with $q_{1k}^{-1}\Delta q_{1k}' = \Delta\Lambda_{1k}'$:

$$\Delta\Lambda_{1k}' = \Delta\Lambda_{1k} + \Delta\lambda - \left(\Delta p - q_{1k}^{-1}\Delta p q_{1k}\right)$$

3.

Split up Δp into two parts Δp_a and Δp_b, introducing the datum point P_2 :

$$\Delta p = \underbrace{\frac{1}{2}\left(\Delta p - q_{12}^{-1}\Delta p q_{12}\right)}_{\Delta p_a} + \underbrace{\frac{1}{2}\left(\Delta p + q_{12}^{-1}\Delta p q_{12}\right)}_{\Delta p_b}$$

Δp_a = vector $\perp q_{12}$, hence $q_{12}^{-1}\Delta p_a = -\Delta p_a q_{12}^{-1}$

$Ve\{\Delta p_b\}$ = vector // q_{12}, hence $q_{12}^{-1}\ \Delta p_b q_{12} = 0$

Or:

$$\begin{cases} \Delta\Lambda'_{1k} = \Delta\Lambda_{1k} + \frac{1}{2}(\Delta\lambda - \Delta p_a) + \frac{1}{2}q_{1k}^{-1}(\Delta\lambda + \Delta p_a)q_{ik} + \\ \qquad -\frac{1}{2}\left(\Delta p_b - q_{1k}^{-1}\Delta p_b q_{ik}\right) \\ \Delta\Lambda'_{12} = \Delta\Lambda'_{12} + (\Delta\lambda - \Delta p_a) \end{cases}$$

With $(\Delta p_a)^T = -\Delta p_a$ the last equation results in:

$$\begin{cases} \Delta\Lambda'_{12} - \Delta\Lambda_{12} = \Delta\lambda - \Delta p_a \\ \Delta\Lambda'^{T}_{12} - \Delta\Lambda^{T}_{12} = \Delta\lambda + \Delta p_a \end{cases}$$

Substitute this into the equation for $\Delta\Lambda'_{1k}$ and transport all quantities with a prime to the left hand side:

$$\Delta\Lambda'_{1k} - \frac{1}{2}\left(\Delta\Lambda'_{12} + q_{1k}^{-1}\Delta\Lambda'^{T}_{12}\, q_{1k}\right) =$$

$$= \Delta\Lambda_{1k} - \frac{1}{2}\left(\Delta\Lambda_{12} + q_{1k}^{-1}\ \Delta\Lambda^{T}_{12}\, q_{1k}\right) - \frac{1}{2}\left(\Delta p_b - q_{1k}^{-1}\Delta p_b\, q_{1k}\right)$$

Multiply by q_{1k}, $q_{1k}\Delta\Lambda'_{1k} = \Delta q'_{1k}$:

$$\Delta q'_{1k} - \frac{1}{2}\left(Q_{21k}\Delta q'_{12} + \Delta q'_{12}\, Q^{T}_{21k}\right) =$$

$$= \Delta q_{1k} - \frac{1}{2}\left(Q_{21k}\Delta q_{12} + \Delta q_{12} Q^{T}_{21k}\right) - \frac{1}{2}(q_{1k}\Delta p_b - \Delta p_b q_{1k})$$

With $q_{1k} = q_{2k} - q_{21}$, $q_{12} = -q_{21}$ one obtains:

$$\boxed{\begin{aligned}&\Delta q_{1k} - \frac{1}{2}\left(Q_{21k}\Delta q_{12} + \Delta q_{12}Q_{21k}^{T}\right) = \\ &= \Delta q_{2k} - \frac{1}{2}\left[\left(1 - Q_{21k}\right)\Delta q_{21} + \Delta q_{21}\left(1 - Q_{21k}^{T}\right)\right] = \\ &= \Delta q_{2k} - \frac{1}{2}\left(Q_{12k}\Delta q_{21} + \Delta q_{21}Q_{12k}^{T}\right) = \\ &\underset{\text{put}}{=} \Delta q_k^{(1,2)}\end{aligned}}$$

Because indices 1 and 2 can be exchanged, and with $k \to 1$ or 2 one gets:

$$\Delta q_1^{(1,2)} = \Delta q_2^{(1,2)} = 0$$

Hence:

$$\Delta q_k'^{(1,2)} = \Delta q_k^{(1,2)} - \frac{1}{2}\left(q_{1k}\Delta p_b - \Delta p_b q_{1k}\right)$$

For the elimination of Δp_b we use a third "datum point" P_3 .

Because $Ve\{\Delta p_b\} \mathbin{/\!/} q_{12}$ we introduce a scalar c and put:

$$\Delta p_b = Sc\{\Delta p_b\} + q_{12}^{-1}\cdot c$$

or:

$$\frac{1}{2}\left(q_{1k}\Delta p_b - \Delta p_b q_{1k}\right) = \frac{1}{2}\left(Q_{21k} - Q_{21k}^{T}\right)c = v_{21k}\sin\alpha_{21k}\cdot e_{21k}\cdot c$$

$$\begin{cases}\Delta q_k'^{(1,2)} = \Delta q_k^{(1,2)} - \frac{1}{2}\left(Q_{21k} - Q_{21k}^{T}\right)c \\ \Delta q_3'^{(1,2)} = \Delta q_3^{(1,2)} - \frac{1}{2}\left(Q_{213} - Q_{213}^{T}\right)c\end{cases}$$

Since the points P_1 , P_2 and P_3 are assumed to be points on the surface of the earth, c must be such that no singularity can result. A safe way is to determine c from the component of $\Delta q_3'^{(1,2)}$ perpendicular to the plane through P_1 , P_2 and P_3, hence parallel to e_{213}:

$$\begin{aligned}&\Delta q_3'^{(1,2)} + e_{231}\Delta q_3'^{(1,2)}e_{213}^{-1} = \\ &= \Delta q_3^{(1,2)} + e_{213}\Delta q_3^{(1,2)}e_{213}^{-1} - \left(Q_{213} - Q_{213}^{T}\right)c\end{aligned}$$

After substitution into $\Delta q_k'^{(1,2)}$ this results in:

$$\Delta q_k'^{(1,2)} - \frac{1}{2}\left(Q_{21k} - Q_{21k}^T\right)\left(Q_{213} - Q_{213}^T\right)^{-1}\left(\Delta q_3'^{(1,2)} + e_{213}\,\Delta q_3'^{(1,2)}\,e_{213}^{-1}\right) =$$

$$= \Delta q_k^{(1,2)} - \frac{1}{2}\left(Q_{21k} - Q_{21k}^T\right)\left(Q_{213} - Q_{213}^T\right)^{-1}\left(\Delta q_3^{(1,2)} + e_{213}\,\Delta q_3^{(1,2)}\,e_{213}^{-1}\right) =$$

$$\underset{\text{put}}{=} \Delta q_k^{(1,2;3)}$$

by which the form element wanted has been found, which is invariant with respect to a differential similarity transformation. One may also refer to this as an S-transformation to the $S_{1,2;3}$ -system.

Now we have for P_3 :

$$\Delta q_3^{(1,2;3)} = \frac{1}{2}\left(\Delta q_3^{(1,2)} - e_{213}\,\Delta q_3^{(1,2)}\,e_{213}^{-1}\right)$$

$$= \left(\text{component } \Delta q_3^{(1,2)} \perp e_{213}\right)$$

or:

$$\left(\text{component } \Delta q_3^{(1,2)} \parallel e_{213}\right) = 0$$

Because $\Delta q_1^{(1,2)} = \Delta q_2^{(1,2)} = 0$, we have for P_1 and P_2 :

$$\Delta q_1^{(1,2;3)} = \Delta q_2^{(1,2;3)} = 0$$

The S-transformation consequently is a form-preserving connection to:

$$q_1^{(\text{appr})},\ q_2^{(\text{appr})} \text{ and } \left(q_3^{(\text{appr})} + e_{213}\,q_3^{(\text{appr})}\,e_{213}^{-1}\right)$$

For a comparison, one can refer back to the discussion on the X, Y, Z -frame in Section 3.1.

3.2.2.2

In order to establish a better connection between the three-dimensional and the two-dimensional S-coordinates, the formulas obtained are written in a different way.
With:

$$Q_{21k} - Q_{21k}^{T} = 2\,\frac{s_{1k}}{s_{12}}\,\sin\alpha_{21k} \cdot e_{21k}$$

one obtains:

$$\Delta q_k^{(1,2;3)} = \Delta q_k^{(1,2)} - \frac{1}{2}\,\frac{s_{1k}\,\sin\alpha_{21k}}{s_{13}\,\sin\alpha_{213}}\,e_{21k}\,e_{213}^{-1}\left(\Delta q_3^{(1,2)} + e_{213}\,\Delta q_3^{(1,2)}\,e_{213}^{-1}\right)$$

In subsequent derivations repeated use is made of the fact that the following is valid for orthogonal unit vectors:

$$e_1 e_2 = -\,e_2 e_1 \quad , \quad \text{hence :} \quad e \cdot \Delta(e) = -\,\Delta(e) \cdot e$$

With:

$$Q_{21k} = q_{1k}\,q_{12}^{-1} \quad \text{and} \quad q_{1k}^{-1}\Delta q_{1k} = \Delta\Lambda_{1k} \quad , \quad \Delta\Lambda_{1k} - \Delta\Lambda_{12} = \Delta\Pi_{21k}$$

one obtains:

$$\Delta q_k^{(1,2)} = \Delta q_{1k} - \frac{1}{2}\left(Q_{21k}\,\Delta q_{12} + \Delta q_{12}\,Q_{21k}^{T}\right)$$

or:

$$\Delta q_k^{(1,2)} = \Delta q_{1k} - \frac{1}{2}\left(q_{1k}\,\Delta\Lambda_{12} + \Delta\Lambda_{12}^{T}\,q_{1k}\right)$$

with:

$$0 = \Delta q_{1k} - \frac{1}{2}\left(q_{1k}\Delta\Lambda_{1k} + \Delta\Lambda_{1k}^{T}\,q_{1k}\right)$$

results in:

$$\Delta q_k^{(1,2)} = \frac{1}{2}\left(q_{1k}\,\Delta\Pi_{21k} + \Delta\Pi_{21k}^{T}\,q_{1k}\right)$$

With:

$$\left\{\begin{aligned} \Delta\Pi_{21k} &= \Delta\left(\ln\frac{s_{1k}}{s_{12}}\right) - e_{21k}\,\Delta(\alpha_{21k}) + \sin\alpha_{21k}\,e_{1k}^{-1}\,\Delta(e_{21k})\,e_{12} \\ \Delta\Pi_{21k}^{T} &= \Delta\left(\ln\frac{s_{1k}}{s_{12}}\right) + e_{21k}\Delta(\alpha_{21k}) - \sin\alpha_{21k}\,e_{12}\,\Delta(e_{21k})\,e_{1k}^{-1} \end{aligned}\right.$$

this becomes:

3.

$$\boxed{\begin{aligned}\Delta q_k^{(1,2)} &= q_{1k}\left[\Delta\left(\ln\frac{s_{1k}}{s_{12}}\right) - e_{21k}\,\Delta(\alpha_{21k})\right] + \\ &+ \frac{1}{2}s_{1k}\sin\alpha_{21k}\left[\Delta(e_{21k})\,e_{12} - e_{12}\Delta(e_{21k})\right]\end{aligned}}$$

Making use of $e_{213}\,q_{13} = -\,q_{13}\,e_{213}$ and $e_{213}\Delta(e_{213})\,e_{12}\,e_{213}^{-1} = \Delta(e_{213})\,e_{12}$
this results in:

$$\Delta q_3^{(1,2)} + e_{213}\,\Delta q_3^{(1,2)}\,e_{213}^{-1} =$$

$$= q_{13}\left[\left[\Delta\left(\ln\frac{s_{13}}{s_{12}}\right) - e_{213}\Delta(\alpha_{213})\right] - \left[\Delta\left(\ln\frac{s_{13}}{s_{12}}\right) - e_{213}\Delta(\alpha_{213})\right]\right] +$$

$$+ \frac{1}{2}s_{13}\sin\alpha_{213}\left[\left[\Delta(e_{213})e_{12} - e_{12}\Delta(e_{213})\right] + \left[\Delta(e_{213})e_{12} - e_{12}\Delta(e_{213})\right]\right] =$$

$$= s_{13}\sin\alpha_{213}\left[\Delta(e_{213})e_{12} - e_{12}\Delta(e_{213})\right]$$

It follows that:

$$\begin{aligned}\Delta q_k^{(1,2;3)} &= q_{1k}\left[\Delta\left(\ln\frac{s_{1k}}{s_{12}}\right) - e_{21k}\Delta(\alpha_{21k})\right] + \\ &+ \frac{1}{2}s_{1k}\sin\alpha_{21k}\left[\left[\Delta(e_{21k}) - e_{21k}e_{213}^{-1}\Delta(e_{213})\right]e_{12} + \right. \\ &\left. - e_{12}\left[\Delta(e_{21k}) - e_{21k}e_{213}^{-1}\Delta(e_{213})\right]\right]\end{aligned}$$

With: $\quad e_{213}^{-1}\Delta(e_{213})e_{213}^{-1} = \Delta\left(e_{213}^{-1}\right)$

one obtains:

$$\Delta(e_{21k}) - e_{21k}\,e_{213}^{-1}\Delta(e_{213}) =$$

$$= \left[\Delta(e_{21k})e_{213}^{-1} - e_{21k}\,e_{213}^{-1}\Delta(e_{213})e_{213}^{-1}\right]e_{213} = \Delta\left(e_{21k}\,e_{213}^{-1}\right)e_{213}$$

Define the angle v_{3k} between e_{21k} and e_{213} by means of the quaternion with norm 1:

$$e_{21k}e_{213}^{-1} \underset{\text{put}}{=} p_{3k} = \cos\nu_{3k} + e_{12}\sin\nu_{3k}$$

Then:

$$\Delta\left(e_{21k}e_{213}^{-1}\right)e_{213} = e_{21k}e_{21k}^{-1}\ \Delta p_{3k}e_{213} = e_{21k}\,\Delta\Pi_{3k}$$

with: $$\Delta\Pi_{3k} = -\ e_{12}\Delta\left(\nu_{3k}\right) + \sin\nu_{3k}e_{21k}^{-1}\,\Delta\left(e_{12}\right)e_{213}\quad .$$

The second term in the right hand member of $\Delta q_k^{(1,2;3)}$ now becomes:

$$\frac{1}{2}s_{1k}\sin\alpha_{21k}e_{21k}\left[\Delta\Pi_{3k}e_{12} + e_{12}\Delta\Pi_{3k}\right] =$$

$$= \frac{1}{2}s_{1k}\ \sin\alpha_{21k}e_{21k}\left[\left[\Delta\left(\nu_{3k}\right) - \sin\nu_{3k}e_{21k}^{-1}\,\Delta\left(e_{12}\right)e_{12}e_{213}\right] + \right.$$

$$\left. + \left[\Delta\left(\nu_{3k}\right) - \sin\nu_{3k}\,e_{21k}^{-1}\,e_{12}\Delta\left(e_{12}\right)e_{213}\right]\right] = s_{1k}\ \sin\alpha_{21k}e_{21k}\Delta\left(\nu_{3k}\right)$$

Finally one gets:

$$\boxed{\begin{aligned}\Delta q_k^{(1,2;3)} &= q_{1k}\left[\Delta\left(\ln\frac{s_{1k}}{s_{12}}\right) - e_{21k}\Delta\left(\alpha_{21k}\right)\right] + \\ &\quad + s_{1k}\ \sin\alpha_{21k} - e_{21k}\Delta\left(\nu_{3k}\right)\end{aligned}}$$

With $\nu_{33} = 0$ we obtain:

$$\boxed{\Delta q_3^{(1,2;3)} = q_{13}\left[\Delta\left(\ln\frac{s_{13}}{s_{12}}\right) - e_{213}\Delta\left(\alpha_{213}\right)\right]}$$

Which result can also be obtained directly.

The first term in the right hand member is exactly the expression for the two-dimensional S-coordinate $\Delta z_k^{(1,2)}$ in the theory using complex numbers [Baarda 1973, 1981] if $e_{21k} = e_{21l}$ for all k and l. The second term is the addition for the third dimension.

It is clear that $\Delta q_k^{(1,2;3)}$ is determined by **three intrinsic quantities**:

$$\ln\frac{s_{1k}}{s_{12}} \quad , \quad \alpha_{21k} \quad \text{and} \quad \nu_{3k}$$

In a certain sense one may therefore call $\Delta q_k^{(1,2;3)}$ a form element as well.

3.2.3

Our focus of attention is the analysis of the covariance matrix of the variates $\Delta q_k^{(1,2;3)}$. It is remarkable that I had developed the relevant quaternion algebra already 25 years ago, but then I could not see through the formula system because I stuck too long to a pure quaternion theory, trying to follow a course similar to the one developed in the theory with complex numbers at the same time. It was not until the spring of 1991 that I suddenly saw the possibility of obtaining a practicable solution by an earlier transition to isomorphic matrices. Therefore we shall first reformulate the S-transformation in matrices, beginning with a summary of the theory.

If:

$$Q = d + ai + bj + ck$$
$$ii = jj = kk = -1 \quad , \quad ijk = -1$$

then the isomorphic matrix is:

$$(Q) = \begin{pmatrix} d & -a & -b & -c \\ a & d & -c & b \\ b & c & d & -a \\ c & -b & a & d \end{pmatrix} , \; (Q^T) = (Q)^T$$

(Q) is a skew symmetric matrix + d I, which in sequel will be called a skew$^+$ symmetric matrix. For $d = 0$ the quaternion Q becomes a vector, which consequently can be written as a skew symmetric matrix.

In considering the product $(Q\gamma)$ of two quaternions not all matrices have to be fully written out; the first columns of the matrices $(Q\gamma)$ and (γ) suffice, other columns provide no new information.

The sequence of multiplication can be changed as follows:

$$(\gamma)(Q) = (\bar{Q})(\gamma)$$
$$(\gamma)(Q^T) = (\bar{Q})^T(\gamma)$$

with:

$$(\bar{Q}) = \begin{pmatrix} d & -a & -b & -c \\ a & d & c & -b \\ b & -c & d & a \\ c & b & -a & d \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (Q)^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence:

$$(Q)(\gamma) + (\gamma)(Q^T) = (Q + \bar{Q}^T)(\gamma)$$

with:

$$\frac{1}{2}(Q + \bar{Q}^T) = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & -c & b \\ 0 & c & d & -a \\ 0 & -b & a & d \end{pmatrix} \underset{\text{put}}{=} (\hat{Q})$$

and:

$$(Q)(\gamma) + (\gamma)(Q) = (Q + \bar{Q})(\gamma)$$

with:

$$\frac{1}{2}(Q + \bar{Q}) = \begin{pmatrix} d & -a & -b & -c \\ a & d & 0 & 0 \\ b & 0 & d & 0 \\ c & 0 & 0 & d \end{pmatrix} \underset{\text{put}}{=} (\hat{\hat{Q}})$$

In general $\bar{Q}$ satisfies the same quaternion relations as Q, but $\bar{Q}$ and Q belong to different groups. As soon as they appear in combination, like in the present case, the group properties are lost. For example neither $(Q + \bar{Q})$ nor $(Q + \bar{Q}^T)$ is the matrix of a quaternion.

Now apply this to the S-transformation in section 3.2.2.1:

(I) $$\boxed{\begin{aligned} \left(\Delta q_k^{(1,2)}\right) &= (\Delta q_{1k}) - \left(\hat{Q}_{21k}\right)(\Delta q_{12}) \\ \left(\Delta q_l^{(1,2)}\right)^T &= (\Delta q_{1l})^T - (\Delta q_{12})^T\left(\hat{Q}_{21l}\right)^T \end{aligned}}$$

Because:

3.

$$\frac{1}{2}\left(Q_{213} - Q_{213}^{T}\right) = Ve\left\{Q_{213}\right\} \underset{\text{put}}{=} q_{213}$$

and hence:

$$e_{213}\,\Delta q_3^{(1,2)}\,e_{213}^{-1} = q_{213}\,\Delta q_3^{(1,2)}\,q_{213}^{-1}$$

one obtains:

$$\Delta q_k^{(1,2;3)} = \Delta q_k^{(1,2)} - \frac{1}{2} q_{21k}\left(q_{213}^{-1}\,\Delta q_3^{(1,2)} + \Delta q_3^{(1,2)}\,q_{213}^{-1}\right)$$

with:

$$q_{231}^{-1} = \frac{-q_{231}}{s_{213}^{2}} \quad , \quad s_{213}^{2} = N\left\{q_{231}\right\} = \sin^2\alpha_{213}\,N\left\{Q_{231}\right\}$$

Hence:

(II)

$$\boxed{\begin{aligned} \left(\Delta q_k^{(1,2;3)}\right) &= \Delta q_k^{(1,2)} + \frac{1}{s_{213}^{2}}\left(q_{21k}\right)\left(\hat{\hat{q}}_{213}\right)\left(\Delta q_3^{(1,2)}\right) \\ \left(\Delta q_l^{(1,2;3)}\right)^T &= \left(\Delta q_l^{(1,2)}\right)^T + \frac{1}{s_{213}^{2}}\left(\Delta q_3^{(1,2)}\right)^T\left(\hat{\hat{q}}_{213}\right)^T\left(q_{21l}\right)^T \end{aligned}}$$

Now we have a system of equations to which the law of propagation of variances can be applied.

Finally we investigate the conditions under which a covariance matrix will produce spherical standard hyperellipsoids for coordinates of each point and for the coordinate differences of each pair of points of a network.

Let now the symbols a and b successively two of the quantities w, x, y, z, with $b \neq a$ (a and b are now arbitrary symbols and not components of a quaternion).

Then the following must hold for each point P_k :

$$\overline{a_k\,,\,a_k} \underset{\text{put}}{=} d_k^2 \qquad , \qquad \overline{a_k\,,\,b_k} = 0$$

or:

$$\boxed{\overline{\begin{Bmatrix} w_k \\ x_k \\ y_k \\ z_k \end{Bmatrix} , \begin{Bmatrix} w_k \\ x_k \\ y_k \\ z_k \end{Bmatrix}^T} = d_k^2 \cdot \mathrm{I} \quad \text{, with I the 4 x 4 unit matrix}}$$

And for each pair of points P_k , P_l :

$$\overline{(a_l - a_k) , (a_l - a_k)} \underset{\text{put}}{=} 2d_{kl}^2 , \qquad d_{kl}^2 = d_{lk}^2$$

$$= d_l^2 + d_k^2 - 2\, \overline{a_l , a_k} \qquad , \qquad \overline{a_k , a_l} = \frac{d_k^2 + d_l^2}{2} - d_{kl}^2$$

$$\overline{(a_l - a_k) , (b_l - b_k)} = 0 \qquad , \qquad \overline{a_k , b_l} = \overline{-a_l , b_k}$$

or:

(III)
$$\boxed{\begin{array}{l} \overline{\begin{Bmatrix} w_k \\ x_k \\ y_k \\ z_k \end{Bmatrix} , \begin{Bmatrix} w_l \\ x_l \\ y_l \\ z_l \end{Bmatrix}^T} - \left(\dfrac{d_k^2 + d_l^2}{2} - d_{kl}^2 \right) \cdot \mathrm{I} \text{ , with } d_{kk}^2 = 0 \text{ ,} \\ \text{is a skew symmetric matrix.} \end{array}}$$

This condition implies the previous one for $l = k$.

Now introduce the notation $a_l - a_k = a_{kl}$, then:

3.

$$\overline{a_{ik}\,,\,a_{il}} = -\left(\overline{a_i\,,a_i} - \overline{a_k\,,a_l}\right) + \left(\overline{a_i\,,a_i} - \overline{a_k\,,a_i}\right) + \left(\overline{a_i\,,a_i} - \overline{a_i\,,a_l}\right) =$$

$$= -\left(d_i^2 - \frac{d_k^2 + d_l^2}{2} + d_{kl}^2\right) + \left(d_i^2 - \frac{d_k^2 + d_i^2}{2} + d_{ki}^2\right) +$$

$$+ \left(d_i^2 - \frac{d_i^2 + d_l^2}{2} + d_{il}^2\right) =$$

$$= -\,d_{kl}^2 + d_{ki}^2 + d_{il}^2$$

$$\overline{a_{ik}\,,\,b_{il}} = \overline{a_k\,,\,b_l} - \overline{a_k\,,\,b_i} - \overline{a_i\,,\,b_l} + \overline{a_i\,,\,b_i} =$$

$$= -\,\overline{a_l\,,\,b_k} + \overline{a_i\,,\,b_k} + \overline{a_l\,,\,b_i} + \overline{a_i\,,\,b_i}\ , \qquad \overline{a_i\,,\,b_i} = 0$$

$$= -\,\overline{a_{il}\,,\,b_{ik}}$$

hence:

(IV)
$$\boxed{\overline{\begin{Bmatrix} w_{ik} \\ x_{ik} \\ y_{ik} \\ z_{ik} \end{Bmatrix}, \begin{Bmatrix} w_{il} \\ x_{il} \\ y_{il} \\ z_{il} \end{Bmatrix}^T} - \left(-d_{kl}^2 + d_{ki}^2 + d_{il}^2\right) \cdot \mathrm{I} \quad \text{is a skew symmetric matrix.}}$$

Unfortunately (IV) only provides the possibility to establish a Criterion Matrix for w (actually for $\frac{W}{g_1}$, see Section 3.2.1). For the vector of coordinates x, y and z we are faced with products of skew symmetric matrices and these products are not in general skew symmetric. This necessitates a further assumption:

(V') $\qquad \overline{a_k\,,\,b_l} = 0 \quad , \quad$ hence $\overline{a_{ik}\,,\,b_{il}} = 0$

This is not a very hazardous assumption. It is more important that no assumptions need to be made for d_i^2 , d_k^2 and d_l^2 and certainly **not** the assumption that all these d^2-values must be equal. Now it follows from (IV) that:

(V")

$$\overline{\begin{Bmatrix} w_{ik} \\ x_{ik} \\ y_{ik} \\ z_{ik} \end{Bmatrix} , \begin{Bmatrix} w_{il} \\ x_{il} \\ y_{il} \\ z_{il} \end{Bmatrix}^T} = \left(-d_{kl}^2 + d_{ki}^2 + d_{il}^2\right) \cdot \mathrm{I}$$

$w = 0$ gives with $(q) = (0, x, y, z)^T$:

$$\left(q_{ik}, q_{il}^T\right) = \left(-d_{kl}^2 + d_{ki}^2 + d_{il}^2\right) \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix}$$

where now I is a 3 x 3 unit matrix.

This results in:

$$\left(d_{\bullet\bullet}^2\right) \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix} (\underset{\sim}{Q}) = d_{\bullet\bullet}^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d & -c & b \\ 0 & c & d & -a \\ 0 & -b & a & d \end{pmatrix}$$

wich for brevity will be referred to as a "3 x 3 skew$^+$ symmetric matrix".

The final assumption is the adoption of a function for d_{kl}^2. In [Baarda 1973, 1981], the most simple function was chosen:

(VI)

$$\boxed{d_{kl}^2 = c \cdot s_{kl}}$$

in which c is a constant, $s_{kl} = N^{1/2}\{q_{kl}\}$. This choice was supported by the results of research by E. Pinkwart and J.E. Alberda [2]. Other choices are possible.

3.2.4

Now the **construction** of **Criterion Matrices** can be taken up.

We start with the one for $w_k = \dfrac{W_k}{g_1}$.

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[2] See "Notes and References, Section 3.2.3"

$$\frac{W_k}{g_1} \simeq r_1 \frac{r_1}{r_k} \quad , \quad \Delta\left(\frac{W_k}{g_1}\right) = r_1 \frac{r_1}{r_k} \Delta\left(\ln \frac{W_k}{g_1}\right)$$

$$\Delta\left(\frac{W_k}{g_1}\right) - \frac{r_1}{r_k} \Delta\left(\frac{W_1}{g_1}\right) = r_1 \frac{r_1}{r_k} \Delta\left(\ln \frac{W_k}{W_1}\right)$$

or in spherical approximation, R being an average radius of the earth:

$$\Delta\left(\frac{W_{1k}}{g_1}\right) = R\Delta\left(\ln \frac{W_k}{W_1}\right) \quad , \qquad \text{cf. Section 3.2.1.}$$

With equation (V) in section 3.2.3 this results in:

(I)
$$\boxed{\overline{\ln \frac{W_k}{W_1} \, , \, \ln \frac{W_l}{W_1}} = \frac{1}{R}\left(-d_{kl}^2 + d_{k1}^2 + d_{1l}^2\right)}$$

Then, with (I) in Section 3.2.3:

$$\left(\overline{q_k^{(1,2)} \, , \, q_l^{(1,2)^T}}\right) = \left(\overline{q_{1k} \, , \, q_{1l}^T}\right) - \left(\overline{q_{1k} \, , \, q_{12}^T}\right)\left(\hat{Q}_{21l}^T\right) + \\ - \left(\hat{Q}_{21k}\right)\left(\overline{q_{12} \, , \, q_{1l}^T}\right) + \left(\hat{Q}_{21k}\right)\left(\overline{q_{12} \, , \, q_{12}^T}\right)\left(\hat{Q}_{21l}^T\right)$$

and with (V) in Section 3.2.3:

(II)
$$\boxed{\begin{aligned} \left(\overline{q_k^{(1,2)} \, , \, q_l^{(1,2)^T}}\right) = {} & \left(-d_{kl}^2 + d_{k1}^2 + d_{1l}^2\right)\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix} + \\ & - \left(-d_{k2}^2 + d_{k1}^2 + d_{12}^2\right)\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix}\left(\hat{Q}_{21l}^T\right) + \\ & - \left(-d_{2l}^2 + d_{21}^2 + d_{1l}^2\right)\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix}\left(\hat{Q}_{21k}\right) + \\ & + 2d_{12}^2\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix}\left(\hat{Q}_{21k}\right)\left(\hat{Q}_{21l}^T\right) \end{aligned}}$$

Finally, from (II) in Section 3.2.3 follows:

(III)

$$\overline{\left(q_k^{(1,2;3)}, q_l^{(1,2;3)^T}\right)} = \overline{\left(q_k^{(1,2)}, q_l^{(1,2)^T}\right)} + \frac{1}{s_{213}^2}\overline{\left(q_k^{(1,2)}, q_3^{(1,2)^T}\right)}\left(\hat{q}_{213}^T\right)\left(q_{21l}^T\right) + \frac{1}{s_{213}^2}\left(q_{21k}\right)\left(\hat{q}_{213}^T\right)\overline{\left(q_3^{(1,2)}, q_l^{(1,2)^T}\right)} + \frac{1}{s_{213}^4}\left(q_{21k}\right)\left(\hat{q}_{213}\right)\overline{\left(q_3^{(1,2)}, q_3^{(1,2)^T}\right)}\left(\hat{q}_{213}^T\right)\left(q_{21l}^T\right)$$

Interpretation:

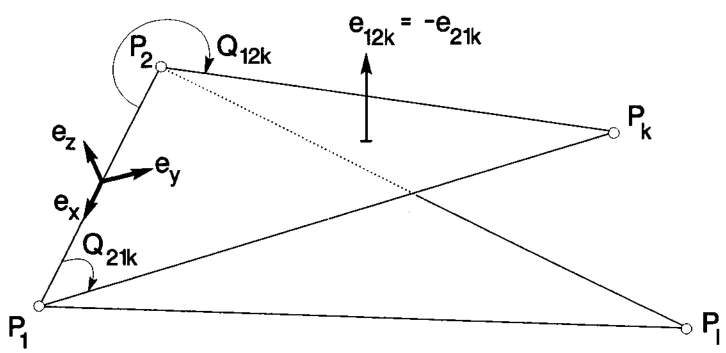
From (I):

$$\overline{\ln\frac{W_k}{W_1}, \ln\frac{W_k}{W_1}} = \frac{2}{R}d_{1k}^2 \quad , \quad \overline{\ln\frac{W_l}{W_1}, \ln\frac{W_l}{W_1}} = \frac{2}{R}d_{1l}^2$$

i.e. the form elements $\ln\frac{W_k}{W_1}$ satisfy (III), Section 3.2.3.

In (II) the first terms in the right hand member are 3 x 3 skew$^+$ symmetric matrices, but the character of the fourth term is not immediately clear and will have to be investigated. The question is if the left hand member is a skew$^+$ symmetric matrix and fulfils (III), Section 3.2.3.
At this stage no statement about (III) can be made.

For the analysis of (II) we choose a special x, y and z system whose x-axis is parallel to q_{21}.



Then:

3.

$$q_{21} = s_{21}e_x = s_{12}e_x \;, \qquad q_{21}^{-1} = -\frac{e_x}{s_{12}}$$

$$q_{12} = -s_{12}e_x \;, \qquad q_{12}^{-1} = +\frac{e_x}{s_{12}}$$

$$\begin{array}{l|l} q_{2k} = x_{2k}e_x + y_{2k}e_y + z_{2k}e_z & q_{1k} = x_{1k}e_x + y_{1k}e_y + z_{1k}e_z \\ q_{2l} = x_{2l}e_x + y_{2l}e_y + z_{2l}e_z & q_{1l} = x_{1l}e_z + y_{1l}e_y + z_{1l}e_z \end{array}$$

$$Q_{12k} = q_{2k}q_{21}^{-1} = \frac{1}{s_{12}}\left(x_{2k} - z_{2k}e_y + y_{2k}e_z\right)$$

$$Q_{21k} = q_{1k}q_{12}^{-1} = \frac{1}{s_{12}}\left(-x_{1k} + z_{1k}e_y - y_{1k}e_z\right)$$

Check:

$$Q_{12k} + Q_{21k} = \frac{1}{s_{12}}\left(x_{21} - z_{21}e_y + y_{21}e_z\right) = 1$$

$$\left(\hat{Q}_{21k}\right) = \frac{1}{s_{12}}\begin{pmatrix} -x_{1k} & 0 & 0 & 0 \\ 0 & -x_{1k} & y_{1k} & z_{1k} \\ 0 & -y_{1k} & -x_{1k} & 0 \\ 0 & -z_{1k} & 0 & -x_{1k} \end{pmatrix}, \quad \left(\hat{Q}_{21l}^{T}\right) = \frac{1}{s_{12}}\begin{pmatrix} -x_{1l} & 0 & 0 & 0 \\ 0 & -x_{1l} & -y_{1l} & -z_{1l} \\ 0 & y_{1l} & -x_{1l} & 0 \\ 0 & z_{1l} & 0 & -x_{1l} \end{pmatrix}$$

$$\left(\hat{Q}_{21k}\right)\left(\hat{Q}_{21l}^{T}\right) =$$

$$= \frac{1}{s_{12}^2}\begin{pmatrix} [x_{1k}x_{1l}] & 0 & 0 & 0 \\ 0 & [x_{1k}x_{1l} + y_{1k}y_{1l} + z_{1k}z_{1l}] & [x_{1k}y_{1l} - y_{1k}x_{1l}] & [x_{1k}z_{1l} - z_{1k}x_{1l}] \\ 0 & [y_{1k}x_{1l} - x_{1k}y_{1l}] & [y_{1k}y_{1l} + x_{1k}x_{1l}] & [y_{1k}z_{1l}] \\ 0 & [z_{1k}x_{1l} - x_{1k}z_{1l}] & [z_{1k}y_{1l}] & [z_{1k}z_{1l} + x_{1k}x_{1l}] \end{pmatrix}$$

Now for the (3, 4)- and (4,3)-elements the following must be valid:

$$y_{1k}z_{1l} = -z_{1k}y_{1l} \;, \text{ or with } y_{1j}\,,\, y_{1k}\,,\, y_{1l} \neq 0:$$

$$\frac{z_{1l}}{y_{1l}} = -\frac{z_{1k}}{y_{1k}}\;, \qquad \text{hence also } \frac{z_{1l}}{y_{1l}} = -\frac{z_{1j}}{y_{1j}}\;,$$

$$\text{hence } \frac{z_{1k}}{y_{1k}} = \frac{z_{1j}}{y_{1j}} \text{ whereas this should be : } \frac{z_{1k}}{y_{1k}} = -\frac{z_{1j}}{y_{1j}}\,.$$

This contradiction can only be solved if for all points P_k we have: $z_{ik} = 0$, and because e_y and e_z are still free to rotate about e_x this means that all points P_k must lie in one plane through P_1 and P_2.

Choose $e_z \perp$ plane $P_1\ P_2\ P_k$, then (II) becomes, with I a 2 x 2 unit matrix:

$$(q) = (0\,, x\,, y\,, 0^{\,T}$$

$$\left(q_k^{(1,2)}\,,\, q_l^{(1,2)^T}\right) = \left(-d_{kl}^2 + d_{k1}^2 + d_{1l}^2\right)\begin{pmatrix}0&0&0\\0&\mathrm{I}&0\\0&0&0\end{pmatrix} + $$
$$- \left(-d_{k2}^2 + d_{k1}^2 + d_{12}^2\right)\begin{pmatrix}0&0&0\\0&\mathrm{I}&0\\0&0&0\end{pmatrix}\left(\hat{Q}_{21l}^{T}\right) + $$
$$- \left(-d_{2l}^2 + d_{21}^2 + d_{1l}^2\right)\begin{pmatrix}0&0&0\\0&\mathrm{I}&0\\0&0&0\end{pmatrix}\left(\hat{Q}_{21k}\right) + $$
$$+ 2d_{12}^2\begin{pmatrix}0&0&0\\0&\mathrm{I}&0\\0&0&0\end{pmatrix}\left(\hat{Q}_{21k}\right)\left(\hat{Q}_{21l}^{T}\right)$$

with for $e_x \,//\, q_{21}$:

$$\left(\hat{Q}_{21k}\right)\left(\hat{Q}_{21l}^{T}\right) = \frac{1}{s_{12}^2}\begin{pmatrix}[x_{1k}x_{1l}] & 0 & 0 & 0\\ 0 & [x_{1k}x_{1l}+y_{1k}y_{1l}] & [x_{1k}y_{1l}-y_{1k}x_{1l}] & 0\\ 0 & [y_{1k}x_{1l}-x_{1k}y_{1l}] & [y_{1k}y_{1l}+x_{1k}x_{1l}] & 0\\ 0 & 0 & 0 & [x_{1k}x_{1l}]\end{pmatrix}$$

This is the Criterion Matrix for the coordinates of points in the plane, as derived in [Baarda 1973, 1981] by means of complex numbers.
In the plane we consequently have the nice property of the Criterion Matrix that the circularity of point- and relative standard ellipses is invariant with respect to an S-transformation. The derivation shows that this is **not** the case in the three-dimensional situation.

To make sure, (III) will be examined.
In addition to $e_x \,//\, q_{21}$, the z-axis is chosen such that $e_z \,//\, e_{123}$, so that $z_{13} = z_{23} = 0$. One obtains:

3.

$$Q_{21k} = \frac{1}{s_{12}} \left(-x_{1k} + z_{1k} e_y - y_{1k} e_z\right)$$

$$\left(q_{21k}\right) = \frac{1}{s_{12}} \begin{pmatrix} 0 & 0 & -z_{1k} & y_{1k} \\ 0 & 0 & y_{1k} & z_{1k} \\ z_{1k} & -y_{1k} & 0 & 0 \\ -y_{1k} & -z_{1k} & 0 & 0 \end{pmatrix}$$

$$Q_{213} = \frac{1}{s_{12}} \left(-x_{13} - y_{13} e_z\right)$$

$$\left(\hat{q}_{213}\right) = \frac{1}{s_{12}} \begin{pmatrix} 0 & 0 & 0 & y_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -y_{13} & 0 & 0 & 0 \end{pmatrix}$$

$$s_{213}^2 = \left(\frac{y_{13}}{s_{12}}\right)^2$$

and hence (II), Section 3.2.3:

$$\left(\Delta q_k^{(1,2;3)}\right) = \left(\Delta q_k^{(1,3)}\right) + \frac{1}{y_{13}} \begin{pmatrix} 0 \\ 0 \\ z_{1k} \\ -y_{1k} \end{pmatrix} \left(\Delta z_3^{(1,2)}\right)$$

Using this, (III) becomes:

$$\left(\overline{q_k^{(1,2;3)}\,,\, q_l^{(1,2;3)^T}}\right) = \left(\overline{q_k^{(1,2)}\,,\, q_l^{(1,2)^T}}\right) +$$

$$+ \frac{1}{y_{13}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \left[z_{1l}\,\overline{x_k\,,z_3}\right] & \left[-y_{1l}\,\overline{x_k\,,z_3}\right] \\ 0 & \left[z_{1k}\,\overline{z_3\,,x_l}\right] & \left[z_{1l}\,\overline{y_k\,,z_3} + z_{1k}\,\overline{z_3\,,y_l}\right] & \left[-y_{1l}\,\overline{y_k\,,z_3} + z_{1k}\,\overline{z_3\,,z_l}\right] \\ 0 & \left[-y_{1k}\,\overline{z_3\,,x_l}\right] & \left[z_{1l}\,\overline{z_k\,,z_3} - y_{1k}\,\overline{z_3\,,y_l}\right] & \left[-y_{1l}\,\overline{y_k\,,z_3} - y_{1k}\,\overline{z_3\,,z_l}\right] \end{pmatrix}^{(1,2)} +$$

$$+\ \frac{1}{y_{13}^{2}}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & [z_{1k}z_{1l}] & [-z_{1k}y_{1l}]\\ 0 & 0 & [-y_{1k}z_{1l}] & [y_{1k}y_{1l}]\end{pmatrix}\overline{z_3^{(1,2)}\ ,\ z_3^{(1,2)}}$$

Assume that $\left(\overline{q_k^{(1,2)}\ ,\ q_l^{(1,2)^T}}\right)$ is a skew$^+$ symmetric matrix. As before, it follows from the third term that for all points P_k: $z_{1k} = 0$.
Then the remaining elements of the matrix in the second term can only comply if for all points P_k:

$$\overline{\begin{pmatrix}x_k^{(1,2)}\\ y_k^{(1,2)}\end{pmatrix},\ z_3^{(1,2)}} = 0$$

This means that the variate $\underline{\Delta z_3^{(1,2)}}$ has no stochasticity:

$$\underline{\Delta z_3^{(1,2)}} = 0$$

But all points of the network can be considered as being P_3, so that the conclusion is that even the **variance** of point coordinates in the direction perpendicular to the plane of P_1, P_2, P_3 is detrimental to the invariance of spherical standard ellipsoids with respect to an S-transformation.

In itself this lack of invariance in a Criterion Matrix is not a serious matter, because computed covariance matrices of networks will probably present the same picture. However, in applications the interpretation will be more complicated.

4.

4.1

Now consider the real earth. Having an eccentricity of $O\ (10^{-3})$, it is almost spherical, and this motivates to a rather rough development of formulas which clearly illustrate the principle of the ideas of my 1979 publication. These formulas lead to further conclusions without much mathematical ballast. Here the proper use of the coordinate frame is extremely important, which must be reflected in the notation.

For the polar coordinates of a terrain point P_1 in our X, Y, Z frame with centre P_M we introduce the radial distance, the latitude and the longitude:

$$r_1\ ,\ \varphi_1\ ,\ \lambda_1$$

and in the parallel coordinate frame with centre P_C :

$$r_{C1}\ ,\ \varphi_{C1}\ ,\ \lambda_{C1}$$

The gravitational potential V is written in condensed form in two different ways with a view to later use (μ is the mass of the earth, R an average radius of the earth):

$$V_k = \frac{\mu}{r_{Ck}} \left[1 + \sum_{n=2}^{\infty} \left(\frac{R}{r_{Ck}} \right)^n Y_k^{(n)} \right]$$

$$= \frac{\mu}{r_{Ck}} \left(1 + \sum_{n=2}^{\infty} B_k^{(n)} \right) = \frac{\mu}{r_{Ck}} \left(1 + \varepsilon V_k \right)$$

W being the gravity potential and Ψ the centrifugal potential, we obtain:

$$W_k = V_k + \Psi_k = V_k \left(1 + \frac{\Psi_k}{V_k} \right) = V_k \left(1 + \varepsilon \Psi_k \right)$$

or:

$$W_k \simeq \frac{\mu}{r_{Ck}} \left(1 + \varepsilon V_k + \varepsilon \Psi_k \right) = \frac{\mu}{r_{Ck}} \left(1 + \varepsilon W_k \right)$$

Analogous:

$$- \frac{\delta W_k}{\delta r_{Ck}} \simeq g_k = \frac{\mu}{r_{Ck}^2} \left(1 + \varepsilon g_k \right)$$

in which always:

$$\varepsilon A = O\left(10^{-3}\right)$$

We now change over to dimensionless quantities having the same order of magnitude viz. 1. Taking again P_1 as the datum point:

$$\frac{W_k}{r_{C1}g_1} = \frac{\mu}{r_{C1}^2 g_1}\frac{r_{C1}}{r_{Ck}}\left(1 + \varepsilon W_k\right) \quad , \quad \frac{W_1}{r_{C1}g_1} = \frac{\mu}{r_{C1}^2 g_1}\left(1 + \varepsilon W_1\right)$$

$$\frac{r_{Ck}g_k}{r_{C1}g_1} = \frac{\mu}{r_{C1}^2 g_1}\frac{r_{C1}}{r_{Ck}}\left(1 + \varepsilon g_k\right) \quad , \quad \frac{r_{C1}g_1}{r_{C1}g_1} = 1 = \frac{\mu}{r_{C1}^2 g_1}\left(1 + \varepsilon g_1\right)$$

The course followed now is aimed at finding for these relationships a uniform way to introduce measured quantities in the form of potential differences (or possibly -ratios), gravity ratios (possibly regionally derived from gravity differences), and length ratios (as known from geometric geodesy). Another aim is the elimination of the nuisance quantity μ. Since the right hand members of the relationships are in fact spherical harmonics series, it is important to introduce differences of functions so that only differences of these series occur.

Denoting the quantities in the left hand members of the relationships by A, all aims are attained by the introduction of the new quantity:

$$A_k - \frac{r_{C1}}{r_{Ck}} A_1$$

By ignoring $(\varepsilon A)^2 = O\left(10^{-6}\right)$, so that:

$$\ln\left(1 + \varepsilon A\right) = \varepsilon A \quad , \quad \left(1 + \varepsilon A\right)^{-1} = 1 - \varepsilon A$$

one finds:

$$\frac{\mu}{r_{C1}^2 g_1} = 1 - \varepsilon g_1$$

$$\frac{W_k}{r_{C1}g_1} = \frac{r_{C1}}{r_{Ck}}\left(1 + \varepsilon W_k - \varepsilon g_1\right) \quad , \quad \frac{W_1}{r_{C1}g_1} = \left(1 + \varepsilon W_1 - \varepsilon g_1\right)$$

$$\frac{r_{Ck}g_k}{r_{C1}g_1} = \frac{r_{C1}}{r_{Ck}}\left(1 + \varepsilon g_k - \varepsilon g_1\right) \quad , \quad \frac{r_{C1}g_1}{r_{C1}g_1} = 1$$

so that the new quantities become:

$$\frac{r_{C1}}{r_{Ck}}\left(\varepsilon W_k - \varepsilon W_1\right) = \frac{W_k}{r_{C1}g_1} - \frac{r_{C1}}{r_{Ck}}\frac{W_1}{r_{C1}g_1} =$$

$$= \frac{r_{C1}}{r_{Ck}} \left(\frac{r_{Ck} W_k}{r_{C1}^2 g_1} - \frac{r_{C1} W_1}{r_{C1}^2 g_1} \right) =$$

$$= \frac{r_{C1}}{r_{Ck}} \left(\frac{r_{Ck} W_k}{r_{C1} W_1} - 1 \right) =$$

$$= \frac{r_{C1}}{r_{Ck}} \left(\ln \frac{W_k}{W_1} + \ln \frac{r_{Ck}}{r_{C1}} \right)$$

or also:

$$= \frac{r_{C1}}{r_{Ck}} \left(\frac{W_k - W_1}{r_{C1} g_1} \left(1 + \frac{W_1}{r_{C1} g_1} - 1 \right)^{-1} + \ln \frac{r_{Ck}}{r_{C1}} \right) =$$

$$= \frac{r_{C1}}{r_{Ck}} \left(\frac{W_k - W_1}{r_{C1} g_1} + \ln \frac{r_{Ck}}{r_{C1}} + \frac{r_{Ck} - r_{C1}}{r_{Ck}} \left(\frac{W_1}{r_{C1} g_1} - 1 \right) \right)$$

$$\frac{r_{C1}}{r_{Ck}} \left(\varepsilon g_k - \varepsilon g_1 \right) = \frac{r_{Ck} g_k}{r_{C1} g_1} - \frac{r_{C1}}{r_{Ck}} \frac{r_{C1} g_1}{r_{C1} g_1} =$$

$$= \frac{r_{C1}}{r_{Ck}} \left(\frac{r_{Ck}^2 g_k}{r_{C1}^2 g_1} - 1 \right) =$$

$$= \frac{r_{C1}}{r_{Ck}} \left(\ln \frac{g_k}{g_1} + 2 \ln \frac{r_{Ck}}{r_{C1}} \right)$$

These mathematical relationships will now be interpreted as relationships between measured quantities or variates. One recognizes the length ratio $\frac{r_{Ck}}{r_{C1}}$, the potential ratio $\frac{W_k}{W_1}$ or the potential difference $(W_k - W_1)$ from levelling, and the gravity ratio $\frac{g_k}{g_1}$.

But these relationships are not very useful if no value is known for r_{C1} , g_1 and W_1 . Of these, only two can be chosen indepently: r_{C1} is a geometrical quantity which can be derived from the geometrical S-system X, Y, Z, although in this system the origin P_M will not coincide with the still unknown position of P_C . Hence the only thing one can do is to take for r_{C1} the distance $\overline{P_M P_1}$ in the X, Y, Z-system. If P_1 is a datum point of the X, Y, Z-system, then its coordinates can be chosen (of course as well as possible), from which will follow $\overline{P_M P_1} = r_1$. Then r_1 actually is part of a consistent choice of approximate values in the X, Y, Z-system, so that the following notation is chosen:

$$\{r_{C1}\}^{\text{approx}} = r_1 \quad , \quad \text{hence also } \{r_{Ck}\}^{\text{approx}} = r_k \quad , \quad \text{etc.}$$

By now choosing a value as good as possible for g_1 too, one in fact establishes a coupling between geometric geodesy and terrestrial physical geodesy by extending the geometric S-system to include g_1 .

Now: $$\frac{r_{Ck} - r_{C1}}{r_{Ck}} = O\left(10^{-3}\right) \quad , \quad \frac{W_1}{r_{C1} g_1} = O\left(10^{-3}\right)$$

so that in their product one can certainly replace r_{Ck} by r_k and r_{C1} by r_1. Considering the limits chosen for neglect, one might even ignore the whole product, but then the transformation to a different datum point (S-transformation) would be less elegant to execute. One obtains:

$$\ln \frac{W_k}{W_1} + \ln \frac{r_{Ck}}{r_{C1}} =$$

$$= \frac{W_k - W_1}{r_1 g_1} + \frac{r_k - r_1}{r_k}\left(\frac{W_1}{r_1 g_1} - 1\right) + \ln \frac{r_{Ck}}{r_{C1}} = \varepsilon W_k - \varepsilon W_1$$

$$\frac{W_1}{r_1 g_1} = 1 + \varepsilon W_1 - \varepsilon g_1$$

$$\ln \frac{g_k}{g_1} + 2 \ln \frac{r_{Ck}}{r_{C1}} = \varepsilon g_k - \varepsilon g_1$$

$$\frac{\mu}{r_1^2 g_1} = 1 - \varepsilon g_1$$

It is curious that the "constants" W_1 and μ are largely determined by the choice of the S-system (in a more complete theory complemented by transforming the respective right hand members into well-known integral formulas, as re-written in [Baarda 1979], Section 4.4).

The chosen limit of neglect would be acceptable if the sharpness of definition (precision **and** reliability) for the measured quantities:

$$\ln \frac{W_K}{W_1} \quad , \quad \ln \frac{g_k}{g_1} \quad \text{and} \quad \ln \frac{r_{Ck}}{r_{C1}}$$

has the order of magnitude 10^{-5} to 10^{-6} . If this order of magnitude is decreased by new techniques of measurement to 10^{-7} or 10^{-8} , so that adjustments to measured values are in the order of 10^{-7} , then one will have to change over to difference equations. The model of relationships for computations will then have to be refined to the same sharpness of definition, with a set of approximate values again satisfying this model to the same sharpness - a consistent system of approximate values - and chosen such that in our case, for example:

$$\Delta(\varepsilon A) = \varepsilon A - (\varepsilon A)^{\text{approx}} = O\,(10^{-5})\,,$$

a value on the safe side in view of the estimate of $\overline{P_M P_C}$ in Section 3.

For our approach the relationships then become:

$$\Delta\left(\ln \frac{W_k}{W_1}\right) + \Delta\left(\ln \frac{r_{Ck}}{r_{C1}}\right) =$$

$$= \Delta\left(\frac{W_k - W_1}{r_1 g_1}\right) + \frac{r_k - r_1}{r_k}\,\Delta\left(\frac{W_1}{r_1 g_1}\right) + \Delta\left(\ln \frac{r_{Ck}}{r_{C1}}\right) =$$

$$= \Delta\left(\varepsilon W_k\right) - \Delta\left(\varepsilon W_1\right)$$

$$\Delta\left(\ln \frac{g_k}{g_1}\right) + 2\Delta\left(\frac{r_{Ck}}{r_{C1}}\right) = \Delta\left(\varepsilon g_k\right) - \Delta\left(\varepsilon g_1\right)$$

and hence:

$$-\,2\Delta\left(\ln \frac{W_k}{W_1}\right) + \Delta\left(\ln \frac{g_k}{g_1}\right) = \left[-\,2\Delta\left(\varepsilon W_k\right) + \Delta\left(\varepsilon g_k\right)\right] + $$

$$-\left[-\,2\Delta\left(\varepsilon W_1\right) + \Delta\left(\varepsilon g_1\right)\right]$$

In the coefficients of difference quantities, one may put $r_k/r_1 = 1$, which means that the earth may be considered as a sphere. Therefore the term:

$$\frac{r_k - r_1}{r_k}\,\Delta\left(\frac{W_1}{r_1 g_1}\right) = O\,(10^{-3})\;O\,(10^{-5}) = O\,(10^{-8})$$

will usually be negligible as well.

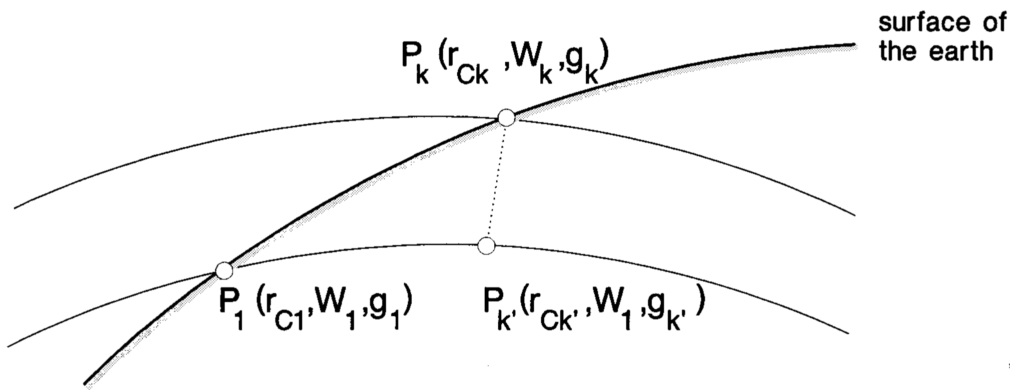
In all right hand members of these relationships the influence of the centrifugal potential practically vanishes so that almost without approximation we have for these members:

$$\Delta\left(\varepsilon W_k\right) - \Delta\left(\varepsilon W_1\right) = \sum_{n=2}^{\infty} \left[\left(\frac{R}{r_k}\right)^n \Delta Y_k^{(n)} - \left(\frac{R}{r_1}\right)^n \Delta Y_1^{(n)}\right]$$

$$\Delta\left(\varepsilon g_k\right) - \Delta\left(\varepsilon g_1\right) = \sum_{n=2}^{\infty} (n+1) \left[\left(\frac{R}{r_k}\right)^n \Delta Y_k^{(n)} - \left(\frac{R}{r_1}\right)^n \Delta Y_1^{(n)}\right]$$

$$\left[-2\Delta(\varepsilon W_k) + \Delta(\varepsilon g_k)\right] - \left[-2\Delta(\varepsilon W_1) + \Delta(\varepsilon g_1)\right] =$$

$$= \sum_{n=2}^{\infty} (n-1)\left[\left(\frac{R}{r_k}\right)^n \Delta Y_k^{(n)} - \left(\frac{R}{r_1}\right)^n \Delta Y_1^{(n)}\right]$$



Now we "forget" for a moment the physical matter which is present, and continue the equipotential surface through P_1 into the earth, choosing on this surface the fictitious point P_k on the plumb line of P_k. By combining our "forgetting" with some more sloppiness we arrive at the well-known interpretations of the left hand members of our relationships:

$$\Delta\left(\ln\frac{W_k}{W_1}\right) + \Delta\left(\ln\frac{r_{Ck}}{r_{C1}}\right) = \Delta\left(\ln\frac{r_{Ck}}{r_{C1}}\frac{W_k}{W_1}\right) \simeq \Delta\left(\ln\frac{r_{Ck}}{r_{C1}}\frac{r_{Ck'}}{r_{Ck}}\right) = \Delta\left(\ln\frac{r_{Ck'}}{r_{C1}}\right)$$

or a "free-air reduction" of $\Delta\left(\ln\dfrac{r_{Ck}}{r_{C1}}\right)$ to the geoid through P_1. If one chooses as the model for approximate values the well-known ellipsoidal model from physical geodesy, N being the height of the geoid above this ellipsoid, then follows:

$$\Delta\left(\ln\frac{r_{Ck'}}{r_{C1}}\right) = \Delta\left(\ln\left(1 + \frac{r_{Ck'} - r_{C1}}{r_{C1}}\right)\right) = \frac{\Delta(r_{Ck'} - r_{C1})}{r_{C1}} = \frac{N_{k'} - N_1}{r_{C1}}$$

Analogous:

$$-2\Delta\left(\ln\frac{W_k}{W_1}\right) + \Delta\left(\ln\frac{g_k}{g_1}\right) = \Delta\left(\ln\frac{g_k}{g_1}\left(\frac{W_1}{W_k}\right)^2\right) =$$

$$\simeq \Delta\left(\ln \frac{g_k}{g_1}\left(\frac{r_{Ck}}{r_{Ck'}}\right)^2\right) \simeq \Delta\left(\ln \frac{g_k}{g_1}\,\frac{g_{k'}}{g_k}\right) = \Delta\left(\ln \frac{g_k'}{g_1}\right) \simeq \frac{\Delta g_{k'} - \Delta g_1}{g_1}$$

or, a "free air reduction" to the geoid through P_1 .

Therefore:

$$\Delta\left(\ln \frac{g_k}{g_1}\right) + 2\Delta\left(\ln \frac{r_{Ck}}{r_{C1}}\right) =$$

$$= \left\{\Delta\left(\ln \frac{g_k}{g_1}\right) - 2\Delta\left(\ln \frac{W_k}{W_1}\right)\right\} + 2\left\{\Delta\left(\ln \frac{r_{Ck}}{r_{C1}}\right) + \Delta\left(\ln \frac{W_k}{W_1}\right)\right\} =$$

$$= \Delta\left(\ln \frac{g_{k'}}{g_1}\right) + 2\Delta\left(\ln \frac{r_{Ck'}}{r_{C1}}\right)$$

One recognizes the well-known anomaly-quantities from physical geodesy.

However it must be realized that this unnecessary and improper interpretation of better defined mixed quantities can never be a good basis for a model theory; it can only lead to confusing discussions.
But if, along with this interpretation, we also reject the fiction of a geoid, what to do with sea-topography? To answer this question, reverse the line of thought. For points P_1 and P_k at sea level:

$$\frac{\Delta(r_{Ck} - r_{Ck'})}{r_{Ck'}} \simeq -\Delta\left(\ln \frac{r_{Ck'}}{r_{Ck}}\right) \simeq -\Delta\left(\ln \frac{W_k}{W_1}\right)$$

with $W_k^{(\mathrm{appr})} = W_1^{(\mathrm{appr})}$ one obtains:

$$\text{relative sea-topography } (r_{Ck} - r_{Ck'}) \simeq -r_k \ln \frac{W_k}{W_1}.$$

This means that it suffices to determine potential ratios (or-differences) of points on the surface of the sea.

In the approach sketched, the important aspect is the inseparable relation between quantities from physical and geometric geodesy. This aspect is even enhanced if all components of the gravity vector and of the gravity gradient vector are included in the analysis, the directions of the vectors being described in the geometric sub-system. This explains why only one extra piece of datum information, viz. g_1, has to be joined to the geometric S-system.

If one also considers the data obtained by satellite methods, as well as the results of computations by various integral formulas such as Stokes's, than the picture of a four-dimensional geodesy appears (of course at a certain epoch in view of the movements of the earth's surface), represented by the quaternion in P_k:

$$\frac{W_k}{g_1} + X_k e_X + Y_k e_Y + Z_k e_Z$$

with components of the same order of magnitude, viz. R.
This might have consequences for a more general choice of an S-system [1]. As it is now, the S-transformation is actually decomposed into a transformation of the scalar part and a transformation of the vector part, the latter also being directive for the construction of a criterion matrix (Section 3.2).

Another very important point is that the choice of the dimensionless quantities practically eliminates, on one hand, the influence of poorly estimable "constants" μ and W_1, and on the other hand, the influence of instrumental units of length and time (of course supposing that the procedure of measurement is aimed at this elimination). This choice, which is not necessarily uniquely determined for a certain field of study, therefore seems to be satisfactory, Finally, reference can be made to further considerations in Chapter 2 of [Baarda 1979].

4.2

In order to elucidate some more aspects of the contents of the previous section, and anticipating Section 5, some consequences of the integral formulas of Stokes and Hotine (without "correction terms") will now be considered. For details, see Chapter 4 of [Baarda 1979].

Use will be made of the elegant line of thought followed in [R. Rummel and P. Teunissen - Height Datum Definition, Height Datum Connection and the Role of the Geodetic Boundary Value Problem - Bull. Géod. **62** (1988) pp. 477-498].

For our purpose we choose the notation of ratios of potential values, with $\frac{r_k}{r_1} = 1$ in the coefficients of the difference equations.

Putting:

$$Y_k^{(n)} = c^{(n)} Y_k'^{(n)} \quad , \quad \Delta Y_k^{(n)} = \Delta c^{(n)} \cdot Y_k'^{(n)}$$

[1] Perhaps by using a four-dimensional simularity transformation instead of a three-dimensional one. In this respect the idea of Grafarend to use a 10 (instead of 7) parameter-datum transformation in 3-D looks more inviting, but I cannot yet find a place for the special conformal part (3 parameters) in my model reasoning. See the paper to be published in the Zeitschrift für Vermessungswesen: E.W. Grafarend, G. Kampmann- C_{10} (B): The ten parameter conformal group as a datum transformation in three-dimensional Euclidean space (communicated by letter of May 24, 1994)

for "Stokes":

$$S_{1,k;j}^{(n-1)} = \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\left(Y_k'^{(n)} - Y_1'^{(n)}\right)Y_j'^{(n)}$$

for "Hotine":

$$S_{1,k;j}^{(n+1)} = \sum_{n=1}^{\infty} \frac{2n+1}{n+1}\left(Y_k'^{(n)} - Y_1'^{(n)}\right)Y_j'^{(n)}$$

then three difference equations are obtained, each with the same left hand member:

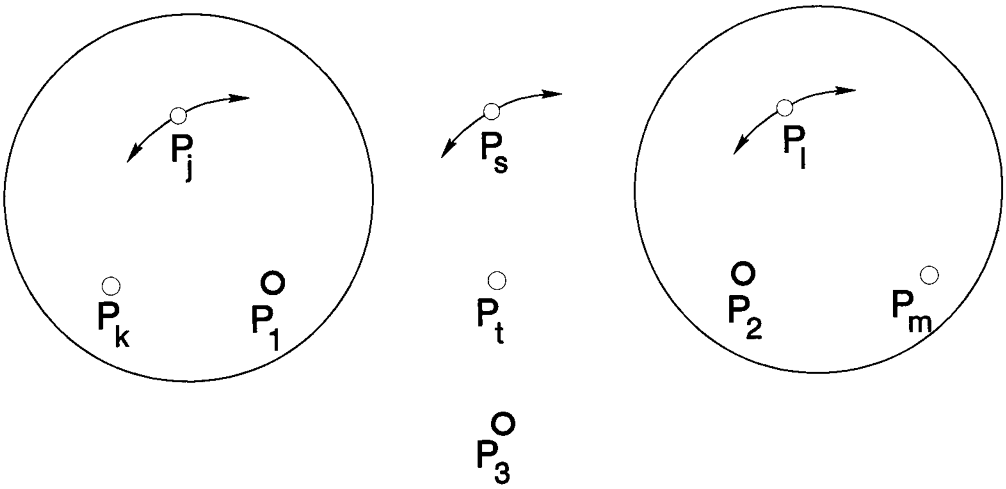
$$\Delta\left(\ln \frac{W_k}{W_1}\right) + \Delta\left(\ln \frac{r_{Ck}}{r_{C1}}\right) = \sum_{n=2}^{\infty} \left(\Delta c^{(n)} \cdot Y_k'^{(n)} - \Delta c^{(n)} \cdot Y_1'^{(n)}\right)$$

$$= \frac{1}{4\pi}\iint S_{1,k;j}^{(n-1)}\left[-2\Delta\left(\ln \frac{W_j}{W_1}\right) + \Delta\left(\ln \frac{g_j}{g_1}\right)\right]d\Omega_j$$

$$= \frac{1}{4\pi}\iint S_{1,k;j}^{(n+1)}\left[\Delta\left(\ln \frac{g_j}{g_1}\right) + 2\Delta\left(\ln \frac{r_{Cj}}{r_{C1}}\right)\right]d\Omega_j$$

Viewed historically, the first equation fits the satellite era with $\frac{r_{Ck}}{r_{C1}}$ and $\Delta c^{(n)}$ from satellite measurements, the second equation fits the pre-satellite era, with $\frac{W_k}{W_1}$ from levelling and $\frac{g_k}{g_1}$ from gravity measurements, the third equation again fits the satellite era, with $\frac{r_{Ck}}{r_{C1}}$ from satellite- and $\frac{g_k}{g_1}$ from gravity measurements. The quantities sought are then $\frac{W_k}{W_1}$, $\frac{r_{Ck}}{r_{C1}}$, $\frac{W_k}{W_1}$ respectively.

It is remarkable that at present the Stokes integral formula is still used almost exclusively. We shall therefore analyse this formula and apply the height datum connection according to Rummel and Teunissen, which is a method to evade the connection by tide gauges between levelling networks and sea level. In this connection I prefer to use the term S-transformations of "vertical quantities". In the case of the occurrence in the formulas of unknown (or partially unknown) compound quantities, these will be framed.



Suppose there are two continental S-systems, and one sea-S-system, with datum points P_1, P_2 and P_3 respectively. Then we have for P_k (with respect to P_1):

$$\Delta\left(\ln\frac{W_k}{W_1}\right) + \Delta\left(\ln\frac{r_{Ck}}{r_{C1}}\right) = \frac{1}{4\pi}\iint S_{1,k;j}^{(n-1)}\left[-2\Delta\left(\ln\frac{W_j}{W_1}\right) + \Delta\left(\ln\frac{g_j}{g_1}\right)\right]d\Omega_j +$$

$$+\frac{1}{4\pi}\iint S_{1,k;l}^{(n-1)}\left\{\left[-2\Delta\left(\ln\frac{W_l}{W_2}\right) + \Delta\left(\ln\frac{g_l}{g_2}\right)\right] + \left[-2\Delta\left(\ln\frac{W_2}{W_1}\right) + \Delta\left(\ln\frac{g_2}{g_1}\right)\right]\right\}d\Omega_j +$$

$$+\frac{1}{4\pi}\iint S_{1,k;s}^{(n-1)}\left\{\left[-2\Delta\left(\ln\frac{W_s}{W_3}\right) + \Delta\left(\ln\frac{g_s}{g_3}\right)\right] + \left[-2\Delta\left(\ln\frac{W_3}{W_1}\right) + \Delta\left(\ln\frac{g_3}{g_1}\right)\right]\right\}d\Omega_s$$

Or:

$$\Delta\left(\ln\frac{W_k}{W_1}\right) + \Delta\left(\ln\frac{r_{Ck}}{r_{C1}}\right) = \frac{1}{4\pi}\iint S_{1,k;j}^{(n-1)}\left[-2\Delta\left(\ln\frac{W_j}{W_1}\right) + \Delta\left(\ln\frac{g_j}{g_1}\right)\right]d\Omega_j +$$

$$+\frac{1}{4\pi}\iint S_{1,k;l}^{(n-1)}\left[-2\Delta\left(\ln\frac{W_l}{W_2}\right) + \Delta\left(\ln\frac{g_l}{g_2}\right)\right]d\Omega_l +$$

(1.k)

$$+\frac{1}{4\pi}\iint S_{1,k;s}^{(n-1)}\left[-2\Delta\left(\ln\frac{W_s}{W_3}\right) + \Delta\left(\ln\frac{g_s}{g_3}\right)\right]d\Omega_s +$$

$$+\boxed{-2\Delta\left(\ln\frac{W_2}{W_1}\right) + \Delta\left(\ln\frac{g_2}{g_1}\right)}\ \frac{1}{4\pi}\iint S_{1,k;l}^{(n-1)}d\Omega_l +$$

4.

$$+ \boxed{-2\Delta\left(\ln\frac{W_3}{W_1}\right) + \Delta\left(\ln\frac{g_3}{g_1}\right)} \; \frac{1}{4\pi}\iint S_{1,k;s}^{(n-1)}\, d\Omega_s$$

(1.m) $$\Delta\left(\ln\frac{W_m}{W_2}\right) + \Delta\left(\ln\frac{r_{Cm}}{r_{C2}}\right) + \boxed{\Delta\left(\ln\frac{W_2}{W_1}\right) + \Delta\left(\ln\frac{r_{C2}}{r_{C1}}\right)} =$$

$=$ right hand member (1.k.) with $k \rightarrow m$

(1.t) $$\Delta\left(\ln\frac{W_t}{W_3}\right) + \Delta\left(\ln\frac{r_{Ct}}{r_{C3}}\right) + \boxed{\Delta\left(\ln\frac{W_3}{W_1}\right) + \Delta\left(\ln\frac{r_{C3}}{r_{C1}}\right)} =$$

$=$ right hand member (1.k) with $k \rightarrow t$

(1.2) $$\boxed{\Delta\left(\ln\frac{W_2}{W_1}\right) + \Delta\left(\ln\frac{r_{C2}}{r_{C1}}\right)} = \text{right hand member (1.k) with } k \rightarrow 2$$

(1.3) $$\boxed{\Delta\left(\ln\frac{W_3}{W_1}\right) + \Delta\left(\ln\frac{r_{C3}}{r_{C1}}\right)} = \text{right hand member (1.k) with } k \rightarrow 3$$

The first two integrals in the right hand members of these equations do not present any difficulty, except that in difficult mountainous terrain levelling will only be possible in the valleys, and gravity measurements will be sparse [2]. The third integral does present difficulties, because one cannot measure ratios or differences of potential at sea, whereas gravity measurements are sparse there too. Consequently one would have to put $2\Delta\left(\ln\frac{W_s}{W_3}\right)$ equal to zero, which assuming a sea topography of some meters causes errors of the order $2 \cdot 5 \cdot 10^{-7} = 10^{-6}$, corresponding to the sharpness of definition of the gravity measurements at sea. Perhaps one might introduce from satellite data:

$$-2\Delta\left(\ln\frac{W_s}{W_3}\right) + \Delta\left(\ln\frac{g_s}{g_3}\right) = \sum_{n=2}^{\infty}(n-1)\left(\Delta c^{(n)} \cdot Y_s'^{(n)} - \Delta c^{(n)} \cdot Y_3'^{(n)}\right)$$

[2] In many respects one therefore meets the same situation as at sea, so that the sea-situation in fact prevails on by far the greater part of the earth.

presumably with a sharpness of definition of 10^{-6}. (An additional advantage is that the index 3 can be replaced by 1, so that in all equations the fifth term vanishes, as well as equations (1.3). But we shall here leave this aside as a speculation).

The core of the line of thought followed by Rummel and Teunissen is the determination of $\frac{r_{C2}}{r_{C1}}$ by satellite measurements, whereas $\frac{g_2}{g_1}$ can be measured. From equation (1.2) one could then solve the height datum connection $\Delta\left(\ln\frac{W_2}{W_1}\right)$ ... if there were no third S-system.

However if one measures by satellite $\frac{r_{C3}}{r_{C1}}$ on land in P_1, and via a ship or platform in P_3 then the equations (1.2) and (1.3) yield both $\Delta\left(\ln\frac{W_2}{W_1}\right)$ and $\Delta\left(\ln\frac{W_3}{W_1}\right)$ because $\frac{g_3}{g_1}$ can be measured as well.

Now one can compute for the continents:

> $\Delta\left(\ln\frac{r_{Ck}}{r_{C_1}}\right)$ from (1.k) and $\Delta\left(\ln\frac{r_{Cm}}{r_{C2}}\right)$ from (1.m), but $\Delta\left(\ln\frac{r_{Ct}}{r_{C3}}\right)$ cannot be computed from (1.t) unless $\Delta\left(\ln\frac{W_t}{W_3}\right) = 0$, hence relative sea topography is assumed to be zero.

One is then led to the derivation of $\Delta\left(\ln\frac{r_{Ct}}{r_{C3}}\right)$ from satellite altimetry (in principle providing ratios of radial distances, see [Baarda 1979]), and then compute from (1.t) the relative sea topography $-\ r_t\Delta\left(\ln\frac{W_t}{W_3}\right)$.

But then the whole approach by the Stokes formula becomes problematic, for if one enters the satellite era anyway, we should not one measure directly all ratios of radial distances?

This leads to the logical conclusion that in the satellite era Stokes's formula should be replaced by the Hotine integral formula, with the possibility of measuring gravity ratios and computing ratios of radial distances from satellite measurements all over the world. Nevertheless, the line of thought of Rummel and Teunissen is once more applicable here, to bridge the transition from land to sea.

To this end the formula will be written out anew, dropping the distinction between the S_1-system and the S_2-system (being the simplest case; of course more systems can be introduced both on land and at sea):

4.

$$\Delta\left(\ln\frac{W_k}{W_1}\right) + \Delta\left(\ln\frac{r_{Ck}}{r_{C1}}\right) = \frac{1}{4\pi}\iint S_{1,k;j}^{(n+1)}\left[+\,2\Delta\left(\ln\frac{r_{Cj}}{r_{C1}}\right) + \Delta\left(\ln\frac{g_j}{g_1}\right)\right]d\Omega_j\; +$$

$$+\;\frac{1}{4\pi}\iint S_{1,k;s}^{(n+1)}\left[+\,2\Delta\left(\ln\frac{r_{Cs}}{r_{C3}}\right) + \Delta\left(\ln\frac{g_s}{g_3}\right)\right]d\Omega_s\; +$$

$(\overline{1.k})$

$$+\;\boxed{+\,2\Delta\left(\ln\frac{r_{C3}}{r_{C1}}\right) + \Delta\left(\ln\frac{g_3}{g_1}\right)}\;\frac{1}{4\pi}\iint S_{1,k;s}^{(n+1)}\,d\Omega_s$$

$(\overline{1.t})$ $$\Delta\left(\ln\frac{W_t}{W_3}\right) + \Delta\left(\ln\frac{r_{Ct}}{r_{C3}}\right) + \boxed{\Delta\left(\ln\frac{W_3}{W_1}\right) + \Delta\left(\ln\frac{r_{C3}}{r_{C1}}\right)} =$$

= right hand member $(\overline{1.k})$ with $k \to t$

$(\overline{1.3})$ $$\boxed{\Delta\left(\ln\frac{W_3}{W_1}\right) + \Delta\left(\ln\frac{r_{C3}}{r_{C1}}\right)} = \text{right hand member } (\overline{1.k}) \text{ with } k \to 3$$

The determination of $\frac{g_j}{g_1}$ and $\frac{g_s}{g_3}\,\frac{g_3}{g_1} = \frac{g_s}{g_1}$ is done in one system, as well as the determination of $\frac{r_{Ck}}{r_{C1}}$ and $\frac{r_{Cj}}{r_{C1}}$ for points on land. The latter system, however, differs from $\frac{r_{Cs}}{r_{C3}}$ for points at sea (satellite altimetry). Consequently the **crucial step** is the determination of $\frac{r_{C3}}{r_{C1}}$, possibly measured by ship. Once this has been solved, one can compute $\Delta\left(\ln\frac{W_3}{W_1}\right)$ from $(\overline{1.3})$, $\Delta\left(\ln\frac{W_k}{W_1}\right)$ and $\Delta\left(\ln\frac{W_t}{W_3}\right)$ and hence relative sea topography from $(\overline{1.k})$ and $(\overline{1.t})$.

Regionally $\Delta\left(\ln\frac{W_k}{W_1}\right)$ can be improved via an adjustment by levelling; globally satellite data can improve $\Delta\left(\ln\frac{W_k}{W_1}\right)$ and $\Delta\left(\ln\frac{W_t}{W_1}\right)$:

$$\Delta\left(\ln\frac{W_p}{W_1}\right) + \Delta\left(\ln\frac{r_{Cp}}{r_{C1}}\right) = \sum_{n=2}^{\infty}\left(\Delta c^{(n)}\cdot Y'^{(n)}_p - \Delta c^{(n)}\cdot Y'^{(n)}_1\right)$$

4.3

Since we have emphasized the distinction between the origin P_M, of the X, Y, Z-system and the centre of mass of the earth P_C, some well-known consequences of this distinction will be exposed anew.

The discussion will be linked up with [G.B. Reed - Application of Kinematical Geodesy for Determining the Short Wave Length Components of the Gravity Field by Satellite Gradiometry - Report No. 201 of the Department of Geodetic Science, O.S.U., Columbus 1973].

Introduce a local reference frame for the point P_k by rotating the X, Y, Z-system, choosing the ζ-axis along r_k, the ξ-axis parallel to local North, and the η-axis parallel to local East. P_M remains the origin, the coordinates of P_C are (η_C, ξ_C, ζ_C).

The rotation gives:

$$\begin{pmatrix} \eta \\ \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} -\sin\lambda_k & \cos\lambda_k & 0 \\ -\sin\varphi_k \cos\lambda_k & -\sin\varphi_k \sin\lambda_k & \cos\varphi_k \\ \cos\varphi_k \cos\lambda_k & \cos\varphi_k \sin\lambda_k & \sin\varphi_k \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} X_k \\ Y_k \\ Z_k \end{pmatrix} = r_k \begin{pmatrix} \cos\varphi_k \cos\lambda_k \\ \cos\varphi_k \sin\lambda_k \\ \sin\varphi_k \end{pmatrix}$$

Since $\overline{P_M P_C}$ is small, simple geometry easily shows:

$$\lambda_{Ck} \cos\varphi_k = \lambda_k \cos\varphi_k - \frac{\eta_C}{r_k}$$

$$\varphi_{Ck} = \varphi_k - \frac{\xi_C}{r_k}$$

$$r_{Ck} = r_k - \zeta_C = r_k\left(1 - \frac{\zeta_C}{r_k}\right) \quad , \quad \frac{1}{r_{Ck}} = \frac{1}{r_k}\left(1 + \frac{\zeta_C}{r_k}\right)$$

In view of the smallness of $(\lambda_k - \lambda_{Ck})$, the difference between $\cos\varphi_{Ck}$ and $\cos\varphi_k$ is negligible.

4.

$$\begin{cases} \dfrac{\zeta_C}{r_k} = \dfrac{R}{r_k}\left(\dfrac{X_C}{R}\cos\varphi_k\cos\lambda_k + \dfrac{Y_C}{R}\cos\varphi_k\sin\lambda_k + \dfrac{Z_C}{R}\sin\varphi_k\right) \\[2ex] \qquad = \dfrac{R}{r_k}\left(\dfrac{X_C}{R}\dfrac{X_k}{r_k} + \dfrac{Y_C}{R}\dfrac{Y_k}{r_k} + \dfrac{Z_C}{R}\dfrac{Z_k}{r_k}\right) = B_k^{(1)} \\[2ex] \dfrac{\eta_C}{r_k} = \dfrac{R}{r_k}\left(-\dfrac{X_C}{R}\sin\lambda_k + \dfrac{Y_C}{R}\cos\lambda_k\right) = \dfrac{\partial B_k^{(1)}}{\cos\varphi_k\,\partial\lambda_k} \\[2ex] \dfrac{\xi_C}{r_k} = \dfrac{R}{r_k}\left(-\dfrac{X_C}{R}\sin\varphi_k\cos\lambda_k - \dfrac{Y_C}{R}\sin\varphi_k\sin\lambda_k + \dfrac{Z_C}{R}\cos\varphi_k\right) = \dfrac{\partial B_k^{(1)}}{\partial\varphi_k} \end{cases}$$

with $B_k^{(1)}$ the spherical harmonics term of degree 1.

With $\dfrac{\partial V_k}{\partial\eta} \underset{\text{put}}{=} V_{k,\eta}$, $\dfrac{\partial^2 V_k}{\partial\eta^2} \underset{\text{put}}{=} V_{k,\eta\eta}$, etc. and $\displaystyle\sum_{n=2}^{\infty} B_k^{(n)} \underset{\text{put}}{=} \Sigma_k$

one obtains:

$$V_k = \frac{\mu}{r_{Ck}}\left(1 + \Sigma_k\right) = \frac{\mu}{r_k}\left(1 + B_k^{(1)} + \Sigma_k\right)$$

$$\begin{cases} r_k V_{k,\eta} = \dfrac{\partial V_k}{\cos\varphi_k\,\partial\lambda_k} = \dfrac{\mu}{r_k}\left(0 + \dfrac{\partial B_k^{(1)}}{\cos\varphi_k\,\partial\lambda_k} + \Sigma_{k,\eta}\right) \\[2ex] r_k V_{k,\xi} = \dfrac{\partial V_k}{\partial\varphi_k} = \dfrac{\mu}{r_k}\left(0 + \dfrac{\partial B_k^{(1)}}{\partial\varphi_k} + \Sigma_{k,\xi}\right) \\[2ex] r_k V_{k,\zeta} = \dfrac{\partial V_k}{\partial\ln r_k} = -\dfrac{\mu}{r_k}\left(1 + 2B_k^{(1)} + \Sigma_{k,\zeta}\right) \end{cases}$$

$$\left\{\begin{array}{l}
r_k^2 V_{k,\eta\eta} = \dfrac{\partial^2 V_k}{\cos^2\varphi_k \partial\lambda_k^2} - \tan\varphi_k \dfrac{\partial V_k}{\partial\varphi_k} + \dfrac{\partial V_k}{\partial \ln r_k} = -\dfrac{\mu}{r_k}\left(1 + 3B_k^{(1)} - \sum_{k,\eta\eta}\right) \\[2ex]
r_k^2 V_{k,\eta\xi} = \dfrac{\partial^2 V_k}{\cos\varphi_k \partial\lambda_k \partial\varphi_k} + \tan\varphi_k \dfrac{\partial V_k}{\cos\varphi_k \partial\lambda_k} = -\dfrac{\mu}{r_k}\left(0 + 0 - \sum_{k,\eta\xi}\right) \\[2ex]
r_k^2 V_{k,\eta\zeta} = \dfrac{\partial^2 V_k}{\cos\varphi_k \partial\lambda_k \partial \ln r_k} - \dfrac{\partial V_k}{\cos\varphi_k \partial\lambda_k} = -\dfrac{\mu}{r_k}\left(0 + 3\dfrac{\partial B_k^{(1)}}{\cos\varphi_k \partial\lambda_k} - \sum_{k,\eta\zeta}\right) \\[2ex]
r_k^2 V_{k,\xi\xi} = \dfrac{\partial^2 V_k}{\partial\varphi_k^2} + \dfrac{\partial V_k}{\partial \ln r_k} = -\dfrac{\mu}{r_k}\left(1 + 3B_k^{(1)} - \sum_{k,\xi\xi}\right) \\[2ex]
r_k^2 V_{k,\xi\zeta} = \dfrac{\partial^2 V_k}{\partial\varphi_k \partial \ln r_k} - \dfrac{\partial V_k}{\partial\varphi_k} = -\dfrac{\mu}{r_k}\left(0 + 3\dfrac{\partial B_k^{(1)}}{\partial\varphi_k} - \sum_{k,\xi\zeta}\right) \\[2ex]
r_k^2 V_{k,\zeta\zeta} = \dfrac{\partial^2 V_k}{\partial(\ln r_k)^2} = 2\dfrac{\mu}{r_k}\left(1 + 3B_k^{(1)} + \sum_{k,\zeta\zeta}\right)
\end{array}\right.$$

For $\Sigma_{k,u}$ and $\Sigma_{k,uv}$ see [Reed 1973] [3)]

As a preparation for the formation of difference equations the coefficients of Δ-quantities are determined from a spherical model of dimensionless quantities:

$$\frac{W_k}{r_1 g_1} \simeq \frac{V_k}{r_1 g_1} \simeq \frac{r_1}{r_k}$$

Let (u, v, w) be a coordinate frame obtained by rotation of the X, Y, Z-frame. Dimensionless differentiation can be done as follows:

$$\frac{\partial\left(\dfrac{W_k}{r_1 g_1}\right)}{\dfrac{1}{r_k}\,\partial u_k} = \frac{r_k \partial\left(\dfrac{W_k}{r_1 g_1}\right)}{\partial u_k} = \frac{r_k \dfrac{\partial W_k}{\partial u_k}}{r_1 g_1}$$

For the spherical model follows:

3) A check by P.J.G. Teunissen disclosed a printing error in Reed: formula (3.28)

$$\frac{\partial W_k}{\partial u_k} = \frac{\partial W_k}{\partial r_k} \cos(r_k, u_k) = \frac{u_k}{r_k} = \frac{\delta_{uv} v_k}{r_k} = \frac{\delta_{uw} w_k}{r_k}$$

$$\text{with } \delta_{uv} \begin{cases} = 1 & \text{for } v = u \\ = 0 & \text{for } v \neq u \end{cases}$$

This gives:

$$\begin{cases} \dfrac{W_k}{r_1 g_1} \simeq \dfrac{r_1}{r_k} \\ \dfrac{-r_k W_{k,u}}{r_1 g_1} \simeq \dfrac{r_1}{r_k} \cos(r_k, u_k) \\ \dfrac{r_k^2 W_{k,uv}}{r_1 g_1} \simeq \dfrac{r_1}{r_k} \left(- \delta_{uv} + 3\cos(r_k, u_k) \cos(r_k, v_k)\right) \\ \dfrac{r_k^3 W_{k,uvw}}{r_1 g_1} \simeq \dfrac{r_1}{r_k} \Big(3\delta_{uv} \cos(r_k, w_k) + 3\delta_{uw} \cos(r_k, v_k) + \\ \qquad + 3\delta_{vw} \cos(r_k, u_k) - 15 \cos(r_k, u_k) \cos(r_k, v_k) \cos(r_k, w_k)\Big) \end{cases}$$

from which all coefficients follow by replacing u , v , w by η and/or ξ and/or ζ.

In table 4.3 an overview of the difference equations is presented; it is to be noted that $\Delta B_k^{(1)} = B_k^{(1)}$ because $\overline{P_M P_C}$ is unknown and its approximate value is zero. Use has been made of:

$$\begin{cases} \Delta\left(\dfrac{- r_k W_{k,u}}{r_1 g_1}\right) = \dfrac{- r_k W_{k,u}}{r_1 g_1} \Delta(\ln r_k) + r_k \Delta\left(\dfrac{- W_{k,u}}{r_1 g_1}\right) \\ \Delta\left(\dfrac{r_k^2 W_{k,uv}}{r_1 g_1}\right) = 2 \dfrac{r_k^2 W_{k,uv}}{r_1 g_1} \Delta(\ln r_k) + r_k^2 \Delta\left(\dfrac{W_{k,uv}}{r_1 g_1}\right) \end{cases}$$

These formulas explain the difference between framed and non-framed Δ-quantities in table 4.3; the deviating coefficients of $\Delta(\ln r_k)$ in the framed Δ-quantities are again framed.

There is a remark to be made about the relation with the compound quantities in Section 4.1:

u	v	Spherical approximation
-	-	$\dfrac{W_k}{r_1 g_1} \simeq +\dfrac{r_1}{r_k}$
η	-	$\dfrac{-r_k W_{k,u}}{r_1 g_1} \simeq 0$
ξ	-	0
ζ	-	$+\dfrac{r_1}{r_k}$
η	η	$\dfrac{r_k^2 W_{k,uv}}{r_1 g_1} \simeq -\dfrac{r_1}{r_k}$
η	ξ	0
η	ζ	0
ξ	ξ	$-\dfrac{r_1}{r_k}$
ξ	ζ	0
ζ	ζ	$+2\dfrac{r_1}{r_k}$

Difference equations

$$
\begin{pmatrix}
\Delta\left(\dfrac{W_k}{r_1 g_1}\right) \\
r_k \Delta\left(-\dfrac{W_{k,u}}{r_1 g_1}\right) \\
\boxed{\Delta\left(\dfrac{-r_k W_{k,u}}{r_1 g_1}\right)} \\
\\
r_k^2 \Delta\left(\dfrac{W_{k,uv}}{r_1 g_1}\right) \\
\boxed{\Delta\left(\dfrac{r_k^2 W_{k,uv}}{r_1 g_1}\right)} \\
\\
\\
\\
\\
\end{pmatrix}
= \frac{r_1}{r_k}
\left[
\begin{pmatrix}
+1 & 0 & 0 & -1 \\
0 & +1 & 0 & 0 \\
0 & 0 & +1 & 0 \\
+1 & 0 & 0 & \underbrace{-2\ \boxed{-1}} \\
-1 & 0 & 0 & \underbrace{+3\ \boxed{+1}} \\
0 & 0 & 0 & 0 \\
0 & +3 & 0 & 0 \\
-1 & 0 & 0 & \underbrace{+3\ \boxed{+1}} \\
0 & 0 & +3 & 0 \\
+2 & 0 & 0 & \underbrace{-6\ \boxed{-2}}
\end{pmatrix}
\cdot
\begin{pmatrix}
\Delta\left(\dfrac{\mu}{r_1^2 g_1}\right) \\
\dfrac{\Delta\eta_k}{r_k} \\
\dfrac{\Delta\xi_k}{r_k} \\
\dfrac{\Delta\zeta_k}{r_k} = \Delta(\ln r_k)
\end{pmatrix}
+
\begin{pmatrix}
+B_k^{(1)} + \Delta\Sigma_k \\
-\dfrac{\partial B_k^{(1)}}{\cos\varphi_k \partial\lambda_k} - \Delta\Sigma_{k,\eta} \\
-\dfrac{\partial B_k^{(1)}}{\partial\varphi_k} - \Delta\Sigma_{k,\xi} \\
+2B_k^{(1)} + \Delta\Sigma_{k,\xi} \\
-3B_k^{(1)} + \Delta\Sigma_{k,\eta\eta} \\
0 + \Delta\Sigma_{k,\eta\xi} \\
-3\dfrac{\partial B_k^{(1)}}{\cos\varphi_k \partial\lambda_k} + \Delta\Sigma_{k,\eta\zeta} \\
-3B_k^{(1)} + \Delta\Sigma_{k,\xi\xi} \\
-3\dfrac{\partial B_k^{(1)}}{\partial\varphi_k} + \Delta\Sigma_{k,\xi\zeta} \\
+6B_k^{(1)} + \Delta\Sigma_{k,\zeta\zeta}
\end{pmatrix}
\right]
$$

Table 4.3

$$A_k - \frac{r_1}{r_k} A_1$$

in which A_k are the respective quantities in column 3 of table 4.3.

Instead of taking $\Delta\left(A_k - \frac{r_1}{r_k} A_1\right)$, as in Section 4.1, whereby $\Delta\left(\frac{\mu}{r_1^2 g_1}\right)$ is directly eliminated, we consider ΔA_k and ΔA_1 separately.

Then we have, with $\Delta\left(\frac{r_1}{r_k}\right) = \frac{r_1}{r_k}\,\Delta\left(\ln\frac{r_1}{r_k}\right) = -\frac{r_1}{r_k}\,\Delta\left(\ln\frac{r_k}{r_1}\right)$:

$$\Delta\left(A_k - \frac{r_1}{r_k}A_1\right) = \Delta(A_k) - \frac{r_1}{r_k}\Delta(A_1) + \frac{r_1}{r_k}A_1\,\Delta\left(\ln\frac{r_k}{r_1}\right)$$

This formulation is used in [Baarda 1979]. After transporting $\frac{r_1}{r_k}A_1\,\Delta\left(\ln\frac{r_k}{r_1}\right)$ to the right hand side of the difference equations obtained from Table 4.3 after subtracting $\frac{r_1}{r_k}\Delta(A_1)$, the framed situation in a modified Table 4.3 is obtained:

$$1^\circ \text{ vector: } \left(\left[\Delta\left(\frac{W_k}{r_1 g_1}\right) - \frac{r_1}{r_k}\,\Delta\left(\frac{W_1}{r_1 g_1}\right)\right], \ldots\ldots\right)^T$$

$$2^\circ \text{ vector: } \left(0\ ,\ \left(\frac{\Delta\eta_k}{r_k} - \frac{\Delta\eta_1}{r_1}\right),\ \left(\frac{\Delta\xi_k}{r_k} - \frac{\Delta\xi_1}{r_1}\right),\ \Delta\left(\ln\frac{r_k}{r_1}\right)\right)^T$$

$$3^\circ \text{ vector: } \left(\left(B_k^{(1)} - B_1^{(1)} + \Delta\textstyle\sum_k - \Delta\sum_1\right), \ldots\ldots\right)^T$$

A clear influence of $\overline{P_M P_C} \neq 0$ is established, although the corresponding terms only make sense if there is a unique S-system valid for the whole earth. In this case Hotine's integral formula is affected by first degree spherical harmonics, so that the lower index C may be omitted from this formula.

But how to deal with the situation of satellite gradiometry, where radial distances for the orbit computation are theoretically computed with respect to P_C (although the effect of $\overline{P_M P_C} \neq 0$ indirectly sneaks into the practical computation), whereas the gradiometer reacts to the gravity field of the earth, perhaps under a direct influence of $\overline{P_M P_C} \neq 0$? In Section 10 a part of the modified table 4.3 is given for this situation, which connects with existing literature, hence with $B^{(1)} = 0$. But is this the right choice?

A second remark pertains to the importance of the formulation of compound quantities

$$\left(\bar{A}_k - \frac{r_1}{r_k}\bar{A}_1\right) \text{ with:}$$

$$\bar{A}_k : \quad \frac{W_k}{r_1 g_1}\ , \quad \frac{-r_k W_{k,\zeta}}{r_1 g_1} = \frac{r_k g_k}{r_1 g_1}\ , \quad \frac{r_k^2 W_{k,\zeta\zeta}}{r_1 g_1} = \frac{r_k^2 \Gamma_k}{r_1 g_1}$$

under the assumption, made throughout in this publication, that the rotational velocity of the earth is sufficiently well known and the small angle between the radial direction and the plumb line is ignored.

With $\frac{r_k}{r_1}\bar{A}_k \simeq 1$ or 2 respectively, one obtains:

$$\Delta\left(\bar{A}_k - \frac{r_1}{r_k}\bar{A}_1\right) = \Delta\left[\frac{r_1}{r_k}\left(\frac{r_k}{r_1}\bar{A}_k - \frac{r_1}{r_1}\bar{A}_1\right)\right] = \Delta\left[\frac{r_1}{r_k}\bar{A}_1\left(\frac{r_k\bar{A}_k}{r_1\bar{A}_1} - 1\right)\right] \simeq$$

$$\simeq \Delta\left[\frac{r_1}{r_k}\bar{A}_1 \ln\frac{r_k\bar{A}_k}{r_1\bar{A}_1}\right] \simeq \frac{r_1}{r_k}\bar{A}_1\,\Delta\left(\ln\frac{r_k\bar{A}_k}{r_1\bar{A}_1}\right)$$

Or, see also Section 4.1:

$$\begin{cases} \Delta\left(\dfrac{W_k}{r_1 g_1} - \dfrac{r_1}{r_k}\dfrac{W_1}{r_1 g_1}\right) = \dfrac{r_1}{r_k}\left[\Delta\left(\ln\dfrac{W_k}{W_1}\right) + \Delta\left(\ln\dfrac{r_k}{r_1}\right)\right] \\ \Delta\left(\dfrac{r_k g_k}{r_1 g_1} - \dfrac{r_1}{r_k}\dfrac{r_1 g_1}{r_1 g_1}\right) = \dfrac{r_1}{r_k}\left[\Delta\left(\ln\dfrac{g_k}{g_1}\right) + 2\Delta\left(\ln\dfrac{r_k}{r_1}\right)\right] \\ \Delta\left(\dfrac{r_k^2\Gamma_k}{r_1 g_1} - \dfrac{r_1}{r_k}\dfrac{r_1^2\Gamma_1}{r_1 g_1}\right) = 2\dfrac{r_1}{r_k}\left[\Delta\left(\ln\dfrac{\Gamma_k}{\Gamma_1}\right) + 3\Delta\left(\ln\dfrac{r_k}{r_1}\right)\right] \end{cases}$$

One recognizes the coefficients 1, 2 and 6 of $\Delta\left(\ln\frac{r_k}{r_1}\right)$ from the modified table 4.3. The importance of this formulation has only recently become clear to me. Section 4.2 already presented an example; in Section 11 we will use it [4)].

Finally we shall now follow the line of thought of Section 1.8 of [Baarda 1979] for the Bruns transformation.

Let u be the direction of differentiation, then:

---

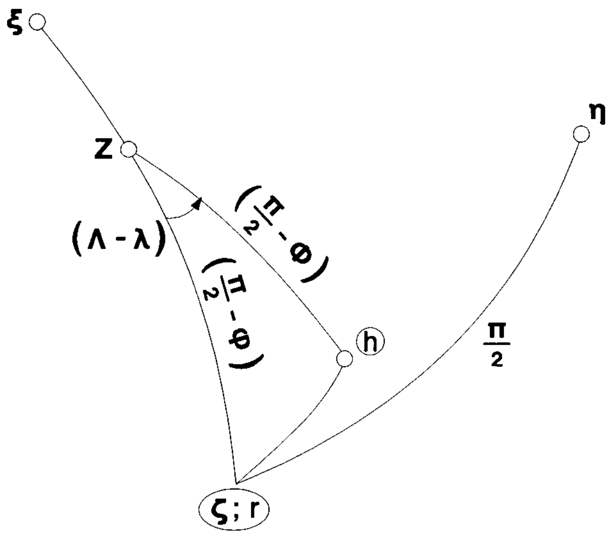
4) Formerly I have been hesitating a long time before choosing between the compound quantities $\left(A_k - \frac{r_1}{r_k}A_1\right)$ and $\left(\frac{r_k}{r_1}A_k - A_1\right)$. I finally chose the first form, but now I would prefer the second because then there is now a coefficient $\frac{r_1}{r_k}$ in the right hand members.

4.

$$\frac{\partial W}{\frac{1}{r}\partial u} = r\frac{\partial W}{\partial u} = r\frac{\partial W}{\partial h}\cos(h,u) = -rg\cos(h,u)$$

in which h_k is the negative plumb line direction in P_k. Transfer all relevant directions to P_k and consider the spherical representation. Provisionally leaving out the index k, one obtains:

$$\cos(h,u) = \cos(h,r)\cos(r,u) + \sin(h,r)\sin(r,u)\cos(h,r,u)$$



Choose $u = \eta$: $\cos(h,r,\eta) = \sin(\xi,r,h)$

hence with the rule of sines: $\sin(h,r)\cos(h,r,\eta) = \cos\Phi\sin(\Lambda - \lambda)$

Further: $\cos(r,\eta) = 0\ ,\ \sin(r,\eta) = 1$

Choose $u = \xi$: $\cos(h,r,\xi) = \cos(\xi,r,h)$

and with the results of cosines and cotangents:

$$\sin\Phi = \sin\varphi\cos(h,r) + \cos\varphi\sin(h,r)\cos(\xi,r,h)$$

$$\cos(h,r) = \sin\Phi\sin\varphi + \cos\Phi\cos\varphi\cos(\Lambda - \lambda)$$

hence: $\sin(h,r)\cos(\xi,r,h) = \sin\Phi\cos\varphi - \cos\Phi\sin\varphi\cos(\Lambda - \lambda)$,

provided $\cos\varphi \neq 0$

Further: $\cos(r,\xi) = 0\ ,\ \sin(r,\xi) = 1$

Choose $u = \zeta$: $\cos(r,\zeta) = 1\ ,\ \sin(r,\zeta) = 0$

Summarizing:

$$\begin{pmatrix} \dfrac{-r_k W_{k,\eta}}{r_1 g_1} \\ \dfrac{-r_k W_{k,\xi}}{r_1 g_1} \\ \dfrac{-r_k W_{k,\zeta}}{r_1 g_1} \end{pmatrix} = \frac{r_k g_k}{r_1 g_1} \begin{pmatrix} \cos\Phi_k \sin(\Lambda_k - \lambda_k) \\ \sin\Phi_k \cos\varphi_k - \cos\Phi_k \sin\varphi_k \cos(\Lambda_k - \lambda_k) \\ \cos(h_k, r_k) \end{pmatrix}$$

Then the difference equations of compound quantities, complemented with the modified table 4.3 become:

$$\frac{r_k}{r_1} \begin{pmatrix} \Delta\left(\dfrac{-r_k W_{k,\eta}}{r_1 g_1}\right) - \dfrac{r_1}{r_k} \Delta\left(\dfrac{-r_1 W_{1,\eta}}{r_1 g_1}\right) \\ \Delta\left(\dfrac{-r_k W_{k,\xi}}{r_1 g_1}\right) - \dfrac{r_1}{r_k} \Delta\left(\dfrac{-r_1 W_{1,\xi}}{r_1 g_1}\right) \\ \Delta\left(\dfrac{-r_k W_{k,\zeta}}{r_1 g_1}\right) - \dfrac{r_1}{r_k} \Delta\left(\dfrac{-r_1 W_{1,\zeta}}{r_1 g_1}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\varphi_k \Delta(\Lambda_k - \lambda_k) - \cos\varphi_1 \Delta(\Lambda_1 - \lambda_1) \\ \Delta(\Phi_k - \varphi_k) - \Delta(\Phi_1 - \varphi_1) \\ \Delta\left(\ln \dfrac{g_k}{g_1}\right) + \Delta\left(\ln \dfrac{r_k}{r_1}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} (\cos\varphi_k \Delta\lambda_k - \cos\varphi_1 \Delta\lambda_1) - \left(\dfrac{\partial B_k^{(1)}}{\cos\varphi_k \partial\lambda_k} - \dfrac{\partial B_1^{(1)}}{\cos\varphi_1 \partial\lambda_1}\right) - (\Delta\Sigma_{k,\eta} - \Delta\Sigma_{1,\eta}) \\ (\Delta\varphi_k - \Delta\varphi_1) - \left(\dfrac{\partial B_k^{(1)}}{\partial\varphi_k} - \dfrac{\partial B_1^{(1)}}{\partial\varphi_1}\right) - (\Delta\Sigma_{k,\xi} - \Delta\Sigma_{1,,\xi}) \\ -\Delta\left(\ln \dfrac{r_k}{r_1}\right) + 2\left(B_k^{(1)} - B_1^{(1)}\right) + (\Delta\Sigma_{k,\zeta} - \Delta\Sigma_{1,\zeta}) \end{pmatrix}$$

From the third elements of the latter two vectors follows the well-known expansion into series:

$$\Delta\left(\ln\frac{g_k}{g_1}\right) + 2\Delta\left(\ln\frac{r_k}{r_1}\right) = 2\left(B_k^{(1)} - B_1^{(1)}\right) + \sum_{n=2}^{\infty}(n+1)\left(\Delta B_k^{(n)} - \Delta B_1^{(n)}\right)$$

The other two elements of these vectors provide, among other things, the influence of $\overline{P_M P_C} \neq 0$ on the plumb line deflections $(\Lambda_k - \lambda_k)$ and $(\Phi_k - \varphi_k)$. The formulation differs from the one given in Section 4 of [Baarda 1989] in the "Festschrift to Torben Krarup".

4.4

We return once more to Hotine's integral formula to point out a way of application which is perhaps feasible.

To begin with it is remarked that in our four-dimensional system, having three geometric coordinates plus the gravity potential per point, the three coordinates which can be determined by satellite positioning and satellite altimetry must be supplemented with a fourth independent measured quantity. For this quantity we choose gravity, which in principle can be measured everywhere on earth. As a consequence, the computation of gravity at sea with the aid of satellite altimetry does not fit in this line of thought, because it does not provide a fourth independent measured quantity per point [5].

This approach makes it possible to apply Hotine's integral formula. As already remarked in Section 4.3, any first degree spherical harmonics in left- and right hand member of this formula cancel each other, so that the application is possible if the origin P_M of the coordinate frame does not coincide with P_C, the centre of mass of the earth.
This is significant for practical use, because in Section 9.2 it is shown that $\overline{P_M P_C}$ may be made small, but it cannot be reduced to zero. In practice the influence [6] of $\overline{P_M P_C} \neq 0$ (but small) on higher degree spherical harmonics can be ignored. For the integral formula this means that in all quantities the lower index C must be dropped.

A practical difficulty is the measurement of gravity at sea where ships or low-flying aircraft can be used. For aircraft we assume that the positions lie on a slightly waving surface, curving with the earth, so that for the outer normal n to this surface we can use, see [Baarda 1979, Sections 1.7 and 1.8]:

---

[5] Although this Section is mainly concerned with the situation at sea, the land situation is not simple either. What is evidently required is a non-reduced gravity observation per stations, whereas in general the presently available data sets contain reduced measurements or anomalies.

[6] See e.g. A. Kleusberg - The Similarity Transformation of the Gravitational Potential Close to the Identity - Man. Geod. **5** (1980) p.p. 241-256

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r}\cos(n, r)$$

If now the sea of the earth are covered by a network of non-overlapping measurements of gravity by ships and aircraft, then the following line of thought is possible:

Let S denote the surface of the earth, both land and sea, leaving changes with time out of consideration. Then, to quote [Baarda 1979, p. 17]: "It is still more essential that in practice measurements are never executed on S itself, but at some distance outside S. Here, a comparison can be made with the spatial geometric networks, by which one determines coordinates of points usually situated at some distance from the earth, on towers, pillars, etc.. For cartographic purposes these points are projected on a reference ellipsoid or on a plane, but this is not essential for spatial gravimetric geodesy. Similarly, the "reduction" of observations in gravimetric geodesy does not belong to the essence of the theory, so that in principle all reductions should be excluded.
The exclusion of reductions is attained by replacing the surface S by the geosurface S^*, containing the observation points on, or near and connected with, the earth's surface. S^* may locally coincide with S but it may also deviate from S. S^* has to fulfil the same requirements as S; the surface may contain a finite number of singular points and a finite number of edges, which divide the surface into a finite number of pieces with continuously changing normal direction. The equations (1.7.4) and (1.7.5) as well as (1.7.1) remain valid if S is replaced by the geosurface S^*. Points P_j, connected with the earth are therefore always situated on S^*. "P_i inside S^*" now assumes a more realistic meaning".

As an example we take the re-written third Green's integral formula [Baarda 1979, (1.7.5)]

$$V_i = \frac{1}{4\pi}\iint_{\Omega}\left[-V_{ij}\,\frac{1+\delta_{ij}}{2} + r_j\left(-\frac{\partial V_j}{\partial r_j}\right)\right]\frac{r_j}{r_{ij}}\,d\Omega_j$$

in which: V is the gravitational potential (for points on S replaced by the gravity potential W minus the centrifugal potential); Ω is the surface of a sphere with unit radius,

$$\delta_{ij} = \frac{r_j^2 - r_i^2}{r_{ij}^2}$$

P_i outside, on or inside S^*; P_j on S^* (P_i cannot be a point inside the matter of the earth).

The integral formulas of Stokes and Hotine are equally valid for S^*; in the approach given it is assumed that the centrifugal potential can be computed with sufficient accuracy from existing data, so that:

$$\Delta V_k = \Delta W_k$$

The geosurface S^* always remains a somewhat vague concept because observations will always be made at discrete points. But this applies to S as well, because one may ask what actually is **the** surface of seas, marshes, fields with drains and ditches, forests, mountains, villages and cities? Considering this, the sharpness of definition of S will therefore have to be appraised at decimetres or metres rather than centimetres, of course with the exception of well-marked points. So there is not much difference in principle, be it that the enclosed mass for S and S^* may be different if S is taken to be "ground level". It is therefore important for practice to develop the formula system in such a way that **the** mass of the earth is eliminated. According to [Teunissen 1980], the effect of the atmosphere is then virtually eliminated as well.

Now **suppose** that modern satellite positioning (also in aircraft) makes it possible to measure gravity in low-flying aircraft with the same order of sharpness of definition for $\ln g$ as by gravity measurement in ships.

Then it is **suggested** to extend S^* to include the aerial stations where gravity is measured, and then apply Hotine's integral formula.
In the situation sketched, the points P_1, P_2, P_3 and P_j, P_l, P_s lie on S^*, whereas P_k, P_m, P_t lie on S (there not necessarily coinciding with S^*). The data for the kernel of the Hotine integral formula are derived from the three-dimensional coordinates of these points. For the analytical expression for the kernel, see [Teunissen 1980]. In the left hand member of Hotine's integral formula the length ratios for points P_t on the sea surface can be determined by satellite altimetry. Values for the potential then follow from the integral formula [7].

Alas, there is a big "but" in the question ... The sharpness of definition of $\ln \frac{g_{\cdot}}{g_{\bullet}}$ and $\ln \frac{r_{\cdot}}{r_{\bullet}}$ from measurements by ship or aircraft will not be better than 10^{-6}. This implies that the sharpness of definition of the integral will be of the same order of magnitude (perhaps somewhat better as a result of damping). Even if altimeter measurements are better defined (in spite of orbit errors), the consequence is that the global sharpness of definition of $\ln \frac{W_{\cdot}}{W_{\bullet}}$ cannot be much better than 10^{-6}. This is the cause that relative sea topography can hardly be determined by this means since the influence on $\ln \frac{W_{\cdot}}{W_{\bullet}}$ is of the order 5.10^{-7}.

Besides it is questionable if ship- and aircraft measurements are worth the trouble now that the results of satellite gradiometry promise a sharpness of definition of 10^{-6} or better. If this promise comes true, the result may be that Hotine's integral formula, like Stokes's, is relegated to the annuals of the history of geodesy.

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[7] There need not be a special convergence problem, for, first, in the situation sketched above equal values can be taken for all radial distances r in the coefficients of the integral formula (See Section 5.1 or [Baarda 1979, Section 1.2]) and, second, the difference compound variates can be considered as being reduced to one and the same equipotential surface (See Section 4.1).

5.

When by the application of division algebras I had succeeded in formulating geometric geodesy as a theory of form elements in geodetic networks, I tried in my 1979 publication to follow the same idea in order to formulate the basic approach of physical geodesy in such a way that the description could be given in terms of form elements. The aim was to make possible a better connection with the theory of geometric networks.

For the solution found, two questions require special attention. The first is the introduction of approximate values for linearization. The theory of adjustment or estimation and the theory of networks show that approximate values and iteration processes have their place within the mathematical model and have no physical meaning. Experience from previous and present measurement processes in observation space leads to the introduction (linking up) of a mathematical model usually containing non-linear relations between measured quantities, as well as to the names given to these quantities. The results of deduction within the mathematical model have to be translated into terms of the observation space, and it is not before this translation - the "unlinking" of the model - has been made, that a physical interpretation is possible. To give an example, the geodetic reference ellipsoid is, on one hand, part of the (mathematical) coordinate system; and on the other hand part of the (consistent) set of approximate values. The assignment of a point in observation space to the approximate coordinates of a physical point, as found in classical derivations, can only lead to confusion. The consequence would be that the ellipsoid (with the approximate direction of the non-central axis of rotation and an approximate angular velocity of rotation) would have to move inside the earth. It is of course correct that this consequence is not drawn in the literature, but it indicates a lack of understanding of the function and meaning of approximate values. In the derivation given here it does not matter how one arrives at a set of approximate values, provided that it is consistent.

The second question concerns the place taken by the introduction of the earth model already used, viz. the homogeneous sphere. Use is made of the fact that for points on the surface of the earth the ratios of moduli of radius vectors, as well as the ratios of moduli of gravity vectors, deviate at most 1/100 from unity [1]. The spherical model may then be used in a number of spherical potential-theoretic relations, **provided** that the relative difference between the approximate value and the estimate of any computed quantity does not exceed the same order of magnitude. This requirement improves the possibility of assessing the reliability of derivations as compared with the usual classical derivations, but it also imposes very high demands of the quality of the set of approximate values. The use of well-known gravimetric methods of reduction - such as the isostatic ones - in the reverse direction is not

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[1] In Section 4.1 a somewhat less cautious estimate was in mind, viz. 3.10^{-3}.

sufficient. Only the use of detailed geological and geophysical knowledge of the substructure can provide an adequate answer here.

Stokes - type integral formula

connection with geometric networks

spirit levelling

gravity measurement

$$\left[\Delta\left(\frac{W_k - W_1}{r_1 g_1}\right) - \Delta_{1k}\right] + \left[\Delta\left(ln\,\frac{r_k}{r_1}\right) - \left(B_k^{(1)} - B_1^{(1)}\right)\right] = \frac{1}{4\pi}\iint_\Omega S_{1,k;j}^{(n-1)}\left\{-2\left[\Delta\left(\frac{W_j - W_1}{r_1 g_1}\right) - \Delta_{1j}\right] + \left[\Delta\left(ln\,\frac{g_i}{g_1}\right) - \frac{1}{2}\Delta_{1j}\right]\right\} d\Omega_j$$

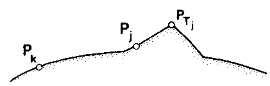
"free-air reduction" to "geoid"

"free-air reduction" to "geoid"

"indirect effect of gravity reductions"

"topografic reduction"

$$\frac{1}{2}\Delta_{1k} = \frac{1}{4\pi}\iint_\Omega \left(C_{kj} - C_{1j}\right)\left[\Delta\left(\frac{W_j - W_1}{r_1 g_1}\right) + \Delta\left(ln\,\frac{r_j}{r_1}\right)\right] d\Omega_j$$



$$C_{kj} \simeq \pm 2\left(\frac{R}{r_{kj}}\right)^3 \frac{h_j - h_{T_j}}{R}, \quad \begin{cases} + : r_{kT_j} > r_{kj} \\ - : r_{kT_j} < r_{kj} \\ h = \text{height} \end{cases}$$

In literature

h_k instead of h_{T_j} in C_{kj}

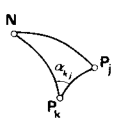
Figure 5.1.a

Vening Meinesz - type integral formula

$$\left[\Delta\left(\Phi_k - \Phi_1\right) - \bar{\Delta}_{1k}\right] - \left[\Delta\left(\varphi_k - \varphi_1\right) + \left(\frac{\delta}{-\delta\varphi_k} B_k^{(1)} - \frac{\delta}{-\delta\varphi_1} B_1^{(1)}\right)\right] = \frac{1}{4\pi}\iint_\Omega \left(\frac{\delta}{-\delta\varphi_k} + \frac{\delta}{-\delta\varphi_1}\right) S_{1,k;j}^{(n-1)} \left\{\text{see Stokes}\right\} d\Omega_j$$

Φ = astronomic latitude
φ = geo(ec)centric latitude

$$\frac{1}{2}\bar{\Delta}_{1k} = \frac{1}{4\pi}\iint_\Omega \left(D_{kj} - D_{1j}\right)\left[\Delta\left(\frac{W_j - W_1}{r_1 g_1}\right) + \Delta\left(ln\,\frac{r_j}{r_1}\right)\right] d\Omega_j$$

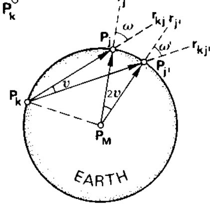


$$D_{kj} \simeq \pm 6\left(\frac{R}{r_{kj}}\right)^4 \cos\alpha_{kj}\,\frac{h_j - h_{T_j}}{R}, \quad \begin{cases} - : r_k\,r_{T_j} > r_{kj} \\ + : r_k\,r_{T_j} < r_{kj} \end{cases}$$

In literature no analogous formula exists

Analogous integral formula for λ:

$$\Delta\Phi_k \rightarrow \cos\varphi_k\,\Delta\Lambda_k$$
$$\Delta\varphi_k \rightarrow \cos\varphi_k\,\Delta\lambda_k$$
$$\cos\alpha_{kj} \rightarrow \sin\alpha_{kj}$$



See Notes

Figure 5.1.b

The form of Stokes's integral formula published in 1979 has been pictured in Fig. 5.1 in a somewhat simplified form, and under the assumption that the angular velocity of the earth is known with a sufficient accuracy. All quantities relate to form elements, those in potential theory are one-dimensional. ΔX denotes the difference measured (or computed) value minus approximate value of X.

However, the formula was not complete because a small error in the mathematics was discovered at the last moment. The correction was found but could not be incorporated. The reader is referred to the treatment of the correction term in the 1979 publication, Section 1.8, in particular formula (1.8.13) in the corrected form by P.J.G. Teunissen (1980). Recently, recognizable "reduction formulas" were found by incorporating the correction term in the solution of the basic integral equation and by applying the method of appraisal which was also used in Section 1.8 mentioned before. In Fig. 5.1 the reduction quantities have been denoted by the symbol Δ, and encircled for clarity.

Besides the figure shows examples of the - always somewhat approximative - **interpretation** of combinations of estimators or measured quantities as "free air reduction" to the "geoid" through P_1. This possibility of interpretation applies to all integral formulas of physical geodesy which I came across and brought into an analogous form. Mass and volume of the earth are eliminated by the introduction of the form elements. It thus appears that the conditions imposed on quantities in earlier derivations of integral formulas, such as Stokes's, are automatically fulfilled in the present formulation. There remains the effect of the topography which has received so much attention in the literature. The newly added encircled correction terms now can be interpretated partly as a "topographic reduction" of gravity ratios, and partly as the so-called "indirect effect" of the omission of the topographic masses outside the "geoid". The correction terms have been elaborated in a (rather rough) approximation in order to facilitate the recognition of similar formulas in the literature. The terms are functions of the slope of the terrain in P_j, as is clear from the formulas shown, but this has been lost in the derivation of existing formulas where the terrain slope has been replaced by the much less harmful slope of $\overrightarrow{P_k P_j}$. The encircled terms, to be introduced in an iteration process, must consequently have an influence that is much more disturbing than has been assumed so far. If the approach followed here is acceptable, geodesy has got rid of the - often mystical - considerations on the treatment of reductions of observations in physical geodesy.

The encircled disturbing term in the Vening Meinesz-like formula proves to have no recognizable counterpart in the existing literature. The fourth power in D_{kj}, compared with the third power in C_{kj}, makes the effect of the disturbing term much more harmful for short distances $\overrightarrow{P_k P_j}$.

But the present formula meets the difficulty that astronomical quantities ϕ and Λ are usually defined in an other system than the geodetic quantities φ and λ, namely by the introduction of differences of quantities. However, if from Vening Meinesz' formula φ- and λ-differences are estimated, similar to the estimation of radius ratios from Stokes's formula, then the question remains if the accuracy of φ- and λ-differences is not much better when taken from geometric networks. Besides, the always awkward computation of the influence of topography should not be forgotten.

But a more important difficulty is that both formulas provide a determination relative to P_C, and not relative to the origin P_M of the operational X, Y, Z-system, such as for instance:

$$\Delta\left(\ln \frac{r_{Ck}}{r_{C1}}\right) = \Delta\left(\ln \frac{r_k}{r_1}\right) - \left(B_k^{(1)} - B_1^{(1)}\right)$$

$$\Delta\left(\varphi_{Ck} - \varphi_{C1}\right) = \Delta\left(\varphi_k - \varphi_1\right) + \left[\frac{\partial}{-\partial\varphi_k} B_k^{(1)} - \frac{\partial}{-\partial\varphi_1} B^{(1)}\right]$$

with unknown $B^{(1)}$- terms.

Now the same basic integral equation yields, apart from "Stokes", another integral formula which is sometimes named after Hotine, as well as its derivatives. The left hand sides are identical with "Stokes" and "Vening Meinesz" respectively, but **without** the $B^{(1)}$- terms. The right hand sides are:

$$\frac{1}{4\pi} \iint_\Omega S_{1,k;j}^{(n+1)} \left\{\left[\Delta\left(\ln \frac{g_j}{g_1}\right) - \frac{1}{2}\Delta_{1j}\right] + 2\Delta\left(\ln \frac{r_j}{r_1}\right)\right\} d\Omega_j$$

and their partial derivatives

$$\left(\frac{\partial}{-\partial\varphi_k} + \frac{\partial}{-\partial\varphi_1}\right) \text{ and } \left(\frac{\partial}{-\cos\varphi_k \cdot \partial\lambda_k} + \frac{\partial}{-\cos\varphi_1 \cdot \partial\lambda_1}\right)$$

But the difficulty here is that the radius ratios sought appear again on the right hand sides. In my 1979 publication the Hotine integral formula was therefore introduced for the determination of potential differences at sea, the radius ratios being taken from satellite altimetry.

Earlier I indicated already that the $B^{(1)}$-terms, too, could only be estimated via satellite geodesy. Why then should one not be consistent and relegate all vertical determination of points on earth to satellite geodesy? The same conclusion was already reached in Section 4.2. Summarizing this means that for the estimation of potential differences it then suffices to use the Hotine approach as follows:

Satellite altimetry, from an approximately circular orbit (in order to eliminate a length scale error) yields

$$\Delta\left(\ln \frac{r_k}{r_1}\right) \text{ and } \Delta\left(\ln \frac{r_j}{r_1}\right)$$

for points of the surface of the sea, the disturbing term Δ_{1j} being negligible in most cases.

Satellite positioning provides the same terms on the continents; for this, the choice of the coordinate system must be carefully heeded.

In this manner one can also establish connections between continental levellings separated by oceans, including the adjustment of potential differences resulting from the Hotine-type integral formula and those resulting from spirit levelling. The **advantage** is that the same integral formula is valid for the whole earth, the **disadvantage** remains that the earth has to be covered by a regular network of gravity measurements, a requirement which on practical grounds still cannot be met.

In this approach, the geoid occurs only in an indirect way, as the equipotential surface passing through P_1 in the analytical continuation of the external potential. It is seen as a mathematical fiction only. If a realization is desired, this can only be obtained approximately, e.g. by defining the value of the right hand sides of the Stokes- or Hotine integral formula as the radial determination of the geoid in the form $\Delta\left(\ln \frac{r_{k'}}{r_1}\right)$, P'_k being the projection of P_k on the "geoid". The right hand sides of the two forms of the Vening Meinesz integral formula then yield the relevant deviations of the vertical.

6.

In the extended S-system the (dimensionless) gravity potential in P_k becomes (with centrifugal potential C and mass of the earth μ:

$$\frac{W_k}{r_1 g_1} = \frac{C_k}{r_1 g_1} + \frac{\mu}{r_1^2 g_1} \frac{r_1}{r_k} \left(1 + \sum_{n=1}^{\infty} B_k^{(n)}\right), \quad \frac{\mu}{r_1^2 g_1} \simeq 1$$

Let a rotated X, Y, Z-system be denoted by the U, V, W-system. Then we introduce the dimensionless quantities:

$$\frac{-r_k \dfrac{\partial W_k}{\partial U_k}}{r_1 g_1} \underset{\text{say}}{=} \frac{r_k g_{k,U}}{r_1 g_1} = \frac{\dfrac{\partial W_k}{-\dfrac{1}{r_k} \partial U_k}}{r_1 g_1} \rightarrow$$

$$\rightarrow \frac{\dfrac{\partial W_k}{-\cos\varphi_k \cdot \partial \lambda_k}}{r_1 g_1}, \quad \frac{\dfrac{\partial W_k}{-\partial \varphi_k}}{r_1 g_1}, \quad \frac{\dfrac{\partial W_k}{-\partial \ln r_k}}{r_1 g_1}$$

$$\frac{r_k^2 \dfrac{\partial^2 W_k}{\partial U_k \partial V_k}}{r_1 g_1} \underset{\text{say}}{=} \frac{r_k^2 \Gamma_{k,UV}}{r_1 g_1}, \text{ etc.}$$

In order to practically eliminate the quantity $\frac{\mu}{r_1^2 g_1}$, the derivations always contain differences of quantities with respect to the datum point P_1, which are approximately zero (≤ 0.01 ?). An additional effect is that scale factors of measuring instruments can be largely eliminated, because of the near-sphericity of the earth. These difference quantities are:

$$\left(\frac{W_k}{r_1 g_1} - \frac{r_1}{r_k} \cdot \frac{W_1}{r_1 g_1}\right)$$

$$\left(\frac{r_k g_{k,U}}{r_1 g_1} - \frac{r_1}{r_k} \frac{r_1 g_{1,U}}{r_1 g_1}\right)$$

$$\left(\frac{r_k^2 \Gamma_{k,UV}}{r_1 g_1} - \frac{r_1}{r_k} \frac{r_1^2 \Gamma_{1,UV}}{r_1 g_1}\right), \quad \text{etc.}$$

The question now was what other consequences this S-System presented for (terrestrial) mechanics. To examine this, I considered the mechanization equations for the Local Level Inertial Surveying System and in these replaced the ellipsoid by a sphere, so that geographical coordinates were replaced by geocentric ones.
Let v = velocity; $\dot{v}$ = acceleration; f = specific force; ω_E = rotation of the earth; E, N, r the coordinate frame obtained by rotation of the X, Y, Z-frame and parallel to local East, North, radial; $\bar{v}_{k,E} = 2r_k \cos\varphi_k * \omega_E + v_{k,E}$; P_1 starting point with zero velocity and scale factor λ_f with $\lambda_f f_1 = g_1$ (f_1 and g_1 being moduli of vectors).
Then we have:

$$\begin{pmatrix} \frac{r_k \dot{v}_{k,E}}{r_1 g_1} \\ \frac{r_k \dot{v}_{k,N}}{r_1 g_1} \\ \frac{r_k \dot{v}_{k,r}}{r_1 g_1} \end{pmatrix} = \begin{pmatrix} \frac{r_k f_{k,E}}{r_1 f_1} - \frac{r_1}{r_k}\frac{r_1 f_{1,E}}{r_1 f_1} \\ \frac{r_k f_{k,N}}{r_1 f_1} - \frac{r_1}{r_k}\frac{r_1 f_{1,N}}{r_1 f_1} \\ \frac{r_k f_{k,r}}{r_1 f_1} - \frac{r_1}{r_k}\frac{r_1 f_{1,r}}{r_1 f_1} \end{pmatrix} +$$

$$- \begin{pmatrix} 0 & -\frac{\bar{v}_{k,E}}{\sqrt{r_1 g_1}} \tan\varphi_k & +\frac{\bar{v}_{k,E}}{\sqrt{r_1 g_1}} \\ +\frac{\bar{v}_{k,E}}{\sqrt{r_1 g_1}} \tan\varphi_k & 0 & +\frac{v_{k,N}}{\sqrt{r_1 g_1}} \\ -\frac{\bar{v}_{k,E}}{\sqrt{r_1 g_1}} & -\frac{v_{k,N}}{\sqrt{r_1 g_1}} & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{v_{k,E}}{\sqrt{r_1 g_1}} \\ \frac{v_{k,N}}{\sqrt{r_1 g_1}} \\ \frac{v_{k,r}}{\sqrt{r_1 g_1}} \end{pmatrix} +$$

$$+ \begin{pmatrix} \frac{r_k g_{k,E}}{r_1 g_1} - \frac{r_1}{r_k}\frac{r_1 g_{1,E}}{r_1 g_1} \\ \frac{r_k g_{k,N}}{r_1 g_1} - \frac{r_1}{r_k}\frac{r_1 g_{1,N}}{r_1 g_1} \\ \frac{r_k g_{k,r}}{r_1 g_1} - \frac{r_1}{r_k}\frac{r_1 g_{1,r}}{r_1 g_1} \end{pmatrix}$$

This equation has been arranged in such a way that the gravity difference quantities mentioned previously appeared as such.
Hence it appears that new dimensionless quantities fit in this scheme:

$$\frac{r_k \dot{v}_{k,u}}{r_1 g_1} \text{ and } \frac{v_{k,u}}{\sqrt{r_1 g_1}}$$

The integration process then requires the introduction of the dimensionless time quantity:

$$(t_k - t_1)\sqrt{\frac{g_1}{r_1}}$$

It follows that the units of length and time must agree in the realisation of instruments measuring r, g, v, $\dot{v}$ and t. The measure of agreement determines the reliability of computed results, see analogous considerations in Section 2 of my 1979 publication. The choice of the S-system is thus shown to have far-reaching consequences.

In the approach presented, the following holds:

$$\dot{v}_{1,U} = v_{1,U} = 0$$

An unanswered question so far is if, more generally, the introduction of difference quantities

$$\left(\frac{r_k \dot{v}_{k,U}}{r_1 g_1} - \frac{r_1}{r_k}\frac{r_1 \dot{v}_{1,U}}{r_1 g_1}\right) \text{ and } \left(\frac{v_{k,U}}{\sqrt{r_1 g_1}} - \frac{r_1}{r_k}\frac{v_{1,U}}{\sqrt{r_1 g_1}}\right)$$

is meaningful.
We list some orders of magnitude:

$$R = 6.4 * 10^3 \text{km}, \qquad \sqrt{\frac{G}{R}} = 1.25 * 10^{-3} s^{-1} = 4.5 h^{-}$$

$$G = 10^{-2} \text{km } s^{-2} = 1.3 * 10^5 \text{km} h^{-2}, \quad \sqrt{RG} = 8 \text{km } s^{-1} = 3.10^4 \text{km} h^{-1}$$

7.

7.1

The positioning of points on earth by means of satellite observations is characterized by three types of vector bundles: radial vectors of points on the surface of the earth, the vectors of such points to a passing satellite and the radial vectors of points of the satellite orbit. The first bundle can by means of vector ratios be fitted into the terrestrial quaternion theory. The second type originally concerned the measurement of directions, and now mainly distance measurement in various forms, where again the division of vectors can eliminate instrumental scale uncertainties and in particular scale differences between the measuring instrument and terrestrial S-systems. The third type proves to have some remarkable properties which make it desirable to change over to vector ratios.
This explains why an application of the theory of quaternions can clarify many aspects of this method of positioning.
If an attempt in this direction is made, one is, however, confronted with the difficulty - already mentioned in Section 2 - that a quaternion is only in a limited way invariant with respect to the choice of the coordinate frame: the unit vector must be described in an operationally defined coordinate frame.
For the first type of vector bundle this is no problem, here a terrestrial S-system such as our X, Y, Z-system is all we need. For the second type there is no problem if only distances are measured. But for the third type there really is a problem, i.e. the description of inertial space in an operationally defined coordinate frame.

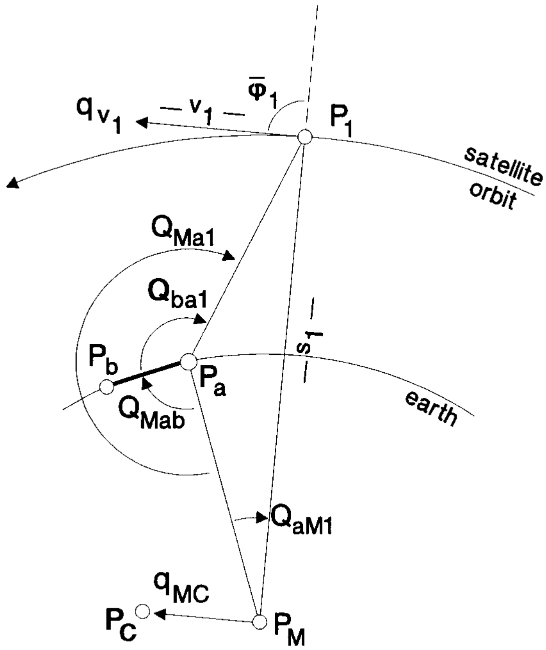
In an attempt to define the latter frame, we will sketch the main features of the determination of satellite orbits. The reader is asked to keep in mind that the author is not an expert in this field, but an attentive spectator who is interested in the connection between methods. In order to avoid confusion with the foregoing discussion of terrestrial situations, for the radial distances to points of the satellite orbit the kernel letter r will be replaced by s.

When a satellite is launched, its orbit is determined by two start vectors, the s_1-vector and the v_1-vector. Together they are customarily called the initial statevector. In principle it has to be assumed that both vectors are by measurement determined in a terrestrial S-system, e.g. our X, Y, Z-system. The s_1-vector is obtained by adding to the radiusvector of the launch point the vector launch point - satellite start point. But this implies that the s_1-vector has its initial point in P_M and not in the centre of mass of the earth, P_C. The consequence is that, for the computation of the satellite orbit according to the usual methods, the s_1-vector, the angle $\bar{\varphi}_1$ between s_1- and v_1-vectors, as well as the spatial orientation of the plane of the two vectors must be corrected for the eccentric position of P_M (being the origin

of the terrestrial X, Y, Z-frame) with respect to P_C. These corrections are unknown and consequently have to be introduced into the computation as unknown quantities. It will have to be investigated to what extent they are estimable.
It is always possible to choose our X, Y, Z-frame so that it is reasonably parallel to the so-called terrestrial geocentric conventional frame. Now assume that the earth rotates around the momentaneous axis with respect to an inertial space.
Denote by $(X,Y,Z)_{t_1}$ the position of the X, Y, Z-frame at the launching time, $(X_C,Y_C,Z_C)_{t_1}$ being the components of the r_C-vector.
Now $(X,Y,Z)_{t_1}$ is subjected to a number of theoretically defined (hence non-stochastic) rotations, resulting in a $\bar{X},\bar{Y},\bar{Z}$-frame - with $(\bar{X}_C,\bar{Y}_C,\bar{Z}_C)$ as the components of the r_C-vector - which describes the inertial space. One sees that the coordinate frame remains defined, because it can be transformed back to the X, Y, Z-frame on account of the transformations mentioned. It is important to recall our statement that the coordinate frame is part of the consistent set of approximate values. For in the course of the computation many stochastic corrections will be introduced, such as the r_C-vector, the correction of the momentaneous rotation vector etc.; this does not impair the definition of the frame.

Now assume that the modulus of the r_C-vector is smaller than $10^{-5}R$, then its influence on the so-called perturbation terms in the orbit computation will be smaller than 10^{-8}, which we will assume to be negligible. However, the computation of the Kepler-ellipse is clearly influenced, so that the vector bundle of the third type is deformed.
Vector bundles of the second type are invariant with respect to the introduction of a r_C-vector. But if (after execution of the necessary rotations) one wishes to fit the three types of vector bundles together, then the first type of vector bundle, too, has to be corrected for the r_C-vector. For a terrestrial point P_k this means the introduction of ($X_k - X_C$, $Y_k - Y_C$, $Z_k - Z_C$), so that the whole covariance matrix of the coordinates is changed and we have lost our S-system. In principle this is the same situation as the one we met in Section 3; again one will have to apply an S-transformation, e.g. by introducing 7 coordinates unchanged in numerical value, but non-stochastic. The origin P_M of this new coordinate frame then becomes an estimate of the centre of mass of the earth, P_C. Just like in Section 3, the estimated position of P_C is fixed relative to the datum points chosen, as well as the three types of vector bundles. In this line of thought the earth seemingly shifts because the frames are held fixed, but this only appears to be so: after appropriate rotation, vector bundles may be subjected to parallel translations for the purpose of fitting them together; the only important thing is the relative position with respect to terrestrial datum points.

7.2



Let a satellite be launched from the point P_a and let it enter its orbit at the point P_1 at time t_1. The positioning of P_1 , like the determination of the velocity vector q_{v_1} can only be effected in the local (regional or sub-continental) terrestrial coordinate frame with the origin P_M not coinciding with P_C , the centre of mass of the earth. Apart from measurements, one needs at least seven coordinates of three coordinated stations.
Assume that measurements and computations result in the quaternion [1] :

$$Q_{ba1} = q_{a1} q_{ab}^{-1} \quad , \quad q_{ik} = \overrightarrow{P_i P_k}$$

Then the relative positioning of P_1 with respect to P_a is:

$$q_{a1} = Q_{ba1} \cdot q_{ab}$$

From coordinates one can compute the quaternion:

$$Q_{Mab} = q_{ab} q_{aM}^{-1}$$

Then follows the computation of the quaternion:

[1] See "Notes and References Section 3"

7.

$$Q_{Ma1} = Q_{ba1} Q_{Mab}$$

By writing out in vectors it is easily verified that in the triangle $P_M P_a P_1$:

$$Q_{Ma1} + Q_{aM1} = 1$$

hence: $$Q_{aM1} = q_{M1} q_{Ma}^{-1} = 1 - Q_{Ma1}$$

or: $$\boxed{q_{M1} = Q_{aM1} \cdot q_{Ma}}$$

However what is wanted is not q_{M1} , but q_{C1} . Theoretically this is simple:

$$q_{C1} = q_{M1} - q_{MC}$$

or: $$q_{C1} = q_{M1}\left(1 - q_{M1}^{-1} q_{MC}\right)$$

Denoting the norm by N one has, see Section 3:

$$N^{1/2}\left\{q_{M1}^{-1} q_{MC}\right\} \leq 10^{-5}$$

(according to estimations in the literature, the value for the USA coordinate system is perhaps $\simeq 2.10^{-7}$).
However, since q_{MC} is unknown one has to work with q_{M1} instead of q_{C1} . This will also influence the angle $\bar{\varphi}_1$ between q_{v_1} and q_{M1} (but not q_{v_1} itself) and the inclination of the Kepler-ellipse plane through q_{v_1} and q_{M1} .

In order to quantify this effect we consider the quaternion:

$$Q_{v1M} = q_{1M} q_{v_1}^{-1} =$$

$$= \frac{v_1}{s_{1M}} \left(\cos\left(\bar{\varphi}_1 + \pi\right) + e_{v1M} \sin\left(\bar{\varphi}_1 + \pi\right)\right) =$$

$$= - \frac{v_1}{s_{M1}} \left(\cos\bar{\varphi}_1 + e_{v1M} \sin\bar{\varphi}_1\right)$$

Then the following is valid:

$$\Delta\Pi_{v1M} = q_{1M}^{-1}\Delta q_{1M} - q_{v_1}^{-1}\Delta q_{v_1}$$

With: $q_{M1} = - q_{1M}$, $\Delta q_{1M} = -\Delta q_{M1} = q_{MC}$, $\Delta q_{v_1} = 0$, $\Delta\Pi_{v1M}$ provides:

$$-q_{M1}^{-1} q_{MC} = \Delta(\ln s_{M1}) - e_{v1M}\Delta(\bar{\varphi}_1) + \frac{s_{1M}}{v_1}\sin(\bar{\varphi}_1 + \pi) q_{1M}^{-1}\Delta(e_{v1M}) q_{v_1}$$

or with $\boxed{\bar{\varphi} \simeq \frac{\pi}{2}}$, $s_{1M} q_{1M}^{-1} = -e_{M1}^{-1}$, $\frac{1}{v_1} q_{v_1} = e_{v_1}$:

$$\boxed{-q_{M1}^{-1} q_{MC} = \Delta(\ln s_{M1}) - e_{v1M}\Delta(\bar{\varphi}_1) + e_{M1}^{-1}\Delta(e_{v1M}) e_{v_1}}$$

From the general solution for components of $\Delta\Pi_{v1M}$ follows:

$$\Delta(\ln s_{M1}) = Sc\left\{-q_{M1}^{-1} q_{MC}\right\} = Sc\left\{-\frac{q_{M1}^{T} q_{MC}}{s_{M1} s_{M1}}\right\} = Sc\left\{\frac{q_{M1} q_{MC}}{s_{M1} s_{M1}}\right\}$$

In the special case always assumed here, viz. $\bar{\varphi}_1 \simeq \frac{\pi}{2}$, - hence an almost circular satellite orbit - we deviate from the general solution. Then we have for the coefficients in the difference equation:

$$Q_{v1M} = -q_{M1} q_{v_1}^{-1} \simeq -\frac{s_{M1}}{v_1} e_{v1M}$$

or:

$$e_{M1} e_{v_1}^{-1} \simeq e_{v1M}$$

With this, the difference equation gives:

$$-e_{M1} q_{M1}^{-1} q_{MC} e_{v_1}^{-1} = e_{v1M}\Delta(\ln s_{M1}) - \Delta(\bar{\varphi}_1) + \Delta(e_{v1M})$$

hence:

$$\Delta(\bar{\varphi}_1) = Sc\left\{e_{M1} q_{M1}^{-1} q_{MC} e_{v_1}^{-1}\right\} = Sc\left\{-\frac{q_{MC} q_{v_1}}{s_{M1} v_1}\right\}$$

and with:

$$e_{\nu 1M} \simeq e_{M1} e_{\nu_1}^{-1} = -\frac{q_{M1} q_{\nu_1}}{s_{M1} \nu_1} ,$$

$$\Delta(e_{\nu 1M}) = Ve\left\{+\frac{q_{MC} q_{\nu_1}}{s_{M1} \nu_1}\right\} + Ve\left\{+\frac{q_{M1} q_{\nu_1}}{s_{M1} \nu_1}\right\} \Delta(\ln s_{M1})$$

The formulas can now be written out in coordinates, during which the index M can be omitted because the origin of the coordinate system is P_M. An index C is added in order to indicate that the corrections obtained are caused by $q_{MC} \neq 0$. The result, see Section 4.3, is:

$$\boxed{\begin{aligned}
\Delta_C(\ln s_{M1}) &= -\left(\frac{X_1}{s_1}\frac{X_C}{s_1} + \frac{Y_1}{s_1}\frac{Y_C}{s_1} + \frac{Z_1}{s_1}\frac{Z_C}{s_1}\right) = -B_1^{(1)} \\
\Delta_C(\bar{\varphi}_1) &= +\left(\frac{\dot{X}_1}{\nu_1}\frac{X_C}{s_1} + \frac{\dot{Y}_1}{\nu_1}\frac{Y_C}{s_1} + \frac{\dot{Z}_1}{s_1}\frac{Z_C}{s_1}\right) \\
\Delta_C(e_{\nu 1M}) &= \frac{1}{s_1 \nu_1}\left(\begin{vmatrix} e_X & e_Y & e_Z \\ X_C & Y_C & Z_C \\ \dot{X}_1 & \dot{Y}_1 & \dot{Z}_1 \end{vmatrix} - B_1^{(1)} \begin{vmatrix} e_X & e_Y & e_Z \\ X_1 & Y_1 & Z_1 \\ \dot{X}_1 & \dot{Y}_1 & \dot{Z}_1 \end{vmatrix} \right) \\
& \boxed{\bar{\varphi}_1 \simeq \frac{\pi}{2}}
\end{aligned}}$$

From the original difference equation follows, again with $e_{\nu 1M} = e_{M1} e_{\nu_1}^{-1} = e_{M1}^{-1} e_{\nu_1}$

$$Ve\left\{-q_{M1}^{-1} q_{MC}\right\} = -\frac{1}{2}\left(q_{M1}^{-1} q_{MC} - q_{MC} q_{M1}^{-1}\right) =$$

$$= e_{M1}^{-1}\left(-\Delta(\bar{\varphi}_1) + \Delta(e_{\nu 1M})\right) e_{\nu_1}$$

or:

$$-\Delta(\bar{\varphi}_1) + \Delta(e_{v1M}) = -\frac{1}{2s_{M1}} e_{M1}\left(e_{M1}^{-1} q_{MC} - q_{MC} e_{M1}^{-1}\right) e_{v_1}^{-1} =$$

$$= -\frac{1}{2s_{M1}} \underbrace{\left(q_{MC} - e_{M1} q_{MC} e_{M1}^{-1}\right)}_{2(\text{component } q_{MC} \perp q_{M1})} e_{v_1}^{-1}$$

$$\Delta(e_{v1M}) = Ve\left\{-\frac{1}{2S_{M1}}\left(q_{MC} - e_{M1} q_{MC} e_{M1}^{-1}\right) e_{v_1}^{-1}\right\}$$

= product of a vector $\perp q_{M1}$ and a vector almost $\perp q_{M1}$

hence:

$$\boxed{\Delta_C(e_{v1M}) \text{ almost } // \; q_{M1}}$$

In the satellite orbit computations one is all the time dealing with vectors q_{Ck}, although in fact one computes with q_{Mk} because q_{MC} is unknown. It is therefore convenient to write the formulas using q_{Ck} and **afterwards** make the substitution:

$$\boxed{\begin{aligned}
q_{C1} &= q_{M1} + \Delta_C(q_{M1}), \quad \Delta_C(q_{M1}) = -q_{MC} \\
q_{C1} &= q_{M1}\left(1 + q_{M1}^{-1} \Delta_C(q_{M1})\right) = q_{M1}\left(1 - q_{M1}^{-1} q_{MC}\right) \\
s_{C1} &= s_1\left(1 + \Delta_C(\ln s_1)\right) \\
\ln s_{C1} &= \ln s_1 + \Delta_C(\ln s_1) \\
\bar{\varphi}_{C1} &= \bar{\varphi}_1 + \Delta_C(\bar{\varphi}_1) \\
e_{v1C} &= e_{v1M} + \Delta_C(e_{v1M})
\end{aligned}}$$

7.3

A second problem, which we shall treat in an analogue way by substitution, is the earth rotation with respect to inertial space. Let q be a vector, defined in the terrestrial X, Y, Z-frame; then the vector rotated over an angle ν in the positive direction is:

$$p \; q \; p^{-1}$$

in which p is the rotation quaternion with Norm 1:

$$p = \cos\frac{\nu}{2} + e\sin\frac{\nu}{2}$$

e being the unit vector of the momentary axis of rotation of the earth, directed towards the North Pole.

It is customary to consider the earth as non-rotating, and therefore let inertial space, described in the $\bar{X}, \bar{Y}, \bar{Z}$-frame, rotate in the opposite direction.

Let $\bar{q}$ be the vector, described in the $\bar{X}, \bar{Y}, \bar{Z}$-frame, then:

$$q = p^{-1}\bar{q}p$$

with:

$$p^{-1} = \cos\frac{\nu}{2} - e\sin\frac{\nu}{2}$$

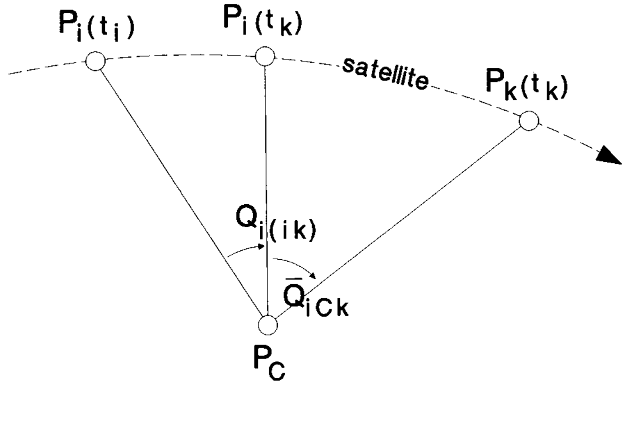
Let $\bar{\omega}$ be the angular velocity of the earth, t_1 the time of launching the satellite, t_i and t_k the points of time when it passes P_i and P_k respectively, then:

$$q_{Ck} = p_k^{-1}\bar{q}_{Ck}p_k \quad , \quad q_{Ci} = p_i^{-1}\bar{q}_{Ci}p_i$$

$$p_k^{-1} = \cos\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right) - e_p \sin\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right)$$

$$p_i^{-1} = \cos\left(\frac{\bar{\omega}}{2}(t_i - t_1)\right) - e_p \sin\left(\frac{\bar{\omega}}{2}(t_i - t_1)\right)$$

The quaternion which can be used for computations in the X, Y, Z-system is then:



$$
\begin{aligned}
Q_{iCk} &= \left(p_k^{-1}\bar{q}_{Ck}\,p_k\right)\left(p_i^{-1}\bar{q}_{Ci}\,p_i\right)^{-1} \\
&= p_k^{-1}\left(\bar{q}_{Ck}\,\bar{q}_{Ci}^{-1}\right)\bar{q}_{Ci}\,p_k\left(p_i^{-1}\bar{q}_{Ci}\,p_i\right)^{-1} \\
&= \left(p_k^{-1}\bar{Q}_{iCk}\,p_k\right)\Big[\underbrace{\left(p_k^{-1}\bar{q}_{Ci}\,p_k\right)\left(p_i^{-1}\bar{q}_{Ci}\,p_i\right)^{-1}}_{\text{put } Q_{i(ik)}}\Big]
\end{aligned}
$$

or, the quaternion $\bar{Q}_{iCk}$ computed in inertial space is first rotated about e_p over the angle $-\frac{\bar{\omega}}{2}\left(t_k - t_i\right)$, and then postmultiplied by a quaternion having Norm 1, $Q_{i\,(ik)}$.

It consequently appears that some of the elegance of working with quaternions is lost, because primarily it is the vectors that rotate and not the quaternions. However, for the difference equations the effect is small and perhaps negligible if the terrestrial measurements concern length ratios.1

Let $\Delta\Lambda = q^{-1}\Delta q$, then:

$$
\begin{aligned}
\Delta\Pi_{iCk} &= \Delta\Lambda_{Ck} - \Delta\Lambda_{Ci} = q_{Ck}^{-1}\Delta q_{Ck} - q_{Ci}^{-1}\Delta q_{Ci} = \\
&= p_k^{-1}\,\bar{q}_{Ck}^{-1}\,p_k\,p_k^{-1}\,\Delta\bar{q}_{Ck}\,p_k - p_i^{-1}\,\bar{q}_{Ci}^{-1}\,p_i\,p_i^{-1}\,\Delta\bar{q}_{Ci}\,p_i = \\
&= p_k^{-1}\,\bar{q}_{Ck}^{-1}\,\Delta\bar{q}_{Ck}\,p_k - p_i^{-1}\,\bar{q}_{Ci}^{-1}\,\Delta\bar{q}_{Ci}\,p_i
\end{aligned}
$$

or:

$$
\begin{aligned}
\Delta\Pi_{iCk} &= p_k^{-1}\,\Delta\bar{\Lambda}_{Ck}\,p_k - p_i^{-1}\,\Delta\bar{\Lambda}_{Ci}\,p_i \\
&= p_k^{-1}\,\Delta\bar{\Pi}_{iCk}\,p_k - \Delta\Pi_{i(ik)} \\
\Delta\Pi_{i(ik)} &= p_k^{-1}\,\Delta\bar{\Lambda}_{Ci}\,p_k - p_i^{-1}\,\Delta\bar{\Lambda}_{Ci}\,p_i \\
&= p_i^{-1}\left(p_i\,p_k^{-1}\,\Delta\bar{\Lambda}_{Ci}\,p_k\,p_i^{-1} - \Delta\bar{\Lambda}_{Ci}\right)p_i
\end{aligned}
$$

Now it follows from the definitions of p_i and p_k^{-1} that:

$$
\left(p_k p_i^{-1}\right)^{-1} = \cos\left(\frac{\bar{\omega}}{2}\left(t_k - t_1\right)\right) - e_p\,\sin\left(\frac{\bar{\omega}}{2}\left(t_k - t_i\right)\right)
$$

or, for small $\left(t_k - t_i\right)$ follows $p_i\,p_k^{-1} \simeq 1$ and hence $\Delta\Pi_{i(ik)} \simeq 0$.

It is therefore possible to compute in the *X*, *Y*, *Z*-system and make substitutions afterwards:

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$$\boxed{\begin{aligned} \Delta_p\, \Pi_{iCk} &= p_k^{-1}\, \Delta\bar{\Pi}_{iCk}\, p_k \;-\; \Delta\Pi_{i(ik)} \\ \Delta\Pi_{i(ik)} &= p_k^{-1}\, \Delta\bar{\Lambda}_{Ci}\, p_k \;-\; p_i^{-1}\, \Delta\bar{\Lambda}_{Ci}\, p_i \\ &\simeq 0 \text{ for } (t_k - t_i) \text{ small} \end{aligned}}$$

For completeness we shall also investigate **the effect of Δ *p*.**

$$\Delta q_{Ck} = \Delta\left(p_k^{-1}\,\bar{q}_{Ck} p_k\right) = -p_k^{-1}\,\Delta p_k q_{Ck} + q_{Ck} p_k^{-1}\,\Delta p_k$$

or:

$$\Delta\Lambda_{Ck} = q_{Ck}^{-1}\,\Delta q_{Ck} = -q_{Ck}^{-1}\left(p_k^{-1}\,\Delta p_k\right) q_{Ck} + \left(p_k^{-1}\,\Delta p_k\right)$$

$$\Delta p_k = \left[-\sin\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right) + e_p\,\cos\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right)\right]\Delta\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right) +$$

$$+ \sin\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right)\Delta\left(e_p\right)$$

hence:

$$\boxed{p_k^{-1}\,\Delta p_k = e_p\,\Delta\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right) + \sin\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right) p_k^{-1}\,\Delta\left(e_p\right)}$$

$$= e_p\,\Delta\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right) + \sin\left(\bar{\omega}(t_k - t_1)\right)\Delta\left(e_p\right) +$$

$$+ \sin^2\left(\frac{\bar{\omega}}{2}(t_k - t_1)\right) e_p^{-1}\,\Delta\left(e_p\right)$$

This outcome is the sum of the three components of the **vector** $\left(p_k^{-1}\,\Delta p_k\right)$, because e_p , $\Delta\left(e_p\right)$ and $e_p^{-1}\,\Delta\left(e_p\right)$ are mutually perpendicular.
Hence also:

$$\boxed{\Delta\Lambda_{Ck} = 2\left(\text{component } \left(p_k^{-1}\,\Delta p_k\right) \perp q_{Ck}\right)}$$

It follows, with: $\Delta\Pi_{iCk} = \Delta\Lambda_{Ck} - \Delta\Lambda_{Ci}$ that:

$$\boxed{\begin{aligned}\Delta_{\Delta p}\,\Pi_{iCk} &= \left[\left(p_k^{-1}\,\Delta p_k\right) - q_{Ck}^{-1}\left(p_k^{-1}\,\Delta p_k\right)\, q_{Ck}\right] + \\ &\quad -\left[\left(p_i^{-1}\,\Delta p_i\right) - q_{Ci}^{-1}\left(p_i^{-1}\,\Delta p_i\right)\, q_{Ci}\right]\end{aligned}}$$

Now write q_{Ck} in the form:

$$q_{Ck} = p_i^{-1}\left(p_k p_i^{-1}\right)^{-1} \bar{q}_{Ck}\left(p_k p_i^{-1}\right) p_i$$

then:

$$\begin{aligned}\Delta\Lambda_{Ck} &= \left[\left(p_i^{-1}\,\Delta p_i\right) - q_{Ck}^{-1}\left(p_i^{-1}\,\Delta p_i\right)\, q_{Ck}\right] + \\ &\quad + \left[p_i^{-1}\left(p_k p_i^{-1}\right)^{-1}\,\Delta\left(p_k p_i^{-1}\right) p_i - q_{Ck} p_i^{-1}\left(p_k p_i^{-1}\right)^{-1}\,\Delta\left(p_k p_i^{-1}\right) p_i q_{Ck}^{-1}\right]\end{aligned}$$

with:

$$\left(p_k p_i^{-1}\right)^{-1}\,\Delta\left(p_k p_i^{-1}\right) = e_p\,\Delta\left(\frac{\bar{\omega}}{2}\left(t_k - t_i\right)\right) + \sin\left(\frac{\bar{\omega}}{2}\left(t_k - t_i\right)\right)\left(p_k p_i^{-1}\right)^{-1}\,\Delta\left(e_p\right)$$

For small $\left(t_k - t_i\right)$ we consequently have:

$$p_k p_i^{-1} \simeq 1 \quad , \qquad \left(p_k p_i^{-1}\right)^{-1}\,\Delta\left(p_k p_i^{-1}\right) \simeq 0$$

hence:

$$\begin{aligned}\Delta_{\Delta p}\Pi_{iCk} &\simeq \left[\left(p_i^{-1}\,\Delta p_i\right) - q_{Ck}^{-1}\left(p_i^{-1}\,\Delta p_i\right) q_{Ck}\right] - \left[\left(p_i^{-1}\,\Delta p_i\right) - q_{Ci}^{-1}\left(p_i^{-1}\,\Delta p_i\right) q_{Ci}\right] \\ &= \text{difference of } \left(\text{component } \left(p_i^{-1}\,\Delta p_i\right) \perp q_{Ck}\right) \text{ and } \left(\text{component } \left(p_i^{-1}\,\Delta p_i\right) \perp q_{Ci}\right)\end{aligned}$$

or:

$$\boxed{\Delta_{\Delta p}\,\Pi_{iCk} \simeq 0 \quad \text{for } \left(t_k - t_i\right) \text{ small}}$$

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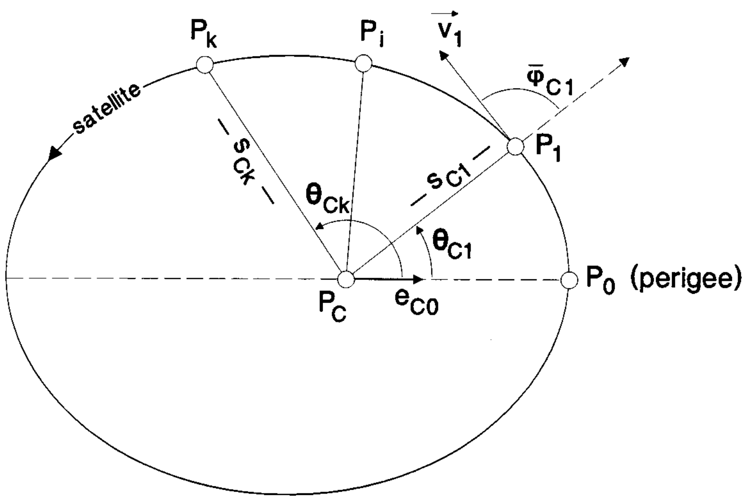
8.1



If all perturbations are left out of consideration, the computation of a satellite orbit is determined by the gravitational potential:

$$V_k = \frac{\mu}{s_{Ck}}\left(1 + \sum_{n=2}^{\infty} B_k^{(n)}\right) \quad \text{in } X, Y, Z\text{-frame}$$

There are no first degree terms in the spherical harmonics expansion because the theory is geocentric, i.e. in the **theory** it is assumed that the origin of the coordinate frame coincides with the centre of mass of the earth P_C. As far as the theory itself is concerned this can be true. Our doubt about this assumption can therefore only concern the **linking-up** of the theory, i.e. the coupling of measurements and theory, which will be treated in the next section.

The theory is classic and will not be questioned, but the formulas will be rewritten in a self-willed form because the author does not believe in absolute lengths and orientations. The argumentation is mainly restricted to the zero degree term in V_k, i.e. we shall investigate the computation of the first order terms of the satellite orbit from the vectors q_{C1} and $q_{v_1} = \vec{v_1}$, resulting in the orientation and the shape of the Kepler ellipse.

Let us here make the reasonable assumption that the relevant seven (component-) quantities are not very well known to begin with, or cannot be determined very well. In the further computations, one consequently has to introduce corrections, the question is: how to do this?

In order to keep the formula system transparent we shall assume the satellite orbit to be approximately circular, with the same approximation as used when the figure of the earth is considered as a sphere. This implies that in the difference equations one may put:

$$\bar{\varphi}_{C1} \simeq \frac{\pi}{2} \;, \quad e^2 \simeq 0 \;, \quad s_k \simeq s_1 \simeq a$$

The thus simplified difference equations, to be used for later interpretation, are marked by framing them.

The **semi-major axis *a*** of the Kepler ellipse follows from the vis viva integral

$$v_1^2 - \mu\left(\frac{2}{s_{C1}} - \frac{1}{a}\right) = 0$$

or:

$$\frac{\mu}{s_{C1} v_1^2}\left(2 - \frac{s_{C1}}{a}\right) = 1 \;, \quad \text{hence} \quad \frac{\mu}{s_{C1} v_1^2} \simeq 1 \quad \text{for } \bar{\varphi}_{C1} \simeq \frac{\pi}{2}$$

$$\Delta\left(\ln \frac{a}{s_{C1}}\right) \simeq -\frac{2a - s_{C1}}{s_{C1}}\,\Delta\left(\ln \frac{\mu}{s_{C1} v_1^2}\right)$$

$$\boxed{\Delta\left(\ln \frac{a}{s_{C1}}\right) \simeq -\,\Delta\left(\ln \frac{\mu}{s_{C1} v_1^2}\right)}$$

It is seen that a dimensionless mass of the earth $\dfrac{\mu}{s_{C1} v_1^2}$ is introduced; compare the analogous quantity $\dfrac{\mu}{r_1^2 g_1}$ which was introduced in the terrestrial gravimetric theory. As was done there, one must here modify the formula for the gravitational potential:

$$\frac{V_k}{v_1^2} = \frac{\mu}{s_{C1} v_1^2}\,\frac{s_{C1}}{s_{Ck}}\left(1 + \sum_{n=2}^{\infty} \bar{B}_k^{(n)}\right) , \qquad \frac{s_{Ck} V_k}{s_{C1} v_1^2} \simeq 1 \text{ for } \bar{\varphi}_{C1} \simeq \frac{\pi}{2}$$

in which $\bar{B}_k^{(n)}$ is the symbolic notation for the transformation of $B_k^{(n)}$ from the (X, Y, Z)-frame to the $(\bar{X}, \bar{Y}, \bar{Z})$-frame.

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The difference equation becomes:

$$\boxed{\begin{aligned}\Delta\ln\left(\frac{V_k}{v_1^2}\right) &= \Delta(\ln V_k) - 2\Delta(\ln v_1) = \\ &= \Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right) - \Delta\left(\ln\frac{s_{Ck}}{s_{C1}}\right) + \sum_{n=2}^{\infty}\Delta\left(\bar{B}_k^{(n)}\right)\end{aligned}}$$

The **mean angular velocity** (mean motion) $\bar{n}$, as a dependent quantity, follows from $\bar{n} = \left(\frac{\mu}{a^3}\right)^{½}$. Or rewritten as a dimensionless quantity:

$$\frac{s_{C1}\bar{n}}{v_1} = \left(\frac{\mu}{s_{C1}v_1^2}\left(\frac{a}{s_{C1}}\right)^{-3}\right)^{½}, \qquad \frac{s_{C1}\bar{n}}{v_1} \simeq 1 \text{ for } \bar{\varphi}_{C1} \simeq \frac{\pi}{2}$$

$$\begin{aligned}\Delta\left(\ln\frac{s_{C1}\bar{n}}{v_1}\right) &= \frac{1}{2}\left[\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right) - 3\Delta\left(\ln\frac{a}{s_{C1}}\right)\right] = \\ &= \frac{3a - s_{C1}}{s_{C1}}\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right)\end{aligned}$$

$$\boxed{\Delta\left(\ln\frac{s_{C1}\bar{n}}{v_1}\right) \simeq 2\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right)}$$

The (first) eccentricity ***e*** follows from:

$$\frac{s_{C1}}{a}\left(2 - \frac{s_{C1}}{a}\right)\sin^2\bar{\varphi}_{C1} = 1 - e^2, \qquad e^2 \simeq 0 \text{ for } \bar{\varphi}_{C1} \simeq \frac{\pi}{2}$$

or:

$$\left(\frac{\mu}{s_{C1}v_1^2}\right)^{-1}\sin^2\bar{\varphi}_{C1} = \frac{a}{s_{C1}}(1 - e^2)$$

$$2e\Delta(e) = \Delta(e^2) = -2(1-e^2)\left[\frac{a - s_{C1}}{s_{C1}}\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right) + \cot\bar{\varphi}_{C1}\Delta(\bar{\varphi}_{C1})\right]$$

$$\boxed{\Delta(e^2) \simeq 0}$$

The true anomaly θ_{C1} follows from:

$$\begin{cases} \dfrac{a}{s_{C1}}(1-e^2) - (1 + e\cos\theta_{C1}) = 0 \\ \dfrac{a}{s_{C1}}(1-e^2)\cos\bar{\varphi}_{C1} - e\sin\theta_{C1} = 0 \end{cases}$$

or:

$$\begin{cases} \left(\dfrac{\mu}{s_{C1}v_1^2}\right)^{-1}\sin^2\bar{\varphi}_{C1} = 1 + e\cos\theta_{C1} \\ \left(\dfrac{\mu}{s_{C1}v_1^2}\right)^{-1}\dfrac{\sin(2\bar{\varphi}_{C1})}{2} = e\sin\theta_{C1} = (1 + e\cos\theta_{C1})\cot\bar{\varphi}_{C1} \end{cases}$$

$$\begin{cases} \Delta(e\cos\theta_{C1}) = (1 + e\cos\theta_{C1})\left[-\Delta\left(\ln\dfrac{\mu}{s_{C1}v_1^2}\right) + 2\cot\bar{\varphi}_{C1}\,\Delta(\bar{\varphi}_{C1})\right] \\ \Delta(e\sin\theta_{C1}) = (1 + e\cos\theta_{C1})\cot\bar{\varphi}_{C1}\left[-\Delta\left(\ln\dfrac{\mu}{s_{C1}v_1^2}\right) + 2\cot(2\bar{\varphi}_{C1})\Delta(\bar{\varphi}_{C1})\right] \end{cases}$$

$$\Delta(\tan\theta_{C1}) = \Delta\left(\frac{e\sin\theta_{C1}}{e\cos\theta_{C1}}\right)$$

$$e\Delta(\theta_{C1}) = \cos\theta_{C1}\Delta(e\sin\theta_{C1}) - \sin\theta_{C1}\Delta(e\cos\theta_{C1})$$

$$e\Delta(\theta_{C1}) = \frac{1 + e\cos\theta_{C1}}{\sin\bar{\varphi}_{C1}}\left[\sin\left[\theta_{C1} + \left(\bar{\varphi}_{C1} - \frac{\pi}{2}\right)\right]\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right) + \right.$$

$$\left. - \cos\left[\theta_{C1} + 2\left(\bar{\varphi}_{C1} - \frac{\pi}{2}\right)\right]\frac{\Delta(\bar{\varphi}_{C1})}{\sin\bar{\varphi}_{C1}}\right]$$

$$\boxed{e\Delta(\theta_{C1}) \simeq \sin\theta_{C1}\,\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right) - \cos\theta_{C1}\,\Delta(\bar{\varphi}_{C1})}$$

The **eccentric anomaly** E_{C1} follows from:

$$\tan\frac{E_{C1}}{2} = \left(\frac{1 - e}{1 + e}\right)^{½}\tan\frac{\theta_{C1}}{2}$$

$$e\left(\frac{\Delta(E_{C1})}{\sin E_{C1}} - \frac{\Delta(\theta_{C1})}{\sin\theta_{C1}}\right) = -\frac{\Delta(e^2)}{2(1 - e^2)}$$

$$\boxed{e\Delta(E_{C1}) \simeq e\Delta(\theta_{C1})}$$

The **mean anomaly** $\bar{M}_{C1}$ follows from Kepler's equation:

$$\bar{M}_{C1} = E_{C1} - e\sin E_{C1}$$

$$\frac{e\Delta(\bar{M}_{C1})}{\sin E_{C1}} = (1 - e\cos E_{C1})\frac{e\Delta(E_{C1})}{\sin E_{C1}} - \frac{1}{2}\Delta(e^2)$$

$$e\Delta(\bar{M}_{C1}) = \frac{(1 - e\cos E_{C1})^2}{(1 - e^2)^{½}}\,e\Delta(\theta_{C1}) +$$

$$- \sin E_{C1}\,\frac{1 - \frac{1}{2}e(e + \cos E_{C1})}{1 - e^2}\,\Delta(e^2)$$

$$\boxed{e\Delta(\bar{M}_{C1}) \simeq e\Delta(\theta_{C1})}$$

With T the epoch of the passage through perigee:

$$\bar{M}_{C1} = \bar{n}(t_1 - T)$$

or:

$$\bar{M}_{C1} = \frac{s_{C1}\bar{n}}{v_1} \cdot \frac{v_1}{s_{C1}}(t_1 - T)$$

$$\boxed{\Delta(\bar{M}_{C1}) = \bar{M}_{C1}\left[2\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right) + \left[\Delta\left(\ln\frac{v_1}{s_{C1}}\right) + \frac{\Delta(t_1 - T)}{t_1 - T}\right]\right]}$$

θ_{C1} , E_{C1} , $\bar{M}_{C1}$ and hence also T are poorly determined because of the factor e of the Δ-quantities in the respective left hand members of the difference equations.
The introduction of the dimensionless quantities makes it compulsory to introduce a dimensionless time interval

$$\frac{v_1}{s_{C1}}(t_k - t_1)$$

which is analogous to the dimensionless time quantity introduced in Section 6. In the difference equations $\Delta\left(\ln\frac{v_1}{s_{C1}}\right)$ then occurs as a (provisionally unknown) scale factor.

For the computation of points of the satellite orbit we use relative quantities; the order of computation is opposite to the computation just completed. We start with:

$$\bar{M}_{Ck} - \bar{M}_{C1} = \frac{s_{C1}\bar{n}}{v_1} \cdot \frac{v_1}{s_{C1}}(t_k - t_1)$$

or, more general:

$$\bar{M}_{Ck} - \bar{M}_{Ci} = \frac{s_{C1}\bar{n}}{v_1} \cdot \frac{v_1}{s_{C1}}(t_k - t_i)$$

Like before we have:

$$\Delta(\theta_{Ck} - \theta_{Ci}) \simeq \Delta(\bar{M}_{Ck} - \bar{M}_{Ci})$$

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$$\boxed{\Delta(\theta_{Ck} - \theta_{Ci}) \simeq (\theta_{Ck} - \theta_{Ci})\left[2\Delta\left(\ln\frac{\mu}{s_{C1}v_1^2}\right) + \left[\Delta\left(\ln\frac{v_1}{s_{C1}}\right) + \frac{\Delta(t_k - t_i)}{t_k - t_i}\right]\right]}$$

Ratios of radial distances follow from:

$$\begin{cases} \dfrac{a}{s_{C1}}(1 - e^2) = 1 + e\cos\theta_{C1} \\ \dfrac{a}{s_{Ci}}(1 - e^2) = 1 + e\cos\theta_{Ci} \\ \dfrac{a}{s_{Ck}}(1 - e^2) = 1 + e\cos\theta_{Ck} \end{cases}$$

$$\frac{s_{Ck}}{s_{C1}} = \frac{1 + e\cos\theta_{C1}}{1 + e\cos\theta_{Ck}} =$$

$$= \frac{1 + \cos(\theta_{Ck} - \theta_{C1})(e\cos\theta_{C1}) - \sin(\theta_{Ck} - \theta_{C1})(e\sin\theta_{C1})}{1 + e\cos\theta_{C1}}$$

$$\Delta\left(\ln\frac{s_{Ck}}{s_{C1}}\right) = -\frac{\Delta(e\cos\theta_{Ck})}{1 + e\cos\theta_{Ck}} + \frac{\Delta(e\cos\theta_{C1})}{1 + e\cos\theta_{C1}}$$

$$\Delta(e\cos\theta_{Ck}) =$$

$$= \left[-e\cos\theta_{C1}\sin(\theta_{Ck} - \theta_{C1}) - e\sin\theta_{C1}\cos(\theta_{Ck} - \theta_{C1})\right]\Delta(\theta_{Ck} - \theta_{C1}) +$$

$$+ \cos(\theta_{Ck} - \theta_{C1})\,\Delta(e\cos\theta_{C1}) - \sin(\theta_{Ck} - \theta_{C1})\,\Delta(e\sin\theta_{C1}) =$$

$$= -e\sin\theta_{Ck}\cdot\Delta(\theta_{Ck} - \theta_{c1}) + \frac{1 + e\cos\theta_{C1}}{\sin\bar{\varphi}_{C1}} *$$

$$* \left[\cos\left(\theta_{Ck} - \theta_{C1}\right) \left[- \sin \bar{\varphi}_{C1} \, \Delta\left(\ln \frac{\mu}{s_{C1} v_1^2} \right) + \sin 2\bar{\varphi}_{C1} \, \frac{\Delta\left(\bar{\varphi}_{C1}\right)}{\sin \bar{\varphi}_{C1}} \right] + \right.$$

$$\left. - \sin\left(\theta_{Ck} - \theta_{C1}\right) \left[- \cos \bar{\varphi}_{C1} \, \Delta\left(\ln \frac{\mu}{s_{C1} v_1^2} \right) + \cos 2\bar{\varphi}_{C1} \, \frac{\Delta\left(\bar{\varphi}_{C1}\right)}{\sin^2 \bar{\varphi}_{C1}} \right] \right]$$

$$\frac{\Delta\left(e \cos \theta_{Ck}\right)}{1 + e \cos \theta_{C1}} = -e\left(1 + e \cos \theta_{C1}\right) \sin \theta_{Ck} \, \Delta\left(\theta_{Ck} - \theta_{C1}\right) +$$

$$+ \frac{1}{\sin \bar{\varphi}_{C1}} \left[\sin\left(\theta_{Ck} - \theta_{C1} - \bar{\varphi}_{C1}\right) \Delta\left(\ln \frac{\mu}{s_{C1} v_1^2} \right) + \right.$$

$$\left. - \sin\left(\theta_{Ck} - \theta_{C1} - 2\bar{\varphi}_{C1}\right) \frac{\Delta\left(\bar{\varphi}_{C1}\right)}{\sin \bar{\varphi}_{C1}} \right]$$

$$\frac{\Delta\left(e \cos \theta_{Ck}\right)}{1 + e \cos \theta_{Ck}} = \frac{\Delta\left(e \cos \theta_{Ck}\right)}{1 - e \cos \theta_{Ck}} \cdot \frac{s_{Ck}}{s_{C1}}$$

This results in:

$$\Delta\left(\ln \frac{s_{Ck}}{s_{C1}} \right) = - \frac{2 \sin \dfrac{\theta_{Ck} - \theta_{C1}}{2}}{\sin \bar{\varphi}_{C1}} \left[\cos\left(\frac{\theta_{Ck} - \theta_{C1}}{2} - \bar{\varphi}_{C1} \right) \Delta\left(\ln \frac{\mu}{s_{C1} v_1^2} \right) + \right.$$

$$\left. - \cos\left(\frac{\theta_{Ck} - \theta_{C1}}{2} - 2\bar{\varphi}_{C1} \right) \frac{\Delta\left(\bar{\varphi}_{C1}\right)}{\sin \bar{\varphi}_{C1}} \right] + O(e)$$

$$= - \frac{2 \sin \dfrac{\theta_{Ck} - \theta_{C1}}{2}}{\sin \bar{\varphi}_{C1}} \left[\sin\left(\frac{\theta_{Ck} - \theta_{C1}}{2} - \left(\bar{\varphi}_{C1} - \frac{\pi}{2} \right) \right) \Delta\left(\ln \frac{\mu}{s_{C1} v_1^2} \right) + \right.$$

$$\left. + \cos\left(\frac{\theta_{Ck} - \theta_{C1}}{2} - 2\left(\bar{\varphi}_{C1} - \frac{\pi}{2} \right) \right) \frac{\Delta\left(\bar{\varphi}_{C1}\right)}{\sin \bar{\varphi}_{C1}} \right] + O(e)$$

8.

$$\boxed{\Delta\left(\ln \frac{s_{Ck}}{s_{C1}}\right) \simeq -2\sin\frac{\theta_{Ck} - \theta_{C1}}{2}\left[\sin\frac{\theta_{Ck} - \theta_{C1}}{2}\,\Delta\left(\ln \frac{\mu}{s_{C1}v_1^2}\right) + \cos\frac{\theta_{Ck} - \theta_{C1}}{2}\,\Delta\left(\bar{\varphi}_{C1}\right)\right]}$$

Analogous:

$$\Delta\left(\ln \frac{s_{Ck}}{s_{Ci}}\right) = -\frac{\Delta\left(e\ \cos\theta_{Ck}\right)}{1 + e\ \cos\theta_{Ck}} + \frac{\Delta\left(e\ \cos\theta_{Ci}\right)}{1 + e\ \cos\theta_{Ci}} =$$

$$= -\frac{2\sin\frac{\theta_{Ck} - \theta_{Ci}}{2}}{\sin\bar{\varphi}_{C1}}\left[\sin\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - \theta_{C1} - \left(\bar{\varphi}_{C1} - \frac{\pi}{2}\right)\right)\Delta\left(\ln \frac{\mu}{s_{C1}v_1^2}\right) + \cos\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - \theta_{C1} - 2\left(\bar{\varphi}_{C1} - \frac{\pi}{2}\right)\right)\frac{\Delta\left(\bar{\varphi}_{C1}\right)}{\sin\bar{\varphi}_{C1}}\right] + O(e)$$

$$\boxed{\Delta\left(\ln \frac{s_{Ck}}{s_{Ci}}\right) \simeq -2\ \sin\ \frac{\theta_{Ck} - \theta_{Ci}}{2}\left[\sin\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - \theta_{C1}\right)\Delta\left(\ln \frac{\mu}{s_{C1}v_1^2}\right) + \cos\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - \theta_{C1}\right)\Delta\left(\bar{\varphi}_{C1}\right)\right]}$$

$\frac{s_{Ck}}{s_{Ci}}$ and $\left(\theta_k - \theta_i\right)$ are elements of the quaternion $\bar{Q}_{iCk}$ with $\bar{e}_{iCk} = \bar{e}_{v1C}$ (the upper score again denotes the $\bar{X}$, $\bar{Y}$, $\bar{Z}$-frame).

$$\bar{Q}_{iCk} = \frac{s_{Ck}}{s_{Ci}}\left[\cos\left(\theta_{Ck} - \theta_{Ci}\right) + \bar{e}_{v1C}\ \sin\left(\theta_{Ck} - \theta_{Ci}\right)\right]$$

$$\Delta\bar{\Pi}_{iCk} = \Delta\left(\ln \frac{s_{Ck}}{s_{Ci}}\right) - \bar{e}_{v1C}\ \Delta\left(\theta_{Ck} - \theta_{Ci}\right) + \sin\left(\theta_{Ck} - \theta_{Ci}\right)\ \bar{e}_{Ck}^{-1}\ \Delta\left(\bar{e}_{v1M}\right)\ \bar{e}_{Ci}$$

If the unit vector $\bar{e}_{C0}$ in the plane of the Kepler ellipse is taken in the direction $(\overrightarrow{P_C - \text{Perigee}})$, the component vectors are:

$$\begin{cases} \bar{q}_{Ck} = s_{Ck}(\cos\theta_{Ck} + \bar{e}_{v1C} \sin\theta_{Ck})\, \bar{e}_{C0} \\ \bar{q}_{Ci} = s_{Ci}(\cos\theta_{Ci} + \bar{e}_{v1C} \sin\theta_{Ci})\, \bar{e}_{C0} \end{cases}$$

$$\begin{cases} \Delta\bar{\Lambda}_{Ck} = \bar{q}_{Ck}^{-1}\, \Delta\bar{q}_{Ck}^{-1} = \Delta(\ln s_{Ck}) - \bar{e}_{v1C}\, \Delta(\theta_{Ck}) + \sin\theta_{Ck}\, \bar{e}_{Ck}^{-1}\, \Delta(\bar{e}_{v1C})\, \bar{e}_{C0} \\ \Delta\bar{\Lambda}_{Ci} = \bar{q}_{Ci}^{-1}\, \Delta\bar{q}_{Ci} = (\Delta\ln s_{Ci}) - \bar{e}_{v1C}\, \Delta(\theta_{Ci}) + \sin\theta_{Ci}\, \bar{e}_{Ci}^{-1}\, \Delta(\bar{e}_{v1C})\, \bar{e}_{C0} \end{cases}$$

Now we have, with, in particular, $\Delta(\theta_{C1})$ relatively large:

$$\Delta(\ln s_{Ck}) = \Delta\left(\ln \frac{s_{Ck}}{s_{C1}}\right) + \Delta(\ln s_{C1}) \quad , \quad \Delta(\theta_{Ck}) = \Delta(\theta_{Ck} - \theta_{C1}) + \Delta(\theta_{C1})$$

$$\Delta(\ln s_{Ci}) = \Delta\left(\ln \frac{s_{Ci}}{s_{C1}}\right) + \Delta(\ln s_{C1}) \quad , \quad \Delta(\theta_{Ci}) = \Delta(\theta_{Ci} - \theta_{C1}) + \Delta(\theta_{C1})$$

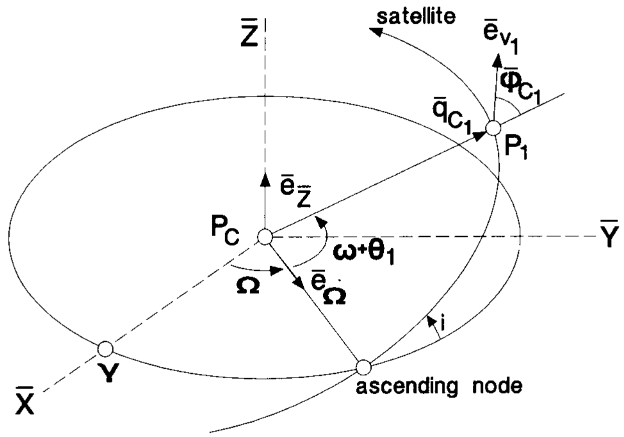
We see that in the difference:

$$\Delta\bar{\Pi}_{iCk} = \Delta\bar{\Lambda}_{Ck} - \Delta\bar{\Lambda}_{Ci}$$

the quantities $\Delta(\ln s_{C1})$ and $\Delta(\theta_{C1})$ vanish, which means that the influence of Δ-quantities in a quaternion is smaller than their influence in the separate component vectors.
It is therefore important to apply methods of (relative) positioning which (as far as possible) do **not** use single vectors but quaternions.

We now treat the remaining Kepler elements i, Ω and $(\omega + \theta_1)$.

8.



In Section 7.2 we used the quaternion:

$$Q_{\nu 1C} = \frac{s_{C1}}{v_1}\,\bar{e}_{1C}\,\bar{e}_{v_1}^{-1} = -\frac{s_{C1}}{w_1}\left(\cos\bar{\varphi}_{C1} + \bar{e}_{\nu 1C}\,\sin\bar{\varphi}_{C1}\right)$$

The angle enclosed by $\bar{e}_{\bar{Z}}$ and $\bar{e}_{\nu 1C}$ is i. Denoting the unit vector in the direction $\left(\overrightarrow{P_C - \text{ascending node}}\right)$ by $\bar{e}_{\Omega}$, with $\bar{e}_{\Omega}$ perpendicular to $\bar{e}_{\bar{Z}}$ and $\bar{e}_{\nu 1C}$, we have:

$$\bar{e}_{\nu 1C}\,\bar{e}_{\bar{Z}}^{-1} = \cos i + \bar{e}_{\Omega}\sin i\,, \qquad e_{\bar{Z}}^{-1} = -e_{\bar{Z}}$$

$$\bar{e}_{\nu 1C}\,\bar{e}_{\bar{Z}} = -\left(\cos i + \bar{e}_{\Omega}\sin i\right), \qquad \bar{e}_{\Omega} = \bar{e}_{\bar{X}}\cos\Omega + \bar{e}_{\bar{Y}}\sin\Omega$$

hence:

$$\cos i = -Sc\left\{\bar{e}_{\nu 1C}\,\bar{e}_{\bar{Z}}\right\} = -\frac{1}{2}\left(\bar{e}_{\nu 1C}\,\bar{e}_{\bar{Z}} + \bar{e}_{\bar{Z}}\,\bar{e}_{\nu 1C}\right)$$

$$\bar{e}_{\Omega}\sin i = \left(\bar{e}_{\bar{X}}\cos\Omega + \bar{e}_{\bar{Y}}\sin\Omega\right)\sin i =$$

$$= -Ve\left\{\bar{e}_{\nu 1C}\,\bar{e}_{\bar{Z}}\right\} = -\frac{1}{2}\left(\bar{e}_{\nu 1C}\,\bar{e}_{\bar{Z}} - \bar{e}_{\bar{Z}}\,\bar{e}_{\nu 1C}\right)$$

$$\bar{e}_{C1}\,\bar{e}_{\Omega}^{-1} = \cos(\omega + \theta_{C1}) + \bar{e}_{\nu 1C}\sin(\omega + \theta_{C1})\,, \qquad \bar{e}_{\Omega}^{-1} = -\bar{e}_{\Omega}$$

hence:

$$\cos(\omega + \theta_{C1}) = -Sc\left\{\bar{e}_{C1}\,\bar{e}_{\Omega}\right\} = -\frac{1}{2}\left(\bar{e}_{C1}\,\bar{e}_{\Omega} + \bar{e}_{\Omega}\,\bar{e}_{C1}\right)$$

$$\boxed{\begin{aligned} \sin i \cdot \Delta(i) &= \frac{1}{2}\left[\Delta(\bar{e}_{\nu 1C})\,\bar{e}_{\bar{Z}} + \bar{e}_{\bar{Z}}\,\Delta(\bar{e}_{\nu 1C})\right] \\ \Delta(\bar{e}_{\Omega}) &= \left(-\bar{e}_{\bar{X}}\sin\Omega + \bar{e}_{\bar{Y}}\cos\Omega\right)\Delta(\Omega) \\ &= \bar{e}_{\Omega+\frac{\pi}{2}}\,\Delta(\Omega) \\ \sin i.\Delta(\bar{e}_{\Omega}) + \bar{e}_{\Omega}\cos i.\Delta(i) &= -\frac{1}{2}\left[\Delta(\bar{e}_{\nu 1C})\,\bar{e}_{\bar{Z}} - \bar{e}_{\bar{Z}}\,\Delta(\bar{e}_{\nu 1C})\right] \\ -\sin(\omega + \theta_{C1})\,\Delta(\omega + \theta_{C1}) &= -\frac{1}{2}\left[\bar{e}_{C1}\,\Delta(\bar{e}_{\Omega}) + \Delta(e_{\Omega})\,\bar{e}_{C1} + \right. \\ &\quad \left. + \Delta(\bar{e}_{C1})\,\bar{e}_{\Omega} + \bar{e}_{\Omega}\,\Delta(\bar{e}_{C1})\right] \end{aligned}}$$

$$\bar{e}_{C1} = \frac{1}{s_{C1}}\,\bar{q}_{C1}\,, \qquad \Delta(\bar{e}_{C1}) = \frac{1}{s_{C1}}\left[-\bar{q}_{C1}\,\Delta(\ln s_{C1}) + \Delta(q_{C1})\right]$$

$$\boxed{\Delta(\bar{e}_{C1}) = \bar{e}_{C1}\left[\Delta\bar{\Lambda}_{C1} - \Delta(\ln s_{C1})\right]}$$

This concludes the theory of the Kepler ellipse. Except for the last equation, it is seen that all Δ-quantities depend on $\Delta\left(\ln \dfrac{\mu}{s_{C1}v_1^2}\right)$, $\Delta(\bar{\varphi}_{C1})$ and $\Delta(e_{\nu 1C})$, the three basic unknowns in the determination of the satellite orbit.

8.2.1

Now the linking-up of the theory of Section 8.1, i.e. the coupling with measurements, has to be effectuated. This concerns the measurement of q_{M1} and q_{ν_1} in the terrestrial *X*, *Y*, *Z*-frame with the origin $P_M \neq P_C$.

This implies that in the formulas of Section 8.1 q_{C1} must be replaced according to Section 7.2 by

$$q_{C1} = q_{M1} - q_{MC}$$

If, on the analogy of the contents of Sections 4-6, s_{M1} is replaced by s_1 this results in:

$$\begin{cases} \ln s_{C1} = \ln s_1 + \Delta_C(\ln s_1)\,, & \Delta_C(\ln s_1) = -B_1^{(1)} \\ \bar{\varphi}_{C1} = \bar{\varphi}_1 + \Delta_C(\bar{\varphi}_1) \\ e_{v1C} = e_{v1M} + \Delta_C(e_{v1M}) \end{cases}$$

In the coefficients of the difference equations it does not matter whether quantities have and index M or C, but for the Δ-quantities the following substitutions have to be made:

$$\left.\begin{array}{l} \begin{cases} \Delta(\ln s_{C1}) = \Delta(\ln s_1) + \Delta_C(\ln s_1) \\ \Delta(\bar{\varphi}_{C1}) = \Delta(\bar{\varphi}_1) + \Delta_C(\bar{\varphi}_1) \\ \Delta(e_{v1C}) = \Delta(e_{v1M}) + \Delta_C(e_{v1M}) \end{cases} \\ \\ \Delta\left(\ln \dfrac{\mu}{s_{C1} v_1^2}\right) = \Delta\left(\ln \dfrac{\mu}{s_1 v_1^2}\right) + B_1^{(1)} \end{array}\right\} \text{the three basic unknowns}$$

hence:

It is now clear why the influence of $q_{MC} \neq 0$ has not been discovered; it is hidden in the basic unknowns. However there is a relationship between the Δ_C-quantities which will deform the results of computations if it is not taken into account.

A good linking-up of the theory therefore requires the introduction of **six** unknowns

$$\begin{cases} \Delta\left(\ln \dfrac{\mu}{s_1 v_1^2}\right) \\ \Delta(\bar{\varphi}_1) \\ \Delta(e_{v1M}) \end{cases} \quad \text{and} \quad \begin{cases} \text{three} \\ \text{components} \\ \text{of } q_{MC} \end{cases}$$

To effectuate this, the difference formulas of Section 8.1 will be rewritten.

$$\boxed{\Delta\left(\ln \frac{a}{s_{C1}}\right) \simeq -B_1^{(1)} - \Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right)}$$

$$\boxed{\begin{aligned}\Delta\left(\ln \frac{V_k}{v_1^2}\right) &= \Delta(\ln V_k) - 2\Delta(\ln v_1) = \\ &= B_1^{(1)} + \Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right) - \Delta\left(\ln \frac{s_{Ck}}{s_{C1}}\right) + \sum_{n=2}^{\infty} \Delta\left(\bar{B}_k^{(n)}\right)\end{aligned}}$$

Note that the right hand member contains $B_1^{(1)}$ and **not** $B_k^{(1)}$. For $\Delta \ln\left(\frac{s_{Ck}}{s_{C1}}\right)$ see the sequel.

$$\boxed{\Delta\left(\ln \frac{s_{C1}\,\bar{n}}{v_1}\right) \simeq 2\left[B_1^{(1)} + \Delta\left(\ln \frac{\mu}{s_1 v^2}\right)\right]}$$

$$\boxed{\Delta(e^2) \simeq 0}$$

$$\begin{aligned}e\,\Delta(\theta_{C1}) \simeq &- \sin\theta_{C1}\,\Delta_C(\ln s_1) - \cos\theta_{C1}\,\Delta_C(\bar{\varphi}_1) + \\ &+ \sin\theta_{C1}\,\Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right) - \cos\theta_{C1}\,\Delta(\bar{\varphi}_1)\end{aligned}$$

According to Section 7.2 we have:

$$\begin{cases}\Delta_C(\ln s_1) = \bar{e}_{M1}\,\dfrac{\bar{q}_{MC}}{s_1} + \dfrac{\bar{q}_{MC}}{s_1}\,\bar{e}_{M1} \\ \Delta_C(\bar{\varphi}_1) = -\bar{e}_{v_1}\,\dfrac{\bar{q}_{MC}}{s_1} - \dfrac{\bar{q}_{MC}}{s_1}\,\bar{e}_{v_1}\end{cases}$$

$$\begin{cases}\bar{e}_{M1} \simeq \bar{e}_{C1} = (\cos\theta_{C1} + \bar{e}_{v1C}\sin\theta_{C1})\,\bar{e}_{C0} \\ \bar{e}_{v_1} \qquad = \left[\cos\left(\theta_{C1} + \dfrac{\pi}{2}\right) + \bar{e}_{v1C}\sin\left(\theta_{C1} + \dfrac{\pi}{2}\right)\right]\bar{e}_{C0} \\ \qquad\quad = (-\sin\theta_{C1} + \bar{e}_{v1C}\cos\theta_{C1})\,\bar{e}_{C0}\end{cases}$$

Hence:

$$e\Delta(\theta_{C1}) \simeq \left(-\sin 2\theta_{C1} + \bar{e}_{\nu 1C} \cos 2\theta_{C1}\right) \bar{e}_{C0} \frac{\bar{q}_{MC}}{s_1} +$$

$$+ \frac{\bar{q}_{MC}}{s_1} \bar{e}_{C0}\left(-\sin 2\theta_{C1} - \bar{e}_{\nu 1C} \cos 2\theta_{C1}\right) +$$

$$+ \sin\theta_{C1} \Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right) - \cos\theta_{C1} \Delta\left(\bar{\varphi}_1\right)$$

Now:

$$\left(-\sin 2\theta_{C1} + \bar{e}_{\nu 1C} \cos 2\theta_{C1}\right) \bar{e}_{C0} \quad \text{with} \qquad - \bar{e}_{\nu 1C} \bar{e}_{C0}^{-1} = \bar{e}_{C0 + \frac{\pi}{2}}$$

$$= \bar{\theta}_{C0} \cos\left(2\theta_{C1} + \frac{\pi}{2}\right) + \bar{e}_{C0 + \frac{\pi}{2}} \sin\left(2\theta_{C1} + \frac{\pi}{2}\right) = \bar{e}_{C0 + 2\theta_{C1} + \frac{\pi}{2}}$$

$$\boxed{\begin{aligned} e\Delta(\theta_{C1}) &\simeq 2\,Sc\left\{\bar{e}_{C0 + 2\theta_{C1} + \frac{\pi}{2}} \cdot \frac{\bar{q}_{MC}}{s_1}\right\} + \\ &+ \sin\theta_{C1} \Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right) - \cos\theta_{C1} \Delta\left(\bar{\varphi}_1\right) \end{aligned}}$$

$$\boxed{\begin{aligned} \Delta\left(\bar{M}_{C1}\right) &\simeq \bar{M}_{C1}\left[3B_1^{(1)} + 2\Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right) + \right. \\ &\left. + \left[\Delta\left(\ln \frac{v_1}{s_1}\right) + \frac{\Delta(t_1 - T)}{t_1 - T}\right]\right] \end{aligned}}$$

$$\boxed{\begin{aligned} \Delta\left(\theta_{Ck} - \theta_{Ci}\right) &\simeq \left(\theta_{Ck} - \theta_{Ci}\right)\left[3B_1^{(1)} + 2\Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right) + \right. \\ &\left. + \left[\Delta\left(\ln \frac{v_1}{s_1}\right) + \frac{\Delta(t_k - t_1)}{t_k - t_i}\right]\right] \end{aligned}}$$

Analogous to the derivation of $e\Delta\left(\theta_{C1}\right)$ one obtains:

$$-\sin\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - \theta_{C1}\right) \Delta_C(\ln s_1) + \cos\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - \theta_{C1}\right) \Delta_C(\bar{\varphi}_1) =$$

$$= -2Sc\left\{\left[\sin\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - 2\theta_{C1}\right) + \bar{e}_{\nu 1C}\cos\left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - 2\theta_{C1}\right)\right] \bar{e}_{C0} \frac{\bar{q}_{MC}}{s_1}\right\} =$$

$$= +2Sc\left\{\bar{e}_{C0 + \frac{3\pi}{2} - \left(\frac{\theta_{Ck} + \theta_{Ci}}{2} - 2\theta_{C1}\right)} (y \cdot e\,\Delta(\theta_{C1})) \frac{\bar{q}_{MC}}{s_1}\right\}$$

$$\boxed{\begin{aligned}\Delta\left(\ln \frac{s_{Ck}}{s_{Ci}}\right) \simeq -2\sin\frac{\theta_{Ck} - \theta_{Ci}}{2}\Bigg[2Sc\left\{\bar{e}_{C0 + \frac{3\pi}{2} - (\frac{\theta_k + \theta_i}{2} - 2\theta_{C1})} \cdot \frac{\bar{q}_{MC}}{s_1}\right\} + \\ + \sin\left(\frac{\theta_k + \theta_i}{2} - \theta_{C1}\right) \Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right) + \cos\left(\frac{\theta_k + \theta_i}{2} - \theta_{C1}\right) \Delta(\bar{\varphi}_1)\Bigg]\end{aligned}}$$

These formulas show that the coefficients of the three components of $\bar{q}_{MC}$ are entirely different from the coefficients of $\Delta\left(\ln \frac{\mu}{s_1 v_1^2}\right)$ and $\Delta(\bar{\varphi}_1)$.

The same applies to the coefficients of $\Delta(\bar{e}_{\nu 1M})$ in the formulas for $\Delta(i)$, $\Delta(\Omega)$, and $\Delta(\omega + \theta_{C1})$. We here restrict ourselves to the formula:

$$\boxed{\begin{aligned}\Delta(\bar{e}_{c1}) = \bar{e}_{C1}\big[-\bar{q}_{C1}^{-1}\bar{q}_{MC} + B_1^{(1)} + \\ + \Delta\bar{\Lambda}_{M1} - \Delta(\ln s_1)\big]\end{aligned}}$$

The (relative) influence of q_{MC} on the satellite orbit can be read:

along track from $\Delta(\theta_{Ck} - \theta_{Ci})$

cross track from $\Delta(i)$

radial from $\Delta\left(\ln \frac{s_{Ck}}{s_{Ci}}\right)$

This influence is clearly **not** negligible. Even if $\frac{1}{s_1}\overline{MC}$ is of the order 2.10^{-7}, an appraisal which is sometimes mentioned in relevant literature.

In the computation of a satellite orbit q_{MC} remains unknown and consequently must be taken to be zero. In Section 8.3 possible appraisals for q_{MC} will be discussed.

8.2.2

In the second part of the difference equations just treated, i.e. the part which is independent of $q_{MC} \neq 0$, there is another problem. This problem concerns the compound quantities, occurring as such in the computation,

$$\frac{\mu}{s_1 v_1^2}\,, \quad \frac{V_k}{v_1^2}\,, \quad \frac{v_1}{s_1}\left(t_k - t_i\right)$$

in which μ, V and t are scaled by the respective coefficients in the compound.

Splitting up the Δ-quantities results in:

$$\begin{cases} \Delta\left(\ln\dfrac{\mu}{s_1 v_1^2}\right) & = \Delta(\ln\mu) - \Delta(\ln s_1) - 2\,\Delta(\ln v_1) \\ \Delta\left(\ln\dfrac{V_k}{v_1^2}\right) & = \Delta(\ln V_k) - 2\Delta(\ln v_1) \\ \Delta\left(\ln\dfrac{v_1}{s_1}\left(t_k - t_i\right)\right) & = \dfrac{\Delta\left(t_k - t_1\right)}{t_k - t_i} - \Delta(\ln s_1) + \Delta(\ln v_1) \end{cases}$$

If now **non-zero** quantities Δ (ln s_1) and Δ (ln v_1) are introduced, this means that the scaling of μ, V and t is changed. It is an arbitrary interference which is not determined by the computation of the satellite orbit, because only the Δ's of the compound quantities occur in this computation.

Therefore it seems reasonable to put:

$$\boxed{\Delta(\ln s_1) = \Delta(\ln v_1) = 0}$$

This means that the computation of satellite orbit segments between updating epochs is executed in an $S_{s_1 v_1}$- system, on the analogy of the $S_{r_1 g_1}$- system in terrestrial physical geodesy.

Dimensionless quantities

Terrestrial gravimetric system		Satellite orbit system	
$S_{r_1 g_1}$		$S_{s_1 v_1}$	
$\frac{W_k}{r_1 g_1} - \frac{r_1}{r_k}\frac{W_1}{r_1 g_1}$		$\frac{V_k}{v_1^2}$	
$\frac{\mu}{r_1^2 g_1}$		$\frac{\mu}{s_1 v_1^2}$	
$\frac{v_{k,u}}{\sqrt{r_1 g_1}} - \frac{r_1}{r_k}\frac{v_{1,u}}{\sqrt{r_1 g_1}}$	?	$\frac{v_{k,u}}{v_1} - \frac{s_1}{s_k}\frac{v_{1,u}}{v_1}$	?
$\frac{r_k \dot{v}_{k,u}}{r_1 g_1} - \frac{r_1}{r_k}\frac{r_1 \dot{v}_{1,u}}{r_1 g_1}$	?	$\frac{s_k \dot{v}_{k,u}}{v_1^2} - \frac{s_1}{s_k}\frac{s_1 \dot{v}_{1,u}}{v_1^2}$	?
$\frac{r_k g_{k,u}}{r_1 g_1} - \frac{r_1}{r_k}\frac{r_1 g_{1,u}}{r_1 g_1}$			
$\frac{r_k^2 \Gamma_{k,uv}}{r_1 g_1} - \frac{r_1}{r_k}\frac{r_1^2 \Gamma_{1,uv}}{r_1 g_1}$		$\frac{s_k^2 \Gamma_{k,uv}}{v_1^2} - \frac{s_1}{s_k}\frac{s_1^2 \Gamma_{1,uv}}{v_1^2}$	
$(t_k - t_1)\sqrt{\frac{g_1}{r_1}}$		$(t_k - t_1)\frac{v_1}{s_1}$	

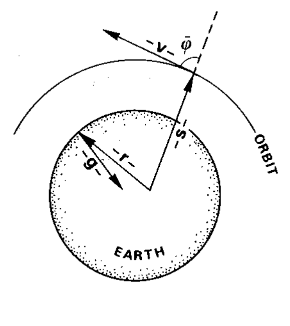


Figure 8.2.2

If, when updating, the values s_1 and/or v_1 are altered, then in principle all the dimensionless quantities which have been introduced are altered as well. This is a reason to be very cautious with interpretations.

There is an interesting similarity between analogous quantities in the two S- systems, as shown in figure 8.2.2.

But a curious problem remains, for which we return to the situation of measurement and computation of Section 7.2.
In P_a one is dealing with the terrestrial gravimetric r, g- system. Assume that P_a is also the datum point of this system. Then we have:

$$\frac{W_a}{r_a g_a} = \frac{\mu}{r_a^2 g_a}\left(1 + \sum_{n=1}^{\infty} B_a^{(n)}\right)$$

In this system the following is valid for the satellite point P_1:

$$\frac{V_1}{r_a g_a} = \frac{\mu}{r_a^2 g_a}\frac{r_a}{s_1}\left(1 + \sum_{n=1}^{\infty} B_1^{(1)}\right)$$

Now according to Section 7.2, $\dfrac{s_1}{r_a}$ is determined by measurement and computation in the terrestrial geometric S-system. Or:

$$\frac{V_1}{s_1 g_a} = \frac{\mu}{s_1^2 g_a}\left(1 + \sum_{n=1}^{\infty} B_1^{(n)}\right)$$

If now one assumes the fiction that g_1/g_a might be measured, then in the terrestrial system one can write:

$$\frac{V_1^{(\mathrm{terr})}}{s_1 g_1} = \frac{\mu^{(\mathrm{terr})}}{s_1^2 g_1}\left(1 + \sum_{n=1}^{\infty} B_1^{(n)}\right)$$

By now applying the correction for $q_{MC} \neq 0$ one obtains in the satellite system:

$$\frac{V_1^{(\mathrm{sat})}}{v_1^2} = \frac{\mu^{(\mathrm{sat})}}{s_1 v_1^2}\left(1 + \sum_{n=1}^{\infty} B_1^{(n)}\right)$$

If it is to be valid that:

$$V_1^{(\mathrm{terr})} = V_1^{(\mathrm{sat})}\ ,\ \mu^{(\mathrm{terr})} = \mu^{(\mathrm{sat})}$$

then the following condition must be fulfilled:

$$s_1 g_1 = v_1^2 , \qquad \text{or:} \qquad \sqrt{\frac{g_1}{s_1}} = \frac{v_1}{s_1}$$

so that also: $t^{(\mathrm{terr})} = t^{(\mathrm{sat})}$.

Unfortunately, g_1/g_a cannot be measured (or it cannot be measured directly). Therefore one has to be aware of a difference [1] in scale between, on one hand $\mu^{(\mathrm{terr})}$, $V^{(\mathrm{terr})}$, $t^{(\mathrm{terr})}$, and on the other hand $\mu^{(\mathrm{sat})}$, $V^{(\mathrm{sat})}$, $t^{(\mathrm{sat})}$.

Remark (added in proof)

The choice of the satellite orbit system was made some ten years ago. Since, direct measurement of the modulus of the velocity vector $\vec{v}$ seems feasible and it is interesting to look for the consequences. For the modulus v_k holds:

$$v_k^2 = \mu\left(\frac{2}{s_{Ck}} - \frac{1}{a}\right)$$

or rewritten:

$$\left(\frac{v_k}{v_1}\right)^2 = \frac{\mu}{s_{C1} v_1^2}\left(2\,\frac{s_{C1}}{s_{Ck}} - \frac{s_{C1}}{a}\right)$$

Hence one may conclude that in satellite orbit computations the ratio of moduli of velocity vectors takes the place of the ratio of moduli of gravity vectors in terrestrial computations.

The other important ratio, the ratio of (radial) distances, forms part of both the satellite orbit system and the terrestrial system.

Comparing the denominator of $\dfrac{v_k^2}{v_1^2}$ with the denominators of most A_k- and $\left(A_k - \dfrac{s_1}{s_k} A_1\right)$-quantities (see Sections 4.1 and 4.3) in the second column of Table 8.2.2 shows that $\left(\dfrac{v_k}{v_1}\right)^2$,

---

[1] The difference cannot be large for, according to Section 4.1 and 8.1:

$$\frac{\mu}{s_1^2 g_1} \simeq 1 \simeq \frac{\mu}{s_1 v_1^2} , \quad \text{hence: } s_1 g_1 \simeq v_1^2$$

but **not** $\frac{v_k}{v_1}$, is an A_k-quantity. The corresponding $\left(A_k - \frac{s_1}{s_k} A_1\right)$-quantity becomes (the equation relates to P_C as origin):

$$\left(\frac{v_k}{v_1}\right)^2 - \frac{s_{C1}}{s_{Ck}} \cdot 1 = \frac{\mu}{s_{C1} v_1^2} \frac{s_{C1}}{a} \frac{s_{C1} - s_{Ck}}{s_{Ck}}$$

so that in the situation of a near-circular orbit the influence of $\frac{\mu}{s_{C1} v_1^2}$ $\left(\text{and } \frac{s_{C1}}{a}\right)$ is almost negligible.
This is reflected in the difference equation:

$$\boxed{\Delta\left(\ln \frac{v_k}{v_1}\right) = -\Delta \ln\left(\frac{s_{CK}}{s_{C1}}\right)}$$

which gives a connection with relevant equations in Section 8.1.

A possible consequence of the quadratic character of $\left(\frac{v_k}{v_1}\right)^2$ may be that in Table 8.2.2 one has to replace:

$$\frac{v_{k,u}}{v_1} - \frac{s_1}{s_k} \frac{v_{1,u}}{v_1}$$

by:

$$\frac{v_{k,u} v_{k,w}}{v_1^2} - \frac{s_1}{s_k} \frac{v_{1,u} v_{1,w}}{v_1^2}$$

But this means that in the first column of Table 8.2.2 and in Section 6 one has to replace:

$$\frac{v_{k,u}}{\sqrt{r_1 g_1}} - \frac{r_1}{r_k} \frac{v_{1,u}}{\sqrt{r_1 g_1}}$$

by:

$$\frac{v_{k,u} v_{k,w}}{r_1 g_1} - \frac{r_1}{r_k} \frac{v_{1,u} v_{1,w}}{r_1 g_1}$$

in agreement with the rewritten mechanization equations.
Yet the significance of the new quantities remains unclear.

8.2.3

It is an interesting exercise to rewrite some formulas for higher order terms in satellite orbit computation by means of the dimensionless quantities used in the foregoing. In doing so, the difference between P_M and P_C can be ignored. For simplicity, the index 1 is left out as well. The following notation is introduced:

$$s_1 = s \ , \ v_1 = v \ , \qquad \frac{\mu}{s_1 v_1^2} = \mu' \ , \qquad \bar{\varphi}_1 = \bar{\varphi}$$

$$\frac{a}{s_1} = a' \qquad \frac{V}{v_1^2} = V' \ , \qquad \frac{s_1 \bar{n}}{v_1} = \bar{n}' \ , \qquad e = e \ ,$$

$$\theta_1 = \theta \ , \ E_1 = E \ , \ \bar{M}_1 = \bar{M} \ , \ \frac{v_1}{s_1} t = t' \ , \ i = i \ , \ \Omega = \Omega \ , \ \omega = \omega$$

One then obtains:

$$\mu' a' = \frac{\mu}{s v^2} \frac{a}{s} = \frac{\sin^2 \bar{\varphi}}{1 - e^2}$$

$$\bar{n}' (a')^2 = \left(\frac{\mu}{s v^2} \frac{a}{s} \right)^{½} = (\mu' a')^{½} = \frac{\sin \bar{\varphi}}{\sqrt{1 - e^e}}$$

$$\frac{d \ln a'}{dt'} = a' \frac{da'}{dt'} = \frac{s}{av} \frac{da}{dt}$$

$$\frac{de^2}{dt} = 2e \frac{de}{dt}$$

Put: $\qquad V = \dfrac{\mu}{s} + R$

in which R is the perturbative or disturbing function, again with:

$$\frac{R}{v^2} = R'$$

then e.g. the following holds:

$$\frac{2}{\bar{n} a} \frac{\partial R}{\partial \bar{M}} = \frac{2}{\bar{n}' (a')^2} \frac{\partial R'}{\partial \bar{M}} \frac{av}{s} = \frac{da}{dt} = \frac{d \ln a'}{dt'} \frac{av}{s}$$

or:

8.

$$\frac{d\ln a'}{dt'} = \frac{2}{\bar{n}'(a')^2}\frac{\partial R'}{\partial \bar{M}} = \frac{2(1-e^2)^{1/2}}{\sin\bar{\varphi}}\frac{\partial R'}{\partial \bar{M}}$$

Similarly one can write all Lagrange planetary equations in dimensionless quantities. Using an elegant notation due to J. Kovalevsky, one obtains:

$$\frac{d}{dt'}\begin{pmatrix} \ln a' \\ e^2 \\ i \\ \Omega \\ \omega \\ \bar{M} - \bar{n}'t' \end{pmatrix} = \frac{1}{\sin\bar{\varphi}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & [2(1-e^2)^{1/2}] \\ 0 & 0 & 0 & 0 & [-2(1-e^2)] & [2(1-e^2)^{3/2}] \\ 0 & 0 & 0 & [-\mathrm{cosec}\, i] & [\cot i] & 0 \\ 0 & 0 & [\mathrm{cosec}\, i] & 0 & 0 & 0 \\ 0 & [2(1-e^2)] & [-\cot i] & 0 & 0 & 0 \\ [-2(1-e^2)^{1/2}] & [-2(1-e^2)^{3/2}] & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta R'}{\delta \ln a'} \\ \frac{\delta R'}{\delta e^2} \\ \frac{\delta R'}{\delta i} \\ \frac{\delta R'}{\delta \Omega} \\ \frac{\delta R'}{\delta \omega} \\ \frac{\delta R'}{\delta \bar{M}} \end{pmatrix}$$

It is seen that a' has been replaced by $\ln a'$ and e by e^2, like was done in the difference equations of the elements of the Kepler ellipse. Note the nice anti-symmetry of the matrix elements:

$$(1.6) = -(6.1)$$

$$(2.5) = -(5.2)\,, \qquad (2.6) = -(6.2)$$

$$(3.4) = -(4.3)\,, \qquad (3.5) = -(5.3)$$

A second example raise questions. They concern the replacement of the set of Kepler elements by sets of canonical variables, viz. the Delaunay set and the Hill set. We follow here E.M. Gaposchkin in his "1973 Smithsonian Standard Earth (III). SAO Special Report No. 353", p. 127. The curious aspect of these sets is that the variables have different dimensions. Their replacement by dimensionless elements, as literally as possible for comparison, results in the following:

for **Delaunay variables:**

$$\begin{cases} l = \bar{M} \;, & L' = (\mu' a')^{1/2} = \dfrac{\sin\bar{\varphi}}{(1-e^2)^{1/2}} \\ g = \omega \;, & G' = [\mu' a'(1-e^2)]^{1/2} = \sin\bar{\varphi} \\ h = \Omega \;, & H' = [\mu' a'(1-e^2)]^{1/2}\cos i = \sin\bar{\varphi}\,\cos i \end{cases}$$

for **Hill variables:**

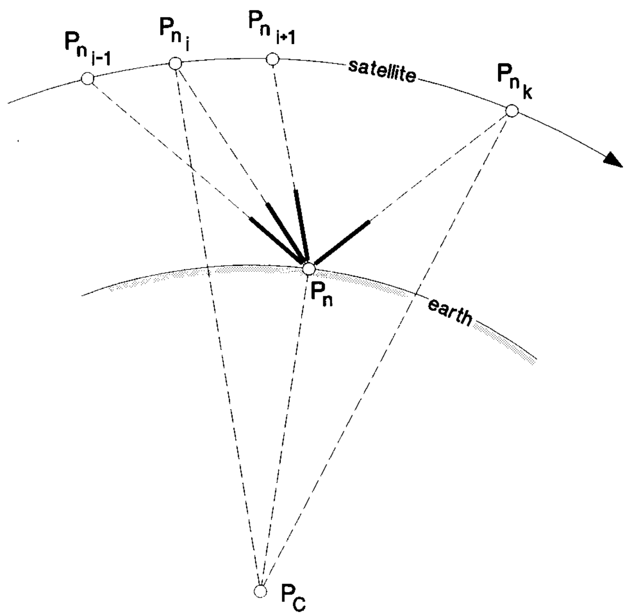
$$\begin{cases} \dfrac{1}{a'} = 1 - e\,\sin E & , \quad \dfrac{-d\ln a'}{dt'} = e a' L' \sin E = \\ & \quad = \dfrac{e}{(1-e^2)^{½}}\, a' \sin\bar{\varphi}\, \sin E \\ \mu = \omega + \theta & , \quad G' = G' \\ h = \Omega & , \quad H' = H' \end{cases}$$

The dimensioning is now in perfect order; it is a question whether this has any practical significance.

9.

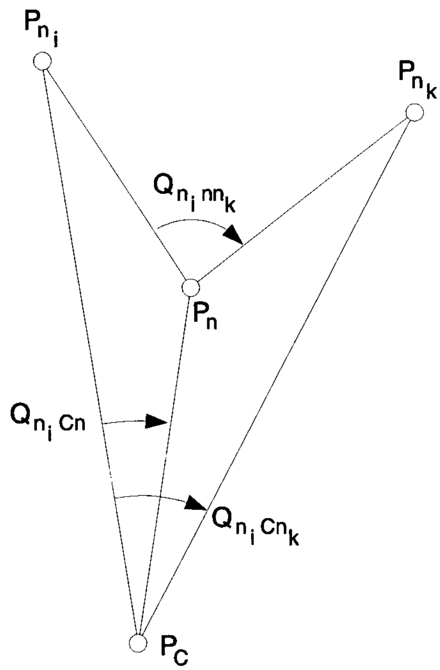
9.1

We shall now sketch the positioning of points on earth by satellite measurements, in order to examine the influence of $\overline{P_M P_C} \neq 0$ on this operation. To begin with, we consider a single satellite; the indices i and k from Sections 7 and 8 for satellite positions are maintained as lower indices to the index number of the terrestrial point involved in the measurements.



Now assume that in P_n a series of pseudo-distances (i.e. distances with an unknown scale factor $\simeq 1$) has been measured, as well as a series of spatial directions to the same satellite points. Because nowadays direction measurements are no more practised this is a temporary assumption, with, for simplicity, the assumption that these directions are defined in the X, Y, Z-frame. Consequently we in fact assume in P_n the measurement of a series of pseudo-vectors to satellite points P_{i-1} , P_i , P_{i+1} , ... , P_k .
A further assumption is that this series of pseudo-vectors is somehow reduced to two pseudo-vectors q_{nn_i} and q_{nn_k} , preferably with approximately equal norms, hence $s_{nn_i} \approx s_{nn_k}$.

Thus one obtains a bird's-tail construction (the bird a swallow).



$Q_{n_i nn_k} = q_{nn_k} q_{nn_i}^{-1}$ follows from measurements, $Q_{n_i Cn_k} = q_{Cn_k} q_{Cn_i}$ follows from the orbit computation.

$Q_{n_i Cn}$ can be computed, and the position of P_n follows from:

$$q_{Cn} = Q_{n_i Cn} q_{Cn_i}$$

For the computation of $Q_{n_i Cn}$ one starts from the central condition with centre P_n :

$$Q_{n_k nC} Q_{n_i nn_k} Q_{Cnn_i} = 1$$

or:

$$Q_{n_i nn_k} \; Q_{Cnn_i} = Q_{n_k nC}^{-1} = Q_{Cnn_k} \tag{a}$$

Then the net-condition in triangle Cnn_k^i

$$q_{Cn} + q_{nn_k} + q_{n_k C} = 0 \quad , \quad q_{nn_k} q_{nC}^{-1} = 1 - q_{Cn_k} q_{Cn}^{-1}$$

or:

$$Q_{Cnn_k} = 1 - Q_{nCn_k} \tag{b}$$

Similarly the net-condition in triangle Cnn_i :

$$Q_{Cnn_i} = 1 - Q_{nCn_i} \tag{c}$$

The central condition with centre P_C gives:

$$Q^{-1}_{n_kCn} = Q_{nCn_k} = Q_{n_iCn_k}Q_{nCn_i} \tag{d}$$

Substitution of (d) into (b) and then with (c) into (a) gives:

$$Q_{n_inn_k}\left(1 - Q_{nCn_i}\right) = 1 - Q_{n_iCn_k}Q_{nCn_i}$$

or:

$$\left(Q_{n_iCn_k} - Q_{n_inn_k}\right) Q_{nCn_i} - \left(1 - Q_{n_inn_k}\right) = 0$$

or:

$$\boxed{\left(Q^{(\text{sat})}_{n_iCn_k} - Q^{(\text{instr})}_{n_inn_k}\right) - \left(1 - Q^{(\text{instr})}_{n_inn_k}\right) Q^{(?)}_{n_iCn} = 0}$$

This is the basic relation of the bird's-tail construction.

(sat) means: computed from the satellite orbit,
(instr) means: computed from measurements,
(?) means: sought.

But there is a snake in the grass. If one starts from $Q^{(\text{sat})}_{n_iCn_k}$ then automatically:

$$e^{(\text{sat})}_{n_iCn_k} \perp e^{(\text{sat})}_{n_in_k}$$

Now $Q^{(\text{instr})}_{n_inn_k}$ may be freely translated, and rotated about $e^{(\text{instr})}_{n_inn_k}$. But this is not sufficient for the spatial connection to $P^{(\text{sat})}_{n_i}$ and $P^{(\text{sat})}_{n_k}$. This connection is subject to the condition:

$$e^{(\text{instr})}_{n_inn_k} \perp e^{(\text{sat})}_{n_in_k}$$

or:

$$e^{(\mathrm{instr})}_{n_i n n_k} + e^{(\mathrm{sat})}_{n_i n_k} e^{(\mathrm{instr})}_{n_i n n_k} e^{(\mathrm{sat})^{-1}}_{n_i n_k} = 0$$

or:

$$\boxed{e^{(\mathrm{instr})}_{n_i n n_k} e^{(\mathrm{sat})}_{n_i n_k} + e^{(\mathrm{sat})}_{n_i n_k} e^{(\mathrm{instr})}_{n_i n n_k} = 0} \qquad ^{1)}$$

With this conditional equation (four real equations having the rank 1), the rank of the above basic relation is three, exactly enough to compute the three components of q_{Cn}.
For our purpose we choose a slightly different way:

If the angle enclosed by $e^{(\mathrm{sat})}_{n_i C n_k}$ and $e^{(\mathrm{instr})}_{n_i n n_k}$ is ν, then one must also have:

$$e^{(\mathrm{instr})}_{n_i n n_k} e^{(\mathrm{sat})^{-1}}_{n_i C n_k} = \cos\nu + e^{(\mathrm{sat})}_{n_i n_k} \sin\nu$$

or:

$$\boxed{e^{(\mathrm{instr})}_{n_i n n_k} = \left(\cos\nu + e^{(\mathrm{sat})}_{n_i n_k} \sin\nu\right) e^{(\mathrm{sat})}_{n_i C n_k}}$$

It is further assumed that this expression is substituted into the basic relation, with as an extra (nuisance) parameter the angle ν. The basic relation then provides four real equations with four unknowns. The indices (instr) and (sat) can then be omitted.
Now proceed to the difference equation of the basic relation, with a reduction to $\Delta\Pi$-quantities:

$$\Delta Q_{n_i C n_k} - \Delta Q_{n_i n n_k}\left(1 - Q_{n_i C n}\right) - \left(1 - Q_{n_i n n_k}\right) \Delta Q_{n_i C n} = 0$$

with:

$$1 - Q_{n_i C n} = 1 - q_{Cn} q^{-1}_{Cn_i} = \left(q_{Cn_i} - q_{Cn}\right) q^{-1}_{Cn_i} = q_{nn_i} q^{-1}_{Cn_i}$$

$$\left(1 - Q_{n_i n n_k}\right) = 1 - q_{nn_k} q^{-1}_{nn_i} = \left(q_{nn_i} - q_{nn_k}\right) q^{-1}_{nn_i} = q_{n_k n_i} q^{-1}_{nn_i} q_{Cn} q^{-1}_{Cn}$$

Premultiply by $q^{-1}_{Cn_k}$ and postmultiply by q_{Cn_i}:

1) When applying quaternion theory to analytical photogrammetry in the sixties we met a similar problem, in which attention was focused on direction measurement. M. Molenaar finally found the cause.

9.

$$\left(q_{Cn_k}^{-1} \vartriangle Q_{n_iCn_k} q_{Cn_i}\right) - \left(q_{Cn_k}^{-1} q_{nn_k}\right)\left(q_{nn_k}^{-1} \vartriangle Q_{n_inn_k} q_{nn_i}\right) +$$
$$-\left(q_{Cn_k}^{-1} q_{n_in_k}\right)\left(q_{n_in}^{-1} q_{Cn}\right)\left(q_{Cn}^{-1} \vartriangle Q_{n_iCn} q_{Cn}\right) = 0$$

or:

$$\boxed{\vartriangle \Pi_{n_iCn_k} - Q_{Cn_kn}^{T} \vartriangle \Pi_{n_inn_k} - Q_{Cn_kn_i}^{T} Q_{n_inC}^{T} \vartriangle \Pi_{n_iCn} = 0}$$

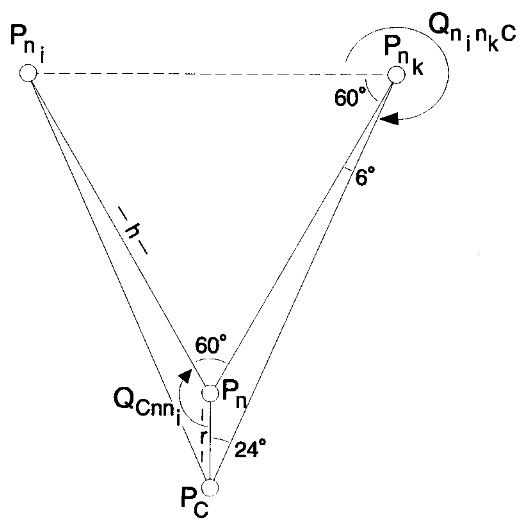
Solve for $\vartriangle \Pi_{n_iCn}$:

$$\vartriangle \Pi_{n_iCn} = Q_{Cnn_i}^{T} Q_{n_in_kC}^{T}\left(\vartriangle \Pi_{n_iCn_k} - Q_{Cn_kn}^{T} \vartriangle \Pi_{n_inn_k}\right)$$

or:

$$\boxed{\vartriangle \Pi_{n_iCn} = Q_{Cnn_i}^{T}\left(Q_{n_in_kC}^{T} \vartriangle \Pi_{n_iCn_k} - Q_{n_in_kn}^{T} \vartriangle \Pi_{n_inn_k}\right)}$$

As an example we take a GPS-satellite at an altitude of three times the earth radius, in the zenith of P_n .



$$Q^T_{Cnn_i} = \frac{h}{r}\left[\cos 150^\circ - e_{n_iCn_k}\sin 150^\circ\right]$$

$$Q^T_{n_in_kC} = \frac{r+h}{h}\left[\cos(-66^\circ) - \left(-e_{n_iCn_k}\right)\sin(-66^\circ)\right]$$

$$\simeq Q^T_{n_in_kn}$$

and with: $\frac{h}{r} + \frac{r+h}{h} = 1 + \frac{h}{r} \simeq 4$:

$$Q^T_{Cnn_i}\, Q^T_{n_in_kC} = 4\left[-0.87 - e_{n_iCn_k} \,.\, 0.50\right]\left[0.41 - e_{n_iCn_k} \,.\, 0.91\right]$$

$$= 4\left[-0.81 + e_{n_iCn_k} \,.\, 0.59\right]$$

Now we have:

$$\Delta\Pi_{n_iCn_k} = \Delta\left(\ln\frac{s_{Cn_k}}{s_{Cn_i}}\right) - e_{n_iCn_k}\,\Delta\left(\theta_{Cn_k} - \theta_{Cn_i}\right) +$$

$$+ \sin\alpha_{n_iCn_k}\, e^{-1}_{Cn_k}\,\Delta e_{n_iCn_k}\, e_{Cn_i}$$

Combination of the last two formulas gives, with

$$e_{n_iCn_k}\, e^{-1}_{Cn_k}\,\Delta e_{n_iCn_k}\, e_{Cn_i} = -e^{-1}_{Cn_k}\, e_{n_iCn_k}\,\Delta e_{n_iCn_k}\, e_{Cn_i} :$$

$$\boxed{\begin{aligned} \frac{1}{4}\Delta\Pi_{n_iCn} = & \left[-0.81\,\Delta\left(\ln\frac{s_{Cn_k}}{s_{Cn_i}}\right) + 0.59\,\Delta\left(\theta_{Cn_k} - \theta_{Cn_i}\right)\right] + \\ & + e_{n_iCn_k}\left[0.59\,\Delta\left(\ln\frac{s_{Cn_k}}{s_{Cn_i}}\right) + 0.81\,\Delta\left(\theta_{Cn_k} - \theta_{Cn_i}\right)\right] + \\ & + \sin\alpha_{n_iCn_k}\, e^{-1}_{Cn_k}\left[-0.81 - 0.59 e_{n_iCn_k}\right]\Delta e_{n_iCn_k}\, e_{Cn_i} + \\ & + \text{influence }\Delta\Pi_{n_inn_k} \end{aligned}}$$

The factor 4 is an extreme case, for e.g. Lageos the factor is 2.
But there is another effect when q_{Cn} is computed:

$$\Delta \Pi_{n_i Cn} = q_{Cn}^{-1} \Delta q_{Cn} - q_{Cn_i}^{-1} \Delta q_{Cn_i}$$

or:

$$\boxed{\Delta q_{Cn} = q_{Cn} \Delta \Pi_{n_i Cn} + Q_{n_i Cn} \Delta q_{Cn_i}}$$

In our example:

$$\boxed{\Delta q_{Cn} = re_{Cn} \cdot \Delta \Pi_{n_i Cn} + \frac{1}{4}\left(0.91 + e_{n_i Cn_k} \cdot 0.41\right) \Delta q_{Cn_i}}$$

i.e. the influence of Δq_{Cn_i} is reduced by a factor 4.

However, the example has been elaborated mainly to appraise the influence of $\overline{P_M P_C} \neq 0$.
From Section 8.2.1 follows:

$$\Delta(\theta_{Ck} - \theta_{Ci}) \simeq (\theta_{Ck} - \theta_{Ci}) \left[3B_1^{(1)} + \ldots\right]$$

$$\Delta\left(\ln \frac{s_{Ck}}{s_{Ci}}\right) \simeq -2\sin \frac{\theta_{Ck} - \theta_{Ci}}{2} \left[2Re\left\{\underset{\cdots}{\bar{e}} \frac{\bar{q}_{MC}}{r_1}\right\} + \ldots\right]$$

with in our example:

$$\begin{cases} (\theta_{Ck} - \theta_{Ci}) \simeq 0.84 \\ 2\sin \dfrac{\theta_{Ck} - \theta_{Ci}}{2} \simeq 0.81 \end{cases}$$

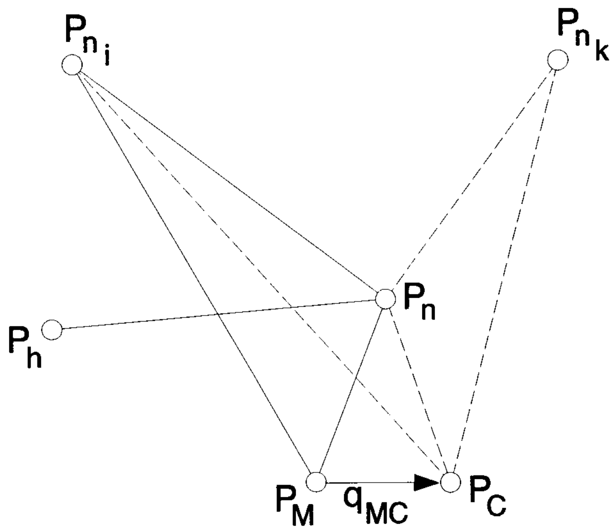
or:

$$\frac{1}{4} \Delta \Pi_{n_i Cn} \gtrsim O\left(\frac{r_{MC}}{r_1}\right) + \ldots$$

The conclusion is that the influence of $\overline{P_M P_C} \neq 0$ is clearly **not** negligible.

9.2

In this section we shall investigate the possibility of determining q_{MC} , leaving **all other unknowns and disturbing influences out of consideration.**



The **first method** consists of fixing the satellite point P_{n_i} from a number of terrestrial points P_n , P_h , ... whose coordinates are known; this results in the quaternion Q_{hnn_i} . Q_{Mnh} is computed from the known coordinates. Q_{n_iCn} follows from the bird's-tail construction of Section 9.1. Then we have:

$$Q_{Mnn_i} = Q_{hnn_i} Q_{Mnh}$$

$$\Delta \Pi_{Mnn_i} = \Delta \Pi_{hnn_i} + \Delta \Pi_{Mnh}$$

Then:

$$Q_{Mnn_i} + Q^{-1}_{n_iMn} = 1$$

$$\Delta Q_{Mnn_i} - Q^{-1}_{n_iMn} \Delta Q_{n_iMn} Q^{-1}_{n_iMn}$$

Premultiply by $q^{-1}_{nn_i}$ and postmultiply by q_{nM} :

$$\left(q^{-1}_{nn_i} \Delta Q_{Mnn_i} q_{nM}\right) - q^{-1}_{nn_i} q_{Mn_i} \left(q^{-1}_{Mn} \Delta Q_{n_iMn} q_{Mn_i}\right) q^{-1}_{Mn} q_{nM} = 0$$

or:

$$\Delta\Pi_{Mnn_i} + Q^T_{nn_iM}\,\Delta\Pi_{n_iMn} = 0$$

Now we have:

$$\Delta\Pi_{n_iMn} = q^{-1}_{Mn}\,\Delta q_{Mn} - q^{-1}_{Mn_i}\,\Delta q_{Mn_i}$$

The approximate value of $q_{MC} = 0$, hence:

$$\begin{cases} q_{Mn} = q_{Cn} + q_{MC} & , \ \Delta q_{Mn} = \Delta q_{Cn} + q_{MC} \\ q_{Mn_i} = q_{Cn_i} + q_{MC} & , \ \Delta q_{Mn_i} = \Delta q_{Cn_i} + q_{MC} \end{cases}$$

In coefficients of Δq-quantities the indices M and C can be interchanged, in fact all approximate values are computed with respect to P_M anyway. Hence:

$$q^{-1}_{Mn}\,\Delta q_{Mn} = q^{-1}_{Cn}\,\Delta q_{Cn} + q^{-1}_{Cn}\,q_{MC} \quad , \ \text{analogous } q^{-1}_{Mn_i}\,\Delta q_{Mn_i}$$

so:

$$\begin{aligned} \Delta\Pi_{n_iMn} &= \Delta\Pi_{n_iCn} + \left(q^{-1}_{Cn} - q^{-1}_{Cn_i}\right) q_{MC} \\ &= \Delta\Pi_{n_iCn} + q^{-1}_{Cn_i}\underbrace{\left(q_{Cn_i} - q_{Cn}\right)}_{q_{nn_i}} q^{-1}_{Cn}\,q_{MC} \end{aligned}$$

and with:

$$Q^T_{nn_iM} \simeq Q^T_{nn_iC} = q^{-1}_{nn_i}\,q_{n_iC} \ :$$

(I) $$\boxed{\Delta\Pi^{(\mathrm{terr})}_{Mnn_i} + Q^T_{nn_iC}\,\Delta\Pi^{(\mathrm{sat})}_{n_iCn} + q^{-1}_{Cn}\,q_{MC} = 0}$$

with (terr) = terrestrial and (sat) = computed from satellite data. This is a condition equation, whose second term contains an indirect influence of q_{MC} and whose third term contains a direct influence of q_{MC}. Both influences have the same order of magnitude because:

$$N^{1/2}\left\{Q^T_{nn_iC}\right\} = 1 + \frac{r}{h}$$

Check:

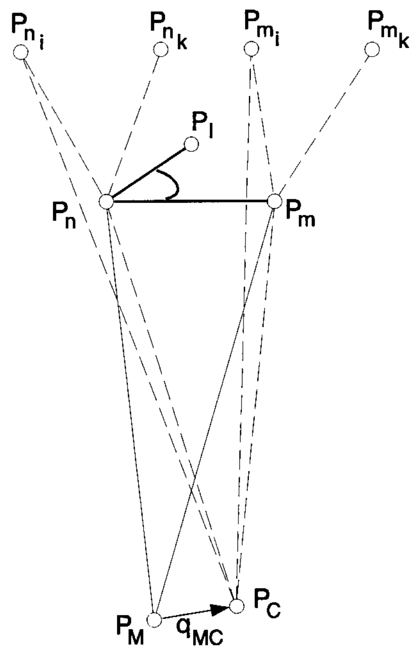
$$\triangle\Pi_{Mnn_i} = \overset{-1}{q_{nn_i}}\triangle q_{nn_i} - \overset{-1}{q_{nC}}\left(\triangle q_{nC} - q_{MC}\right)$$

$$= \triangle\Pi_{Cnn_i} - \overset{-1}{q_{Cn}}q_{MC}$$

or:

$$\triangle\Pi_{Cnn_i} + Q^T_{nn_iC}\triangle\Pi_{n_iCn} = 0\ , \qquad \text{correct!}$$

Yet (I) does not offer a practical possibility for estimating q_{MC} , because the sharpness of definition of $\triangle\Pi_{Mnn_i}$ is certainly not better than 10^{-5} or 10^{-6}, whereas nowadays the order of magnitude of $\frac{r_{MC}}{r_1}$ must be deemed to be the same or smaller.



In the **second method** a bird's-tail construction is applied in two terrestrial points P_n and P_m of the same terrestrial network. This provides the quaternions Q_{n_iCn} and Q_{m_iCm}.

As an example we think of the network of the USA with P_n and P_m at about 40° latitude. The maximum difference in longitude is 40° , so that, with $\cos 40^\circ \simeq 0.77$, $\simeq \alpha_{nCm} \simeq 0.77, * 40^\circ \simeq 30^\circ$.

Q_{nMm} is computed from the known coordinates of P_n and P_m , and compared with Q_{nCm} :

$$\begin{aligned}
\Delta\Pi_{nMm} &= q_{Mm}^{-1}\,\Delta q_{Mm} - q_{Mn}^{-1}\,\Delta q_{Mn} = \\
&= q_{Cm}^{-1}\left(\Delta q_{Cm} + q_{MC}\right) - q_{Cn}^{-1}\left(\Delta q_{Cn} + q_{MC}\right) = \\
&= \Delta\Pi_{nCm} + q_{Cm}^{-1}\underbrace{\left(q_{Cn} - q_{Cm}\right) q_{Cn}^{-1}}_{-q_{nm}} q_{MC} = \\
&= \Delta\Pi_{nCm} - Q_{Cmn}^{T}\, q_{Cn}^{-1}\, q_{MC}
\end{aligned}$$

For the quaternions around P_C we have:

$$Q_{nCm} = Q_{m_iCm}\, Q_{n_iCm_i}\, Q_{nCn_i}\,, \qquad Q_{nCn_i} = Q_{n_iCn}^{-1}$$

with Q_{m_iCm} and Q_{n_iCn} from the bird's-tail construction and $Q_{n_iCm_i}$ from the satellite orbit computation. Consequently we have:

$$\text{(II)} \qquad \boxed{\begin{array}{c} \Delta\Pi_{nCm}^{(\mathrm{sat})} = \left(\Delta\Pi_{m_iCm} - \Delta\Pi_{n_iCn}\right) + \Delta\Pi_{n_iCm_i} \\ \hline \Delta\Pi_{nMm}^{(\mathrm{terr})} - \Delta\Pi_{nCm}^{(\mathrm{sat})} + Q_{Cmn}^{T}\, q_{Cn}^{-1}\, q_{MC} = 0 \end{array}}$$

(II) is again a condition equation; here all terms containing the index C are influenced by q_{MC} in the same order of magnitude because:

$$N^{1/2}\left\{Q_{Cmn}^{T}\right\} \simeq \frac{\tfrac{1}{2}r}{r} = \frac{1}{2}$$

Alas, the conclusion is the same as the one reached for (I).

For the **third method** we add a point P_l belonging to the same terrestrial network as P_n and P_m , with for example $P_l\, P_n \perp P_n\, P_m$ and possibly $\overline{P_lP_n} \simeq \frac{1}{2}\overline{P_nP_m}$. The bird's-tail construction in P_l is once more applied.
For the quaternions around P_n we than have:

$$Q_{lnm} = Q_{Cnm}\, Q_{Cnl}^{-1}\,, \quad \Delta\Pi_{lnm} = \Delta\Pi_{Cnm} - \Delta\Pi_{Cnl}$$

with in the triangles $P_n\, P_C\, P_m$ and $P_n\, P_C\, P_l$ respectively:

$$Q_{Cnm} = 1 - Q_{nCm} \;,\; \Delta\Pi_{Cnm} = Q_{nmC}^{T} \cdot \Delta\Pi_{nCm}$$

$$Q_{Cnl} = 1 - Q_{nCl} \;,\; \Delta\Pi_{Cnl} = Q_{nlC}^{T} \cdot \Delta\Pi_{nCl}$$

or:

$$Q_{lnm} = \left(1 - Q_{nCm}\right)\left(1 - Q_{nCl}\right)^{-1}$$

$$\boxed{\Delta\Pi_{lnm}^{(\mathrm{sat})} = Q_{nmC}^{T} \cdot \Delta\Pi_{nCm} - Q_{nlC}^{T} \cdot \Delta\Pi_{nCl}}$$

For our USA-example we then have:

$$N^{1/2}\left\{Q_{nmC}^{T}\right\} \simeq \frac{r}{\tfrac{1}{2}r} = 2 \;,\quad N^{1/2}\left\{Q_{nlC}^{T}\right\} \simeq \frac{r}{\tfrac{1}{4}r} = 4$$

Again there is an evident deformation of $\Delta\Pi_{lnm}^{(\mathrm{sat})}$ *by* $q_{MC} \neq 0$.

Q_{lnm} can also be directly computed from the terrestrial network, the result being denoted by $Q_{lnm}^{(\mathrm{terr})}$, so that the condition equation becomes:

$$\Delta\Pi_{lnm}^{(\mathrm{terr})} - \Delta\Pi_{lnm}^{(\mathrm{sat})} = 0$$

but for the same reason as before with (I) and (II), this is useless.

However it does make sense to determine Q_{lnm} from VLBI-measurements, for which a sharpness of definition of 10^{-7} to 10^{-8} can be expected, whereas there is no influence of $q_{MC} \neq 0$.

Then the condition equation takes the form:

$$\text{(III)} \qquad \boxed{\Delta\Pi_{lnm}^{(\mathrm{VLBI})} - \Delta\Pi_{lnm}^{(\mathrm{sat})} = 0}$$

An advantage of this approach is that one is not restricted to the use of points P_n , P_m , P_l belonging to the same terrestrial network. Points can be chosen anywhere on earth and their number can be arbitrarily increased. However, a remaining restriction is that one must always operate in the *X, Y, Z*-frame used for the launching of the satellite because otherwise the definition of q_{MC} is lost. One may of course introduce new base points by means of a similarity transformation, but this also implies a change of P_M , so that in fact a new q_{MC} is introduced by the transformation.

The indiscriminate connection of sets of VLBI-, satellite- and possibly terrestrial coordinates which is sometimes met in the literature does not seem to make much sense, and as far as the interpretation of mutual differences is concerned I think it can even be misleading.

Finally it must be noted that there will always be a residual influence of q_{MC}, even if the estimate is corrected for. This residual influence will depend on the variances and covariances of the many unknowns to be estimated in the total adjustment; the order of magnitude of the resulting residual deformation cannot be predicted as yet.

9.3.1

In the Sections 9.1 and 9.2 the building stones have been brought together for examining the influence of $q_{MC} \neq 0$ (and, naturally, other unknowns) on relevant Δ-quantities. We must now focus on more realistic observable quantities, i.e. **length ratios or series of pseudo-distances** to satellites, as the case may be.

Nevertheless we adhere the bird's-tail construction, in which from approximate coordinates approximate values are computed for $\frac{s_{n_k}}{s_{n_i}}$, $\alpha_{n_i n n_k}$ and $e_{n_i n n_k}$.
This implies that $\Delta\alpha_{n_i n n_k}$ and $\Delta e_{n_i n n_k}$ become extra (nuisance) unknowns in our problem.

Now we solve $\Delta\Pi_{n_i n n_k}$ from the difference equation of the bird's-tail construction; the quaternion notation Q will temporarily be used as abbreviation:

$$Q^T_{nn_kC}\,\Delta\Pi_{n_iCn_k} - Q^T_{nn_kn_i}\,Q^T_{n_inC}\,\Delta\Pi_{n_iCn} \underset{\text{put}}{=} Q = \Delta\Pi_{n_inn_k}$$

Elimination of, first, $\Delta e_{n_inn_k}$ and, second, $\Delta\alpha_{n_inn_k}$ is effected by the computation of components of $\Delta\Pi_{n_inn_k}$:

$$\underbrace{\frac{1}{2}\left(Q + e_{n_inn_k}\,Q e^{-1}_{n_inn_k}\right)}_{Sc\{Q\} \,+\, \text{component } Ve\{Q\} \,/\!/\, e_{n_inn_k}} = \Delta\left(\ln \frac{s_{nn_k}}{s_{nn_i}}\right) - e_{n_inn_k}\,\Delta\left(\alpha_{n_inn_k}\right)$$

or:

$$\boxed{Sc\left\{Q^T_{nn_kC}\,\Delta\Pi_{n_iCn_k} - Q^T_{nn_kn_i}\,Q^T_{n_inC}\,\Delta\Pi_{n_iCn}\right\} = \Delta\left(\ln \frac{s_{nn_k}}{s_{nn_i}}\right)}$$

Besides q_{MC} and other unknowns, the three components of $\Delta\ q_{Cn}$ appear in this equation, namely via $\Delta\Pi_{n_iCn}$. If one considers only the latter three unknowns, they require the measurement of three length ratios. The measurement of a number of length ratios to the same segment of a satellite orbit is useless, because the length ratios will result in difference equations having practically the same coefficients for each unknown, so that the solution will be indeterminate.

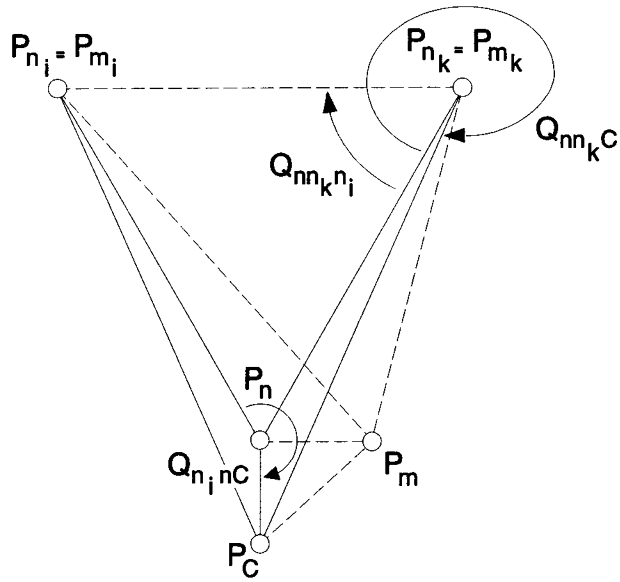
One is therefore compelled to measure length ratios to at least **three non-parallel satellite orbits**, i.e. one length ratio to each orbit. In the case that different satellites are involved it is to be assumed that **the launching data of all these satellites have been determined in the same *X*, *Y*, *Z*- frame**, otherwise there is not a unique q_{MC}. This assumption is valid e.g. for GPS-satellites.

The reason for preferring $s_{nn_k} \approx s_{nn_i}$ in the bird's-tail construction can be found in remarks by P.J.G. Teunissen in his contribution to the Lustrum Book 1990 of the Delft geodetic students' association Snellius. The point is that in this case certain (time-dependent) unknown constants, arising from the measuring procedure, can be eliminated.
Less favourable is the measurement of a series of pseudo-distances to **four** satellites, at least one distance measurement per satellite. From the measurements s_{nn_i} , s_{nn_k} , s_{n,n_l} , s_{n,n_m} one can form three ratios for the computation. But the points P_{n_i} , P_{n_k} , P_{n_l} and P_{n_m} are now situated on different satellite orbits; this causes an inconvenient mixture of orbit errors and besides the above mentioned elimination procedure cannot be applied without complications.

9.3.2

Simultaneous measurements in P_n and P_m result in:

9.



$$Q^T_{nn_kC}\,\Delta\Pi_{n_iCn_k} - Q^T_{nn_kn_i}\,Q^T_{n_inC}\,\Delta\Pi_{n_iCn} = \Delta\Pi_{n_inn_k}$$

$$Q^T_{mm_kC}\,\Delta\Pi_{m_iCm_k} - Q^T_{mm_km_i}\,Q^T_{m_imC}\,\Delta\Pi_{m_iCm} = \Delta\Pi_{m_imm_k}$$

with $P_{n_i} = P_{m_i}$, $P_{n_k} = P_{m_k}$, s_{nm} small compared to s_{nn_i} and s_{mm_i} and s_{nC} and s_{mC}.

In this case a differential approximation can be applied:

$$\left.\begin{array}{lll} q_{mm_i} = q_{nn_i} + \Delta q_{nm_i} & , & \Delta q_{nm_i} = \\ q_{mm_k} = q_{nm_k} + \Delta q_{nn_k} & , & \Delta q_{nn_k} = \\ q_{mC} = q_{nC} + \Delta q_{nC} & , & \Delta q_{nC} = \end{array}\right\} - q_{nm}$$

$$Q^T_{mm_kC} = Q^T_{nn_kC} + \Delta Q^T_{nn_kC} \quad , \quad \text{with:}$$

$$\Delta Q^T_{nn_kC} = q^{-1}_{n_kn}\,\Delta\Pi^T_{nn_kC}\,q_{n_kC} = q^{-1}_{n_kn}\left(\Delta q_{n_kC}\,q^{-1}_{n_kC} - \Delta q_{n_kn}\,q^{-1}_{n_kn}\right) q_{n_kC} =$$

$$= -\,q^{-1}_{n_kn}\,q_{nm}\,q^{-1}_{n_kC}\,\underbrace{\left(q_{n_kn} - q_{n_kC}\right)}_{q_{Cn}}\,q^{-1}_{n_kn}\,q_{n_kC} =$$

$$= -\,Q^T_{n_knm}\,Q^T_{n_kCn}\,Q^T_{nn_kC}$$

or:

$$\boxed{Q^T_{mm_kC} = \left(1 - Q^T_{n_knm}\, Q^T_{n_kCn}\right) Q^T_{nn_kC}}$$

Similarly with $\Delta q_{n_kn_i} = 0$:

$$\Delta\left(Q^T_{nn_kn_i}\, Q^T_{n_inC}\right) = \Delta Q^T_{nn_kn_i}\, Q^T_{n_inC} + Q^T_{nn_kn_i}\, \Delta Q^T_{n_inC} =$$

$$= q^{-1}_{n_kn} \left(\Delta q_{n_kn_i}\, q^{-1}_{n_kn_i} - \Delta q_{n_kn}\, q^{-1}_{n_kn}\right) q_{n_kn_i}\, Q^T_{n_knC} +$$

$$+ Q^T_{nn_kn_i}\, q^{-1}_{nn_i} \left(\Delta q_{nC}\, q^{-1}_{nC} - \Delta q_{nn_i}\, q^{-1}_{nn_i}\right) q_{nC} =$$

$$= - q^{-1}_{n_kn}\, q_{nm}\, q^{-1}_{n_kn}\, q_{n_kn_i}\, Q^T_{n_inC} +$$

$$- Q^T_{nn_kn_i}\, q^{-1}_{nn_i}\, q_{nm}\, q^{-1}_{nC} \underbrace{\left(q^{-1}_{nn_i} - q_{nC}\right)}_{q_{Cn_i}} q^{-1}_{nn_i}\, q_{nC} =$$

$$= + Q^T_{n_knm}\, Q^T_{nn_kn_i}\, Q^T_{n_inC} + Q^T_{nn_kn_i}\, Q^T_{n_inm}\, Q^T_{nCn_i}\, Q^T_{n_inC}$$

or:

$$\boxed{Q^T_{mm_km_i}\, Q^T_{m_imC} = \left(1 + Q^T_{n_knm}\right) Q^T_{nn_kn_i} \left(1 + Q^T_{n_inm}\, Q^T_{nCn_i}\right) Q^T_{n_inC}}$$

Now we have:

$$N^{1/2}\left\{Q^T_{n_knm}\, Q^T_{n_kCn}\right\} \simeq \frac{r_{nm}}{h}\, \frac{r}{h + r} = \frac{\dfrac{r_{nm}}{r}}{\dfrac{h}{r}\left(1 + \dfrac{h}{r}\right)}$$

$$N^{1/2}\left\{Q^T_{n_knm}\right\} \simeq \frac{r_{nm}}{h} = \frac{\dfrac{r_{nm}}{r}}{\dfrac{h}{r}}$$

$$N^{1/2}\left\{Q^T_{n_i nm}\, Q^T_{nCn_i}\right\} \simeq \frac{r_{nm}}{h}\,\frac{h+r}{r} = \frac{\dfrac{r_{nm}}{r}}{\dfrac{h}{r}\,\dfrac{1}{\left(1+\dfrac{h}{r}\right)}}$$

The last norm is largest: $\dfrac{r_{nm}}{r}\left(1+\dfrac{r}{h}\right)$.

The difference of the coefficients may be ignored if after choosing a value for α:

$$\frac{r_{nm}}{r}\,\frac{r+h}{h} \leq 0.01 \cdot \alpha$$

and, with $r \simeq 64.10^2$ km:

$$(r_{nm})km \leq \frac{\dfrac{h}{r}}{1+\dfrac{h}{r}}\, 64 \cdot \alpha$$

h/r \ α	1	5
1	32	160
2	43	215
3	48	240
	r_{nm}	km

In this case the difference of the two equations mentioned first in this section is:

$$-\,Q^T_{nn_kn_i}\, Q^T_{n_inC}\left(\Delta\Pi_{m_iCm} - \Delta\Pi_{n_iCn}\right) = \Delta\Pi_{m_imm_k} - \Delta\Pi_{n_inn_k}$$

so that the influence of the orbit computation is practically completely eliminated. Further we have:

$$\Delta\Pi_{m_iCm} - \Delta\Pi_{n_iCn} = \Delta\Pi_{n_iCm} - \Delta\Pi_{n_iCn} = \Delta\Pi_{nCm} =$$

$$= q^{-1}_{Cm}\,\Delta q_{Cm} - q^{-1}_{Cn}\,\Delta q_{Cn} = q^{-1}_{Cn}\left(\Delta q_{Cm} - \Delta q_{Cn}\right) = q^{-1}_{Cn}\,\Delta q_{nm}$$

Finally the difference of the equations becomes:

$$\boxed{+\ Q^T_{nn_kn_i}\ q^{-1}_{nn_i}\ \Delta q_{nm} = \Delta\Pi_{m_imm_k} - \Delta\Pi_{n_inn_k}}$$

i.e. a relative positioning of P_m with respect to P_n.

For the case that **only pseudo-distance measurement** is applied, we still have to investigate whether $e_{m_imm_k}$ may be replaced by $e_{n_inn_k}$.

$$\text{Put :}\quad Q_{m_imm_k} = Q_{n_inn_k} + \Delta Q_{n_inn_k}$$

$$\begin{aligned}
\Delta Q_{n_inn_k} &= q_{nn_k}\ \Delta\Pi_{n_inn_k}\ q^{-1}_{nn_i} = \\
&= q_{nn_k}\left(q^{-1}_{nn_k}\ \Delta q_{nn_k} - q^{-1}_{nn_i}\ \Delta q_{nn_i}\right) q^{-1}_{nn_i} = \\
&= -\ q_{nn_k}\ q^{-1}_{nn_k}\ \underbrace{q_{nn_i} - q_{nn_k})}_{q_{n_kn_i}}\ q^{-1}_{nn_i}\ q_{nm}\ q^{-1}_{nn_i} = \\
&= -\ Q_{nn_in_k}\ Q_{n_knm}\ Q_{n_inn_k}
\end{aligned}$$

hence:

$$\boxed{Q_{m_imm_k} = \left(1 - Q_{nn_in_k}\ Q_{n_knm}\right) Q_{n_inn_k}}$$

with:

$$N^{1/2}\left\{Q_{nn_in_k}\ Q_{n_knm}\right\} \simeq \frac{h}{h}\ \frac{r_{nm}}{h} = \frac{\;\frac{r_{nm}}{r}\;}{\frac{h}{r}}$$

or, the neglect of $\Delta Q_{n_inn_k}$, and consequently of $\left(e_{m_imm_k} - e_{n_inn_k}\right)$ stays within the limits previously defined.

Now introduce again the temporary notation:

$$Q^T_{nn_kn_i}\ q_{nn_i}\ \Delta q_{nm} = Q'$$

Then we have:

$$\frac{1}{2}\left(Q' + e_{n_i nn_k}\, Q'\, e^{-1}_{n_i nn_k}\right) =$$

$$= \Delta\left(\ln\frac{s_{mm_k}}{s_{mm_i}}\right) - \Delta\left(\ln\frac{s_{nn_k}}{s_{nn_i}}\right) - e_{n_i nn_k}\left(\Delta\alpha_{m_i mm_k} - \Delta\alpha_{n_i nn_k}\right)$$

or:

$$\boxed{Sc\left\{Q^{T}_{nn_k n_i}\, q^{-1}_{nn_i}\, \Delta q_{nm}\right\} = \Delta\left(\ln\frac{s_{mm_k}}{s_{mm_i}}\right) - \Delta\left(\ln\frac{s_{nn_k}}{s_{nn_i}}\right)}$$

being one real equation in the three components of Δq_{nm} .

The right hand member can also be written as:

$$\Delta\left(\ln\frac{s_{mn_k}}{s_{nn_k}}\right) - \Delta\left(\ln\frac{s_{mn_i}}{s_{nn_i}}\right) =$$

$$= \Delta\ln\left(1 + \frac{s_{mn_k} - s_{nn_k}}{s_{nn_k}}\right) - \Delta\ln\left(1 + \frac{s_{mn_i} - s_{nn_i}}{s_{nn_i}}\right) =$$

$$= \Delta\left(\frac{s_{mn_k} - s_{nn_k}}{s_{nn_k}}\right) - \Delta\left(\frac{s_{mn_i} - s_{nn_i}}{s_{nn_i}}\right)$$

being the difference equation for the measurement of distance-differences.

10.

Ever since I introduced the distance or length ratio as a basic quantity alongside the traditional angle quantity, I have been wondering how one could form dimensionless quantities in mechanics. It seems to me that the quantities found now may provide a good point of departure.

The length ratio presented the problem that in society one only uses the concept of "length", but this was easily explained because in daily use one only deals with "carpenter's length" in which the finer differences of scale are not felt.

Something similar applies to the concepts of speed, acceleration and time. Admittedly it is important for the movement of vehicles and aeroplanes that the vectors "follow" the curvature of the earth, i.e. follow local coordinate frames, but then the conceptual construction of the quantities can be greatly simplified. This is particularly evident if only small areas are considered. In daily life one can then put $\frac{r_k}{r_1}$ and $\frac{g_k}{g_1}$ equal to 1, the difference between local coordinate systems can be ignored and the factors

$$\left(\frac{1}{r_1 g_1}\right)^{1/2}, \ \frac{r_k}{r_1 g_1} \text{ and } \left(\frac{g_1}{r_1}\right)^{1/2}$$

can be replaced by some form of calibration. One then arrives at the usual differences of velocity vectors, acceleration vectors and points of time.

In order to illustrate the importance of the new quantities, an imagined example in satellite gradiometry is given, by the way a field of which I cannot claim to have any knowledge.

Choose the local x, y, z-coordinate system as identical with the E, N, r-system from Section 6, except for the eccentricity of the origin of the coordinate system.

Assume the possibility for the gradiometer to realize this system, up to a small systematic rotation error in each of the three axes. Furthermore a small systematic scale error is permitted, separately for the measurement of each tensor component. It is assumed that the following quantity is measured (see Section 6):

$$\frac{s_k^2 \Gamma_{k,UV}}{v_1^2}$$

If $\sum_{k,UV}$ represents the spherical harmonics series starting from degree two, then the following difference equations for measurements in one and the same $S_{s_1 v_1}$-system are valid:

$$\Delta\left(\frac{s_k^2\Gamma_{k,xx}}{v_1^2}\right) - \frac{s_1}{s_k}\,\Delta\left(\frac{s_1^2\Gamma_{1,xx}}{v_1^2}\right) = + \frac{s_1}{s_k}\,\Delta\left(\ln\frac{s_k}{s_1}\right) + \frac{s_1}{s_k}\left(\Delta\Sigma_{k,xx} - \Delta\Sigma_{1,xx}\right)$$

$$\Delta\left(\frac{s_k^2\Gamma_{k,xy}}{v_1^2}\right) - \frac{s_1}{s_k}\,\Delta\left(\frac{s_1^2\Gamma_{1,xy}}{v_1^2}\right) = + \frac{s_1}{s_k}\left(\Delta\Sigma_{k,xy} - \Delta\Sigma_{1,xy}\right)$$

$$\Delta\left(\frac{s_k^2\Gamma_{k,xz}}{v_1^2}\right) - \frac{s_1}{s_k}\,\Delta\left(\frac{s_1^2\Gamma_{1,xz}}{v_1^2}\right) = + 3\,\frac{s_1}{s_k}\left(\frac{\Delta x_k}{s_k} - \frac{\Delta x_1}{s_1}\right) + \frac{s_1}{s_k}\left(\Delta\Sigma_{k,xz} - \Delta\Sigma_{1,xz}\right)$$

$$\Delta\left(\frac{s_k^2\Gamma_{k,yy}}{v_1^2}\right) - \frac{s_1}{s_k}\,\Delta\left(\frac{s_1^2\Gamma_{1,yy}}{v_1^2}\right) = \frac{s_1}{s_k}\,\Delta\left(\ln\frac{s_k}{s_1}\right) + \frac{s_1}{s_k}\left(\Delta\Sigma_{k,yy} - \Delta\Sigma_{1,yy}\right)$$

$$\Delta\left(\frac{s_k^2\Gamma_{k,yz}}{v_1^2}\right) - \frac{s_1}{s_k}\,\Delta\left(\frac{s_1^2\Gamma_{1,yz}}{v_1^2}\right) = + 3\,\frac{s_1}{s_k}\left(\frac{\Delta y_k}{s_k} - \frac{\Delta y_1}{s_1}\right) + \frac{s_1}{s_k}\left(\Delta\Sigma_{k,yz} - \Delta\Sigma_{1,yz}\right)$$

$$\Delta\left(\frac{s_k^2\Gamma_{k,zz}}{v_1^2}\right) - \frac{s_1}{s_k}\,\Delta\left(\frac{s_1^2\Gamma_{1,zz}}{v_1^2}\right) = - 2\,\frac{s_1}{s_k}\,\Delta\left(\ln\frac{s_k}{s_1}\right) + \frac{s_1}{s_k}\left(\Delta\Sigma_{k,zz} - \Delta\Sigma_{1,zz}\right)$$

In a rough approximation we have for the earth as a homogeneous sphere:

$$\frac{s_k^2\Gamma_{k,xx}}{v_1^2} \simeq \frac{s_k^2\Gamma_{k,yy}}{v_1^2} \simeq \frac{1}{2}\,\frac{s_k^2\Gamma_{k,zz}}{v_1^2} \simeq \frac{s_1}{s_k}$$

$$\frac{s_k^2\Gamma_{k,xy}}{v_1^2} \simeq \frac{s_k^2\Gamma_{k,xz}}{v_1^2} \simeq \frac{s_k^2\Gamma_{k,yz}}{v_1^2} \simeq 0$$

i.e. the influence of scale errors of the supposedly measured quantities in the left hand members is negligible.

Furthermore it seems to me that the following is valid:

$$\left(\frac{\Delta x_k}{s_k} - \frac{\Delta x_1}{s_1}\right) \text{ eliminates a systematic rotation error around } y\text{-axes}$$

$$\left(\frac{\Delta y_k}{s_k} - \frac{\Delta y_1}{s_1}\right) \text{ eliminates a systematic rotation error around } x\text{-axes}$$

$$\frac{s_k^2 \Gamma_{k,xx}}{v_1^2} \simeq \frac{s_k^2 \Gamma_{k,yy}}{v_1^2} \text{ contains: negligible systematic rotation error around } z\text{-axes}$$

Conclusion: Under the assumptions made it does not seem to be necessary to calibrate the instrument!

Now it is interesting to compare accuracies with [Rummel, 1986]:
For a near-circular satellite orbit the following holds:

$$\frac{\mu}{s v^2} \simeq 1 \;;\; v^2 \simeq \frac{\mu}{s} \;;\; \frac{v^2}{s^2} \simeq \frac{\mu}{s^3}$$

With $\mu \simeq 39\ 86\ 00$ km s^{-2}; $s \simeq (6400 + 200)$ km:

$$\frac{v^2}{s^2} \simeq 1.4 \cdot 10^{-6}\, s^{-2} \simeq 1400 \cdot 10^{-9}\, s^{-2} = 1400 \text{ E.U.}$$

or:

$$\frac{s^2}{v^2} \simeq 7 \cdot 10^{-4}\ (\text{E.U.})^{-1}$$

With these values the amounts in [Rummel, 1986, p. 356] become:

$$\left.\begin{array}{l}
\dfrac{s^2 \Gamma_{xx}}{v^2} \simeq \dfrac{s^2 \Gamma_{yy}}{v^2} \simeq -1 \\[2ex]
\dfrac{s^2 \Gamma_{zz}}{v^2} \simeq +2 \\[2ex]
\dfrac{s^2 \Gamma_{yz}}{v^2} \simeq +7 \cdot 10^{-3} \\[2ex]
\dfrac{s^2 \Gamma_{xy}}{v^2} \simeq \dfrac{s^2 \Gamma_{xz}}{v^2} \simeq 7 \cdot 10^{-5}
\end{array}\right\}$$

10.

Now write $\sigma\left(\frac{s^2\Gamma_{UV}}{v^2}\right) = \frac{s^2}{v^2}\,\sigma(\Gamma_{UV})$ and put: $\sigma(\Gamma_{UV}) = \text{constant}$

If one now requires: $\frac{s^2}{v^2}\,\sigma\,(\Gamma_{UV}) \leq 10^{-8}$, which corresponds to an accuracy of half a decimeter, then:

$$\sigma(\Gamma_{UV}) \leq 1.4 \cdot 10^{-5}\ \text{E.U.}$$

This requirement cannot be met by instruments presently in use.

11.

11.1

Looking back I get the uneasy feeling that the foregoing sounds very much like a voice from the past. There is little of the dynamism that makes geodesy so different from what it was in my active years. This generation gap is most clearly illustrated by a comparison with the deeper and broader treatises in the book "Theory of Satellite Geodesy and Gravity Field Determination" (F. Sanso and R. Rummel Eds.- Lecture Notes in Earth Sciences Vol. 25, Springer Verlag, Berlin, Heidelberg 1989), which came to my attention thanks to Rummel.

Nevertheless I venture to make three final observations:

I. I should like to be young again and develop this lines of thought on the basis of the General Theory of Relativity. As it is now, the study of this theory is made difficult for me by the loss of invariance of angles and distance ratios in reciprocal observations of systems moving relative to each other, and by the inevitable tensor theory, whereas tensor algebra is not a division algebra. On the other hand there are recognizable problems when I read in Bulletin Géodésique (1992, **66**, page 65 [1)]): "the need to define a barocentric coordinate system with spatial origin at the centre of mass of the solar system and a geocentric coordinate system with spatial origin at the centre of mass of the Earth, and the desirability of defining analogous coordinate systems for other planets and for the Moon". How to execute such things operationally involves questions which have always occupied me in my work, including the previous Sections, be it in a simpler theoretical framework.

II. A further problem that keeps me occupied is the effect of my idea that in satellite computations every coordinate system is in fact an S-system with terrestrial base points. I mention this here in connection with a remark by R. Rummel, observing that the precision from the distance between a terrestrial point and an orbit point as computed from coordinates, is considerably worse than the precision of this distance directly measured by SLR.

As remarked in "Notes and References", Section 3.2", I would not be surprised if in partial networks for satellite orbit determination and in subsequent establishment of control by satellite method, the covariance function $d_{kl}^2 = cs_{kl}$ would give a rough approximation. Therefore this function will be chosen in order to obtain numerical values by means of a

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criterion covariance matrix. Since the three-dimensional matrix has not been sufficiently analysed, the two-dimensional situation will be used.

Assume that P_a and P_b are terrestrial datum points (and usually also the points from which the satellite orbit is controlled) and let P_i be a point of the satellite orbit.
Now consider the two-dimensional situation in the plane through P_a, P_b, P_i. If we denote by σ_i the standard deviation in each coordinate direction in P_i, then we have, with some approximation, according to [Baarda, 1973, p. 161]:

$$(\sigma_i \mathrm{cm})^2 \approx 2[c \ \mathrm{cm}^2 \ \mathrm{km}^{-1}] \frac{[s_{ai} \ \mathrm{km}][s_{bi} \ \mathrm{km}]}{[s_{ab} \ \mathrm{km}]}$$

We choose c from values found in partial nets of the Netherlands triangulation [Baarda, 1973, pp. 145-147], viz. the round value $c = 0.5$. In the terrestrial network containing the datum points this corresponds to a standard deviation of 10 cm for distances of 100 km between points, being the standard deviation of coordinate differences, or 10^{-6} . For $s_{ab} =$ 100 km and $\sigma_a = \sigma_b = 0$ one then obtains:

s_{ia} km	s_{ib} km	s_{ab} km	c cm^2 km^{-1}	σ_i cm
200	200	100	0.5	20
1000	1000	100	0.5	100
2000	2000	100	0.5	200

The order of magnitude for σ_i looks about right, although the first value seems too small. Perhaps this means that the effect of the choice of datum points justifies a closer study.

III. The last problem concerns my doubts about the order of magnitude of the precision in the determination of quantities in physical geodesy.
As an example the beautiful paper:
R. Rummel and M. van Gelderen - Spectral Analysis of the full gravity tensor - Geophys. J. Int. (1992), **111**, pp. 159-169
incidentally mentions that with the precision of 10^{-2} E.U. for Γ_{zz} , Γ_{yz} and Γ_{yy} "the global gravity field can be determined in six months time with a precision of 2.5 mGal in terms of gravity anomalies or 5 - 10 cm in terms of geoid heights".

If we take the example of satellite gradiometry from Section 10, restricting ourselves to the vertical gravity gradient Γ_j as an abbreviation of $\Gamma_{j,zz}$, then in our dimensionless formulation, with $s_j \approx s_1$, and nearly circular satellite orbits (see also Section 4.3):

$$\frac{s_j^2}{v_1^2}\,\Delta(\Gamma_j) - \frac{s_1^2}{v_1^2}\,\Delta(\Gamma_1) = 2\Delta\left(\ln\frac{\Gamma_j}{\Gamma_1}\right) \simeq 7.10^{-6} \qquad 2)$$

(In fact only the influence of Γ_j is taken into account here, but our model requires the combined influence of Γ_j **and** Γ_1, although in the following integral formulas the constant influence of a possible separate $\Delta(\Gamma_1)$ vanishes).

If $P_{k'}$ is the projection of P_k on the "geoid" through $P_{\bar{1}}$, then we have for the other two quantities mentioned in our model (see Section 4.1):

$$\Delta\left(\ln\frac{g_{k'}}{g_{\bar{1}}}\right) = -2\Delta\left(\ln\frac{W_k}{W_{\bar{1}}}\right) + \Delta\left(\ln\frac{g_k}{g_{\bar{1}}}\right) \simeq 3.5\ 10^{-6}$$

$$\Delta\left(\ln\frac{r_{k'}}{r_{\bar{1}}}\right) = \Delta\left(\ln\frac{W_k}{W_{\bar{1}}}\right) + \Delta\left(\ln\frac{r_k}{r_{\bar{1}}}\right) \simeq 10^{-8}$$

(The reader is reminded that $P_{\bar{1}}$ like P_k is a point on the surface of the earth, and that P_1 is the datum point of the satellite orbit.)

My doubts now are concerned with the **difference** between the last two orders of magnitude. I shall try to illustrate this by applying the line of thought of Section 4.4 to satellite gradiometry, i.e. by taking for S^* a surface through the joint satellite orbits.

We consider the following situation: P_k is again a terrestrial point, with an arbitrary terrestrial datum point $P_{\bar{1}}$; P_j is a satellite orbit point, with orbit datum point P_1 ; $P_{k'}$ is the projection of P_k on the equipotential surface ("geoid") through $P_{\bar{1}}$, including first degree terms $B^{(1)}$ so that, as customary in the literature, the count of n can always start at 2.

Then we have (see "Notes and References, Section 11") with:

2) For the planned STEP mission, having the satellite at the height of 550 km and a precision of gradiometer measurements of 10^{-4} E.U. this becomes $8\ .\ 10^{-8}$.

11.

$$\begin{cases} \Delta X_{\bar{1}k} \equiv \Delta\left(\ln \dfrac{W_k}{W_{\bar{1}}}\right) + \Delta\left(\ln \dfrac{r_k}{r_{\bar{1}}}\right) - \left(B_k^{(1)} - B_{\bar{1}}^{(1)}\right) \simeq \Delta\left(\ln \dfrac{r_{k'}}{r_{\bar{1}}}\right) - \left(B_k^{(1)} - B_{\bar{1}}^{(1)}\right) \\[2ex] \Delta \bar{X}_{\bar{1}k} \equiv \Delta\left(\ln \dfrac{g_k}{g_{\bar{1}}}\right) + 2\Delta\left(\ln \dfrac{r_k}{r_{\bar{1}}}\right) - 2\left(B_k^{(1)} - B_{\bar{1}}^{(1)}\right) \\[2ex] \Delta \bar{\bar{X}}_{1j} \equiv 2\Delta\left(\ln \dfrac{\Gamma_j}{\Gamma_1}\right) + 6\Delta\left(\ln \dfrac{s_j}{s_1}\right) - 6\left(B_j^{(1)} - B_1^{(1)}\right) \\[2ex] -2\Delta X_{\bar{1}k} + \Delta \bar{X}_{\bar{1}k} = -2\Delta\left(\ln \dfrac{W_k}{W_{\bar{1}}}\right) + \Delta\left(\ln \dfrac{g_k}{g_{\bar{1}}}\right) \simeq \Delta\left(\ln \dfrac{g_{k'}}{g_{\bar{1}}}\right) \end{cases}$$

$$\begin{cases} \Delta \bar{X}_{\bar{1}k} = \dfrac{1}{4\pi} \displaystyle\iint \sum_{n+2}^{\infty} \dfrac{2n+1}{n+2} \left\{ \left(\dfrac{R}{r_k}\right)^n Y_k^{\prime(n)} - \left(\dfrac{R}{r_{\bar{1}}}\right)^n Y_{\bar{1}}^{\prime(n)} \right\} Y_j^{\prime(n)} \Delta \bar{\bar{X}}_{1j} \, d\Omega_j \\[2ex] \Delta X_{\bar{1}k} = \dfrac{1}{4\pi} \displaystyle\iint \sum_{n=2}^{\infty} \dfrac{2n+1}{(n+1)(n+2)} \left\{ \left(\dfrac{R}{r_k}\right)^n Y_n^{\prime(n)} - \left(\dfrac{R}{\bar{r}_1}\right) Y_{\bar{1}}^{\prime(n)} \right\} Y_j^{\prime(n)} \Delta \bar{\bar{X}}_{1j} \, d\Omega_j \\[2ex] -2\Delta X_{\bar{1}k} + \Delta \bar{X}_{\bar{1}k} = \dfrac{1}{4\pi} \displaystyle\iint \sum_{n=2}^{\infty} \dfrac{(2n+1)(n-1)}{(n+1)(n+2)} \left\{ \left(\dfrac{R}{r_k}\right)^n Y_k^{\prime(n)} - \left(\dfrac{R}{r_{\bar{1}}}\right)^n Y_{\bar{1}}^{\prime(n)} \right\} Y_j^{\prime(n)} \Delta \bar{\bar{X}}_{1j}^{\prime(n)} \, d\Omega_j \end{cases}$$

From these integral formulas one might compute $\Delta\left(\ln \dfrac{W_k}{W_{\bar{1}}}\right)$ and $\Delta\left(\ln \dfrac{g_k}{g_{\bar{1}}}\right)$ if beside $\dfrac{\Gamma_j}{\Gamma_1}$ one also measures $\dfrac{s_j}{s_1}$ and $\dfrac{r_k}{r_{\bar{1}}}$, whereas the $B^{(1)}$-terms must be known. But there is a restriction: In the example of Section 10 we assumed $R \simeq r_j \simeq r_1 \simeq 6600$ km and $r_k \simeq r_{\bar{1}} \simeq 6400$ km, hence $\dfrac{R}{r_k} \simeq \dfrac{R}{r_{\bar{1}}} \simeq \dfrac{6600}{6400} \simeq 1.03$.

This clearly endangers the convergence of the series in the integral formulas, because this factor is greater than was assumed in [Baarda 1979].

Nevertheless we go on and assume that the random influence of $\dfrac{s_j}{s_1}$, $\dfrac{r_k}{r_{\bar{1}}}$ and the $B^{(1)}$-terms is small compared with the influence of $\dfrac{\Gamma_j}{\Gamma_1}$ (too optimistic an assumption if the global situation is considered). Assume that there is no correlation:

$$\sigma_{\Delta\bar{\bar{X}}_{1j},\,\Delta\bar{\bar{X}}_{1\bar{j}}} = \sigma^2 \delta^j_{\bar{j}}\,, \qquad \delta^j_{\bar{j}} \begin{cases} = 1 \,, & j = \bar{j} \\ = 0 \,, & j \neq \bar{j} \end{cases}$$ [3)]

with:

$$\frac{1}{4\pi} \iint Y_j^{\prime(n)}\, Y_j^{\prime(\bar{n})}\, d\Omega_j = \frac{\delta^n_{\bar{n}}}{2n+1}\,, \qquad \delta^n_{\bar{n}} \begin{cases} = 1 \,, & n = \bar{n} \\ = 0 \,, & n \neq \bar{n} \end{cases}$$

and

$$\frac{1}{4\pi} \iint d\Omega_j = 1$$

the law of propagation of variances results in:

$$\sigma^2_{\Delta\bar{X}_{\bar{1}k}} = \sigma^2 \sum_{n=2}^{\infty} \frac{2n+1}{(n+2)^2} \left\{ \left(\frac{R}{r_k}\right)^n Y_k^{\prime(n)} - \left(\frac{R}{r_{\bar{1}}}\right)^n Y_{\bar{1}}^{\prime(n)} \right\}^2$$

$$\sigma^2_{\Delta X_{\bar{1}k}} = \sigma^2 \sum_{n=2}^{\infty} \frac{2n+1}{(n+1)^2\,(n+2)^2} \left\{ \left(\frac{R}{r_k}\right)^n Y_k^{\prime(n)} - \left(\frac{R}{r_{\bar{1}}}\right)^n Y_{\bar{1}}^{\prime(n)} \right\}^2$$

$$\sigma^2_{-2\Delta X_{\bar{1}k} + \Delta\bar{X}_{\bar{1}k}} = \sigma^2 \sum_{n=2}^{\infty} \frac{(2n+1)\,(n-1)^2}{(n+1)^2\,(n+2)^2} \left\{ \left(\frac{R}{r_k}\right)^n Y_k^{\prime(n)} - \left(\frac{R}{r_{\bar{1}}}\right)^n Y_{\bar{1}}^{\prime(n)} \right\}^2$$

---

3) This simple assumption for the variances is not realistic for situations in practice, as is also observed in the geodetic literature, but it suffices for our present purpose. In this literature it is stated that the application of the law of propagation of variances to the integral formulas would not give realistic results because of singularity of the Stokes-like functions. By definition this does not necessarily apply to our formulation. It is true that the Stokes-like functions increase infinitely for $j \rightarrow 1$ (see [Teunissen 1980]), but in conjunction with this $\ln\frac{\Gamma_j}{\Gamma_1}$ and $\ln\frac{s_j}{s_1}$, and consequently $\Delta\bar{\bar{X}}_{1j}$ approach zero. In order to emphasize this we introduce (again ignoring the random influence of $\Delta\left(\ln\frac{s_j}{s_1}\right)$):
$\sigma_{2\Delta(\ln\Gamma_j)} = \sigma_{2\Delta(\ln\Gamma_1)} = ½\sigma$; no correlation

Then the variates $\Delta\bar{\bar{X}}_{1j}$ are correlated; their covariance matrix has elements ½ σ^2 on the main diagonal, the non-diagonal elements are ¼ σ^2 . Applying the law of propagation of variances then gives the same results as the previously mentioned covariance.

Now take the average of these variances over the whole earth, taking for $\frac{R}{r_k}$ and $\frac{R}{r_{\bar{1}}}$ a constant value $\frac{R}{r}$. Then we have, for example:

$$M^2_{\Delta \bar{X}_{\bar{1}k}} = \frac{1}{4\pi} \iint \sigma^2_{\Delta \bar{X}_{\bar{1}k}} \, d\Omega_k =$$

$$= \sigma^2 \left[\sum_{n=2}^{\infty} \frac{1}{(n+2)^2} \left(\frac{R}{r} \right)^{2n} + \sum_{n=2}^{\infty} \frac{2n+1}{(n+2)^2} \left(\frac{R}{r} \right)^{2n} \left\{ Y_1^{\prime(n)} \right\}^2 \right]$$

We now continue by considering only the first term in the right-hand member. (If one wishes to introduce an average value for the second term, this has the same effect as doubling the first term.) Denoting the first term by m^2, we have:

$$\left\{ \begin{aligned} m^2_{\Delta \bar{X}_{\bar{1}k}} &= \sigma^2 \sum_{n+2}^{\infty} \frac{1}{(n+2)^2} \left(\frac{R}{r} \right)^{2n} \\ m^2_{\Delta X_{\bar{1}k}} &= \sigma^2 \sum_{r=2}^{\infty} \frac{1}{(n+1^2(n+2)^2} \left(\frac{R}{r} \right)^{2n} \\ m^2_{-2\Delta X_{\bar{1}k} + \Delta \bar{X}_{\bar{1}k}} &= \sigma^2 \sum_{n+2}^{\infty} \frac{(n-1)^2}{(n+1)^2 (n+2)^2} \left(\frac{R}{r} \right)^{2n} \end{aligned} \right. \tag{a}$$

This follows from our shortened notation. According to [Baarda 1979, Section 1.4] we have, with the fully normalized harmonics $\bar{R}$ and $\bar{S}$:

$$Y_k^{\prime(n)} Y_l^{\prime(n)} = \sum_{m=0}^{\infty} \left\{ \frac{\bar{R}_k^{(nm)}}{\sqrt{1n+1}} \frac{\bar{R}_l^{(nm)}}{\sqrt{2n+1}} + \frac{\bar{S}_k^{(nm)}}{\sqrt{2n+1}} \frac{\bar{S}_l^{(nm)}}{\sqrt{2n+1}} \right\}$$

so that the integral formulas actually contain $(2n+1)$ terms for each n $\left(\bar{S}^{no} = 0 \right)$. This means that (a) has to be replaced by:

$$\left\{ \begin{aligned} \bar{m}^2_{\Delta \bar{X}_{\bar{1}k}} &= \sigma^2 \sum_{n=1}^{\infty} \frac{2n+1}{(n+2)^2} \left(\frac{R}{r} \right)^{2n} \\ \bar{m}^2_{\Delta X_{\bar{1}k}} &= \sigma^2 \sum_{n=2}^{\infty} \frac{2n+1}{(n+1)^2 (n+2)^2} \left(\frac{R}{r} \right)^{2n} \\ \bar{m}^2_{-2\Delta X_{\bar{1}k} + \Delta \bar{X}_{\bar{1}k}} &= \sigma^2 \sum_{n=2}^{\infty} \frac{(2n+1)(n-1)^2}{(n+1)^2 (n+2)^2} \left(\frac{R}{r} \right)^{2n} \end{aligned} \right. \tag{b}$$

Whereas the series (a) clearly are convergent for $\frac{R}{r} \simeq 1$, for the series (b) this already becomes doubtful. If one assigns the value 1.03 to $\frac{R}{r}$, then all series are strongly divergent. This means that the line of thought of Section 4.4 is **not** applicable. But we **can** use the formulas with $\frac{R}{r} = 1$ to make an appraisal for the order of magnitude of the influence of n (this would actually imply gradiometer measurements very near the surface of the earth), see the following table:

$\frac{R}{r} = 1$	$n \le 200$	$P_{k'}$ projection of P_k on "geoid" through $P_{\bar{1}}$				
		Concerning	m/σ	$\bar{m}/\sigma$	$10^6 m$	$10^6 \bar{m}$
	$\Delta\bar{\bar{X}}_{1j}$	$2\Delta\left(\ln \frac{\Gamma_j}{\Gamma_1}\right)$	1	1	7	7
	$\Delta\bar{X}_{\bar{1}k}$	$\Delta\left(\ln \frac{g_k}{g_{\bar{1}}}\right)$	0.53	2.7	3.7	19
	$\Delta X_{\bar{1}k}$	$\Delta\left(\ln \frac{W_k}{W_{\bar{1}}}\right)$, $\Delta\left(\ln \frac{r_{k'}}{r_{\bar{1}}}\right)$	0.11	0.29	0.77	2.0
	$-2\Delta X_{\bar{1}k} + \Delta\bar{X}_{\bar{1}k}$	$\Delta\left(\ln \frac{g_{k'}}{g_{\bar{1}}}\right)$	0.35	2.1	2.5	15

I do not venture to make a statement about the meaning of these numbers themselves. But it is interesting to see the ratio between numbers pertaining to $\Delta\left(\ln \frac{g_{k'}}{g_{\bar{1}}}\right)$ and $\Delta\left(\ln \frac{r_{k'}}{r_{\bar{1}}}\right)$; this ratio is 3 and 7 respectively, which is much smaller than the previously mentioned value $3.5 * 10^2$ taken from the literature. Hence my doubts.

11.2

Two names are only sporadically mentioned in this treatise, they are H. Moritz whose publications were constant companions in my development, and E.W. Grafarend, whose publications I have tried to fathom. I owe much to both, although they will not always find their ideas back in my train of thought. Perhaps my view on the problems of geodesy is too simplistic; it certainly seems so when comparing it with the results of research by the present staff of the Delft Faculty of Geodesy.

Therefore I am grateful to the Netherlands Geodetic Commission for nevertheless publishing my essay and giving it a place in the series I initiated when I was the Commission's secretary.

I am afraid I would never have rounded off this publication if professor Reiner Rummel had not put me under great pressure. I gratefully think back to the many discussions we had, first about matters of university management, later increasingly more about our fields of science, as witnessed by several themes of this essay. These discussions made me forget for a moment the heavy task waiting for me at home. I regret that the finishing of this essay for other reasons coincides with an end to the possibilities for further discussions.

I gratefully acknowledge the contributions of M. van Gelderen, who made the computations for Section 3.1, J.J. Kok, who took care of the computations for Section 11.1 and M.G.G.J. Jutte who made the drawings.

I am thankful to the staff of the secretariat of the Netherlands Geodetic Commission, who transformed the manuscript into a well got-up publication.

And finally I express my warm thanks to my former co-worker and colleague J.E. Alberda, who managed to translate my scribble into readable English and in doing so again continued a cooperation which by now has almost lasted a lifetime.

Notes and references

Section 1

The publications mentioned are:
W. Baarda
- A Connection between Geometric and Gravimetric Geodesy, Netherlands Geodetic Commission **6**, No. 4, Delft, 1979.
- Geodetische aspecten van het werk van Vening Meinesz (Geodetic Aspects of the Work of Vening Meinesz), in: Verslag van de bijzondere zitting van de Afdeling Natuurkunde op 18 december 1987 ter eere van de herdenking van de 100ste geboortedag van F.A. Vening Meinesz, Royal Netherlands Academy of Arts and Sciences, Amsterdam, 1987.
- Some tentative Remarks on Adjustment Models in Geodesy, in: Festschrift to Torben Krarup, Geodeatisk Institut, Meddelelse No. 58, København, 1989.

A valuable compliment is given by:
P.J.G. Teunissen
- Some Remarks on Gravimetric Geodesy, Reports of the Department of Geodesy, No. 80.2, Delft, 1980.

Complex II-quantities are first mentioned in:
W. Baarda
- A Generalization of the Concept Strength of Figure, Report Special Study Group No 1:14, Delft 1962 (IAG-Assembly Helsinki 1960), Appendix to [Baarda 1967] see notes Section 2.

For an elegant treatment of isoparametric mapping see:
E.W. Grafarend
- The Bruns Transformation and a Dual Setup of Geodetic Observational Equations, NOAA Technical Report NOS 85, NGS 16, National Geodetic Survey, Rockville, Md., USA, 1980.

Section 2

The principles of the theory using complex numbers can be found in:
W. Baarda
- Statistical Concepts in Geodesy, Netherlands Geodetic Commission, **2**, No. 4, Delft, 1967.

An indication of the quaternion theory is given in:
W. Baarda
- S-Transformations and Criterion Matrices, Netherlands Geodetic Commission, **5**, No. 1, Delft, 1973, 1981.

Parts of the theory are included and elaborated in the theses:
M. Molenaar
- A further inquiry into the Theory of S-Transformations and Criterion Matrices, Netherlands Geodetic Commission, **7**, No. 1, Delft, 1981.

H. Quee
- Quaternion Algebra applied to Polygon Theory in Three Dimensional Space, Netherlands Geodetic Commission, **7**, No. 2, Delft, 1983.

An overview has been published in:
W. Baarda
- Mathematical Models, in: 25 Years of OEEPE, OEEPE Official Publ. No. 11, Frankfurt a.M., 1979.

The doubts about the value of ellipsoidal computations in geometric geodesy since my paper:
Some Remarks on the Computation and Adjustment of Large Systems of Geodetic Trangulation, Bull. géod. 1957, No. 43 (IAG-Assembly, Rome, 1954)
have been taken away to a large extent in the thesis:
P.J.G. Teunissen
- The Geometry of Geodetic Inverse Linear Mapping and Non-linear Adjustment, Netherlands Geodetic Commission, **8**, No. 1, Delft, 1985,

and other publications of this author, mentioned in his thesis.

Section 3

For the literature used in the development of an algebra for a geodetic quaternion theory, reference is made to Chapter 1 of [Baarda 1973, 1981]; for results and a further elaboration see [Quee, 1983]. A valuable supplement is given in:
E.W. Grafarend and B. Schaffrin
- Vectors, Quaternions and Spinors, A discussion of Algebras Underlying Three-dimensional geodesy, in: Anniversary Volume on the occasion of Prof. Baarda's 65 birthday, Vol. 1, p. 111-134, Department of Geodesy, Delft University of Technology, 1982. 1)

1) I did not succeed in establishing a relationship between the Π-quantity as suggested by Grafarend and Schaffrin in their contribution and the $\Delta\Pi$-quantity I am using.

In the present publication it therefore suffices to give some indications on quaternion algebra regarding the notation used, supplemented by some new findings. We mainly follow Chapter X of [L. Brand - Vector and Tensor Analysis, Willy, New York, 1947, 1964].

If d, a, b, c are real numbers, the quaternion is defined with the quaternion units:

$$1 = (1,0,0,0) \quad , \quad i = (0,i,0,0) \quad , \quad j = (0,0,j,0) \quad , \quad k = (0,0,0,k)$$

$$Q = d1 + ai + bj + ck$$

with:

$$\begin{cases} ii = jj = kk = -1 \ , \\ ij = k \ , jk = i \ , ki = j \ , \\ ji = -k \ , kj = -i \ , ik = -j \end{cases}$$

The (non-commutative) **product** $Q\,Q'$ is then:

$$\begin{aligned} QQ' &= (d + ai + bj + ck)(d' + a'i + b'j + c'k) \\ &= dd' - aa' - bb' - cc' + \\ &\quad + d(a'i + b'j + c'k) + d'(ai + bj + ck) + \\ &\quad + \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix} \end{aligned}$$

i, j, k may be interpreted with a dextral set of orthogonal unit vectors, often using the notation e_1 , e_2 , e_3 (or other indices) for a right-handed orthogonal set of unit vectors. Q is then composed of a scalar part $Sc\ \{Q\} = d$ and a vector part $Ve\ \{Q\} = ai + bj + ck$:

$$Q = Sc\ \{Q\} + Ve\ \{Q\}$$

The **conjugate** of Q is defined by:

$$Q^T = Sc\ \{Q\} - Ve\ \{Q\} = d - ai - bj - ck$$

hence:

$$QQ^T = d^2 + a^2 + b^2 + c^2 = N\ \{Q\}$$

$N\ \{Q\}$ being the norm of Q .

The **inverse** is:

$$Q^{-1} = \frac{Q^T}{N\{Q\}}$$

Likewise we have:

$$\{QQ'\}^T = Q'^T Q^T$$

$$Sc\{Q\} = \frac{1}{2}(Q + Q^T)$$

$$Ve\{Q\} = \frac{1}{2}(Q - Q^T)$$

If $Sc\ \{Q\} = 0$ we speak about a **vector**, denoted by q, hence:

$$Sc\ \{q\} = 0\ ,\ Ve\ \{q\} = q\ ,\ q^T = -q$$

We put:

$$N^{½}\ \{q\} = s_q$$

For the **division** of two vectors q' and q we always choose the order $q'\,q^{-1}$. Then the following is valid:

$$q'q^{-1} = \frac{s_{q'}}{s_q}\ (\cos\theta + e\sin\theta)$$

with: the angle $(q\ ,\ q') = \theta$; the unit vector e is perpendicular to plane q, q'; q , q', e form a right-handed set of vectors.

It is remarked that any unit quaternion can be written in the form:

$$\text{unit quaternion} \longleftrightarrow (\cos\theta + e\sin\theta)$$

Rotation plays an important role.
Let:

$$p = N^{½}\ \{p\}\ \left(\cos\theta_p + e_p\sin\theta_p\right)$$

Then we have for

$$Q' = pQp^{-1}$$

$$Sc\ \{Q'\} = Sc\ \{Q\}\quad ,\quad N\ \{Q'\} = N\ \{Q\}$$

$Ve\ \{Q'\}$ is obtained by revolving $Ve\ \{Q\}$ conically about e_p through an angle $2\ \theta_p$.

If $N\{p\} = 1$, then p will be called a rotation quaternion.

For a vector q with $\theta_q = \dfrac{\pi}{2}$ we consequently have:

$$Q' = qQq^{-1}$$

$Ve\{Q'\}$ is obtained by revolving $Ve\{Q\}$ conically about e_q through an angle π.

Or:

$$\frac{1}{2}(Q - qQq^{-1}) = \text{component } Ve\{Q\} \perp e_q(\text{or } q)$$

$$\frac{1}{2}(Q + qQq^{-1}) = Sc\{Q\} + \text{component } Ve\{Q\} \parallel e_q\ (\text{or } q)$$

There is practical significance in the **isomorphy** of quaternions with a matrix group. The ordering of $Q'' = QQ'$ results in:

$$\begin{aligned} Q'' = d'' + a''i + b''j + c''k = {} & (dd' - aa' - bb' - cc') + \\ & + (ad' + da' - cb' + bc')\, i + \\ & + (bd' + ca' + db' - ac'\ j + \\ & + (cd' - ba' + ab' + dc')\, k \end{aligned}$$

Hence the components of Q'' can be computed from a matrix product:

$$\begin{pmatrix} d'' \\ a'' \\ b'' \\ c'' \end{pmatrix} = \begin{pmatrix} d & -a & -b & -c \\ a & d & -c & b \\ b & c & d & -a \\ c & -b & a & d \end{pmatrix} \begin{pmatrix} d' \\ a' \\ b' \\ c' \end{pmatrix}$$

Then we also have:

$$\begin{pmatrix} d'' & -a'' & -b'' & -c'' \\ a'' & d'' & -c'' & b'' \\ b'' & c'' & d'' & -a'' \\ c'' & -b'' & a & d'' \end{pmatrix} = \begin{pmatrix} d & -a & -b & -c \\ a & d & -c & b \\ b & c & d & -a \\ c & -b & a & d \end{pmatrix} \begin{pmatrix} d' & -a' & -b' & -c' \\ a' & d' & -c' & b' \\ b' & c' & d' & -a' \\ c' & -b' & a' & d' \end{pmatrix}$$

by which the isomorphy has been reached. In an abbreviated notation:

$$(Q'') = (Q)\,(Q')$$

Now the **application to difference quantities.**

$$\boxed{Q = q'q^{-1}} \quad , \quad Q^{-1} = qq'^{-1} \quad , \quad Q^{T} = q^{-1}q'$$

$$qq^{-1} = 1 \qquad , \quad \Delta q \cdot q^{-1} + q \cdot \Delta q^{-1} = 0 \qquad , \quad \Delta q^{-1} = -q^{-1} \cdot \Delta q \cdot$$

$$\Delta Q = \Delta q' \cdot q^{-1} + q' \cdot \Delta q^{-1} = \Delta q' \cdot q^{-1} - q'q^{-1} \cdot \Delta q \cdot q^{-1}$$

$$\boxed{\Delta \Pi_Q = q'^{-1} \cdot \Delta Q \cdot q = q'^{-1} \cdot \Delta q' - q^{-1} \cdot \Delta q = \Delta \Lambda_{q'} - \Delta \Lambda_q}$$

$$\Delta Q^{-1} = \Delta q^{-1} \cdot q'^{-1} - qq'^{-1} \cdot \Delta q' \cdot q'^{-1}$$

$$\boxed{\Delta \Pi_{Q^{-1}} = q^{-1} \Delta Q^{-1} q' = q^{-1} \cdot \Delta q - q'^{-1} \cdot \Delta q' = -\Delta \Pi_Q}$$

$$\Delta Q^{T} = -q^{-1} \cdot \Delta q \cdot q^{-1}\, q' + q^{-1} \cdot \Delta q'$$

$$\boxed{\Delta \Pi_{Q^T} = q \cdot \Delta Q^{T} \cdot q'^{-1} = -\Delta q \cdot q^{-1} + \Delta q' \cdot q'^{-1} = \left(\Delta \Pi_Q\right)^{T}}$$

Rotation:

$$\bar{Q} = p\, Q\, p^{-1}$$

$$\Delta \bar{Q} = p \cdot \Delta Q \cdot p^{-1}$$

$$\bar{Q} = (p\, q'\, p^{-1})(p\, q\, p^{-1})^{-1} = \bar{q}'\, \bar{q}^{-1}$$

$$\boxed{\Delta\Pi_{\bar{Q}} = \bar{q}'^{-1}\,\Delta\bar{Q}\,\bar{q} = p\,q'^{-1}\cdot\Delta Q\cdot q\,p^{-1} = p\cdot\Delta\Pi_{Q'}\cdot p^{-1}}$$

It is seen that $\Delta\Pi$ is not invariant with respect to a rotation.

Finally it follows for a unit vector e, with $ee = -1$, that:

$$\Delta e\cdot e = -e\cdot\Delta e\ \text{, hence}\ \Delta e \perp e$$

Consider the vectors $q = s_q e_q$ and $q' = s_{q'} e_{q'}$ and form the quaternion:

$$Q = q' q^{-1} = \frac{s_{q'}}{s_q}\left(\cos\theta_Q + e_Q\sin\theta_Q\right)$$

with θ_Q = angle (q, q') and e_q, $e_{q'}$, e_Q a right-handed set of unit vectors; $e_Q \perp e_q$ and $e_{q'}$. Then the difference equation is:

$$\Delta Q = Q\cdot\Delta\ln\frac{s_{q'}}{s_q} + \frac{s_{q'}}{s_q}\left(-\sin\theta_Q + e_Q\cos\theta_Q\right)\Delta\theta_Q + \frac{s_{q'}}{s_q}\sin\theta_Q\cdot\Delta e_Q$$

Now:

$$\frac{s_{q'}}{s_q}\left(-\sin\theta_Q + e_Q\cos\theta_Q\right) = \frac{s_{q'}}{s_q}\left(e_Q\sin\theta_Q + \cos\theta_Q\right)e_Q =$$

$$= Q_{e_Q} = q' q^{-1} e_Q = -q' e_Q q^{-1}$$

This results in:

$$\boxed{\Delta\Pi_Q = q'^{-1}\cdot\Delta Q\cdot q = \Delta\ln\frac{s_{q'}}{s_q} - e_Q\cdot\Delta\theta_Q + \sin\theta_Q e_{q'}^{-1}\cdot\Delta e_Q\cdot e_q}$$

Now we have:

$$\begin{aligned}\Delta\Pi_Q + e_Q\cdot\Delta\Pi_Q\cdot e_Q^{-1} = {} & \left(1 + e_Q e_Q^{-1}\right)\Delta\ln\frac{s_{q'}}{s_q} + \\ & -\left(e_Q + e_Q e_Q e_Q^{-1}\right)\Delta\theta_Q + \\ & + \sin\theta_Q\left(e_{q'}^{-1}\cdot\Delta e_Q\cdot e_q + e_Q e_{q'}^{-1}\cdot\Delta e_Q\cdot e_q e_Q^{-1}\right)\end{aligned}$$

and with:

$$e_Q e_{q'}^{-1} \cdot \triangle e_Q \cdot e_q e_Q^{-1} = e_{q'}^{-1} e_Q \cdot \triangle e_Q \cdot e_Q^{-1} e_q =$$

$$= -e_{q'}^{-1} \cdot \triangle e_Q \cdot e_Q e_Q^{-1} e_q = -e_{q'}^{-1} \cdot \triangle e_Q \cdot e_q$$

we obtain:

$$\boxed{\frac{1}{2}\left(\triangle \Pi_Q + e_Q \cdot \triangle \Pi_Q \cdot e_Q^{-1}\right) = \triangle \ln \frac{s_{q'}}{s_q} - e_Q \cdot \triangle \theta_Q}$$

Similarly:

$$\boxed{\frac{1}{2}\left(\triangle \Pi_Q - e_Q \cdot \triangle \Pi_Q \cdot e_Q^{-1}\right) = \sin\theta_Q \cdot e_{q'}^{-1} \cdot \triangle e_Q \cdot e_q}$$

from which $\triangle \ln \dfrac{s_{q'}}{s_q}$, $\triangle \theta_Q$ and $\triangle e_Q$ can be solved.

Occasionally, e.g. in the computation of a satellite orbit in the plane of the Kepler ellipse, it is desirable to develop for $\triangle\Lambda$ a form similar to the one for $\triangle\Pi$.
Choose a unit vector e_0 in the plane of q and q', and put:

$$\text{angle } (e_0, e_q) = \varphi_q \text{ , angle } (e_0, e_{q'}) = \varphi_{q'}$$

Then:

$$\varphi_{q'} - \varphi_q = \theta_Q \text{ and:}$$
$$q e_0^{-1} = s_q(\cos\varphi_q + e_Q \sin\varphi_q)$$

or:

$$q = s_q(\cos\varphi_q + e_Q \sin\varphi_q) e_0$$

$$q' = s_{q'}(\cos\varphi_{q'} + e_Q \sin\varphi_{q'}) e_0$$

Check:

$$q' q^{-1} = \frac{s_{q'}}{s_q}\left[\cos(\varphi_{q'} - \varphi_q) + e_Q \sin(\varphi_{q'} - \varphi_q)\right] = Q$$

$$\triangle q = q - \triangle \ln s_q + s_q e_Q (e_Q \sin\varphi_q + \cos\varphi_q) e_0 \cdot \triangle \varphi_q + s_q \sin\varphi_q \cdot \triangle e_Q \cdot e_0$$

$$\boxed{\triangle \Lambda_q = q^{-1} \cdot \triangle q = \triangle \ln s_q - e_Q \cdot \triangle \varphi_q + \sin\varphi_q \cdot e_q^{-1} \cdot \triangle e_Q \cdot e_0}$$

Some elaboration provides the check:

$$\Delta\Lambda_{q'} - \Delta\Lambda_q = \left(\Delta\ln s_{q'} - \Delta\ln s_q\right) - e_Q\left(\Delta\varphi_{q'} - \Delta\varphi_q\right) +$$

$$+ \sin\left(\varphi_{q'} - \varphi_q\right) e_{q'}^{-1} \cdot \Delta e_Q \cdot e_q = \Delta\Pi_Q$$

Section 3.2.3

For me the idea of the covariance function

$$d_{kl}^2 = c\ s_{kl}$$

finds its origin in a series of interesting papers published in the thirties by the German practical geodesist dr. E. Pinkwart. For example, in the paper "Zur Fehlertheorie der trigonometrischen Punktbestimmung - Zeitschrift für Vermessungswesen 1940, Heft 16, p. 377 ff.", he writes: .."erscheint es mir zweckmässig, den relativen mitleren Punktfehler nach dem Quadratwurzelgesetz zu definieren."

First results of my own research, which also used results of Pinkwart, can be found in Sections 2 and 3 of the report:
W. Baarda and D. de Groot
- Opzet en techniek van kadastrale metingen (Design and technique of cadastral survey & measurements) - Rapporten 12e Congres van de Nederlandse Landmeetkundige Federatie, 7 juni 1952.

Further research concerning the introduction of the above mentioned covariance function was published in [Baarda 1973, 1981], and complemented in:
J.E. Alberda
- Planning and Optimization of Networks: Some General Considerations - Bolletino di Geodesia e Scienze Affini, Anno XXXIII, No. 2, 1974.

In the publication:
W. Baarda
- Measures for the Accuracy of geodetic Networks. Discussion Paper IAG Special Study Group 4.14 - IAG International Symposium on Optimization of Design and Computation of Control Networks, 1977, 3-9 July, Sopron, Hungary.

I suggested to combine the results of research by P. Meissl, K. Borre and others into the more manageable form:

$$d_{kl}^2 = \bar{c}^2 \, s_0 \, \ln\left(1 + \frac{s_{kl}}{s_0}\right)$$

in which s_0 is a reference distance.

Meissl later informed me that this was unacceptable to him. For various reasons the discussion regrettably could not be continued so that a covariance function of this form had better be put on the shelf. Actually the difference with the previously mentioned function is minimal.

My own research, and also the applications by others, have shown that the covariance function first mentioned is practically applicable for all forms of terrestrial networks. Therefore I would not be surprised if this would also apply to newer types of spatial networks.

Section 5

A sketch of the derivation of the formulas is given in a close connection with [Baarda, 1979], with the corrections given in [Teunissen, 1980]. The references to formulas concern my 1979 publication.

Assuming again that the centrifugal potential does not require a correction, we introduce according to (4.2.3) and (4.2.6):

$$\Delta X_{1k} \simeq \Delta\left(\frac{W_{1k}}{r_1 g_1} - \frac{r_1}{r_k}\right) , \quad \Delta \bar{X}_{1k} \simeq \Delta\left(\frac{r_k g_k}{r_1 g_1} - \frac{r_1}{r_k}\right)$$

$$\Delta \bar{\bar{X}}_{1k} = -2 \cdot \Delta X_{1k} + \Delta \bar{X}_{1k}$$

Then the integral equation (1.8.13) becomes:

$$\frac{1}{2} \Delta X_{1k} = \frac{1}{4\pi} \iint \left(-\frac{1}{2} \Delta X_{1j} + \Delta \bar{X}_{1j}\right) \left(\frac{r_j}{r_{kj}} - \frac{r_1}{r_k} \frac{r_j}{r_{1j}}\right) d\Omega_j + \frac{1}{2} \Delta_{1k}$$

with:

$$\Delta_{1k} = \frac{2}{4\pi} \iint \Delta X_{1j} \left(C_{kj} - \frac{r_1}{r_k} C_{1j}\right) d\Omega_j$$

$$C_{kj} = -\left(\frac{r_j}{r_{kj}}\right)^2 \tan(r_j, n_j)_{r_{kj}} \, \sin(r_j, r_{kj})$$

with $(r_j,n_j)_{r_{kj}}$ the angle of slope β_j in P_j of the intersection of the terrain with the plane P_k P_M P_j.

Now solve this integral equation according to Section 4.2, but now including Δ_{1k}. Then the corrected Hotine integral formula (4.2.9) becomes:

$$\Delta X_{1k} = \frac{1}{4\pi} \iint S_{1k;j}^{(n+1)} \, \Delta\bar{X}_{1j} \, d\Omega_j + \left(\Delta_{1k} - \frac{1}{8\pi} \iint S_{1k;j}^{(n+1)} \, \Delta_{1j} \, d\Omega_j \right)$$

Now the integral equation, according to (4.1.11), can also be written as follows (with correction Δ_{1k}):

$$\frac{1}{2}\Delta X_{1k} = \frac{3}{4\pi} \iint \left(\frac{1}{2}\Delta X_{1j} + \frac{1}{3}\Delta\bar{\bar{X}}_{1j} \right) \left(\frac{r_j}{r_{kj}} - \frac{r_1}{r_k}\frac{r_j}{r_{1j}} \right) d\Omega_j + \frac{1}{2}\Delta_{1k}$$

The solution of this integral then gives the corrected Stokes integral formula (4.2.10):

$$\Delta X_{1k} = B^{(1)}\text{-term} + \frac{1}{4\pi} \iint S_{1k;j}^{(n-1)} \, \Delta\bar{\bar{X}}_{1j} \, d\Omega_j +$$

$$+ \left(\Delta_{1k} + \frac{3}{8\pi} \iint S_{1k;j}^{(n-1)} \, \Delta_{1j} \, d\Omega_j \right)$$

The correction term looks much like the first term in the correction-series of Molodensky, see [Heiskanen, Moritz - Physical Geodesy - Freeman, 1967, section 8-7]. Our solution also is in fact the first step in an iteration process.
But a comparison of the two integral formulas turns out to provide the possibility to interpret the correction terms as corrections to potential differences and gravity ratios as follows:

$$\left(\Delta X_{1k} - \Delta_{1k}\right) = \frac{1}{4\pi} \iint S_{1k;j}^{(n+1)} \left(\Delta\bar{X}_{1j} - \frac{1}{2}\Delta_{1j} \right) d\Omega_j$$

$$\left(\Delta X_{1k} - \Delta_{1k}\right) = B^{(1)}\text{-term} +$$

$$+ \frac{1}{4\pi} \iint S_{1k;j}^{(n-1)} \left[-2\left(\Delta X_{1j} - \Delta_{1j}\right) + \left(\Delta\bar{X}_{1j} - \frac{1}{2}\Delta_{1j} \right) \right] d\Omega$$

Now we can give an appraisal for Δ_{1k} by approximating C_{1k}:
From (1.8.12) follows $r_{kj} \leq 1300$ km ; a reasonable approximation is then:
$\sin(r_j,r_{kj}) \simeq 1$, $r_j \simeq R$. Or:

$$C_{kj} \simeq -\left(\frac{R}{r_{kj}}\right)^2 \tan(r_j,n_j)_{r_{kj}}, \qquad (r_j,n_j)_{r_{kj}} = \beta_j$$

In order to establish a link with formulas from the literature, the line of thought of (1.8.8) is followed. Replace a mountain range by a prismatic form and consider two points P_j and $P_{j'}$, one on each side of the intersection with the plane $P_k\,P_M\,P_j$, on equal geoid height h, hence $\Delta X_{1j'} \simeq \Delta X_{1j}$, $\tan(r_{j'},n_{j'})_{r_{kj'}} = -\tan(r_j,n_j)_{r_{kj}}$.

Put:

$$r_1 \simeq r_k \simeq r_j \simeq r_{j'} \simeq R$$

$$(r_{kj},r_{kj'}) = \upsilon \quad , \; \upsilon \text{ small on account of (1.8.12)}$$

$$(r_{j'}r_{kj}) = \omega \quad , \; (r_{j'},r_{kj'}) = \omega' = \omega - \upsilon$$

Using:

$$\frac{R}{r_{kj}} = \frac{1}{2\cos\omega} \;,\; \frac{R}{r_{kj'}} = \frac{1}{2\cos\omega'} \;,\; \cos\omega' = \cos\omega + \upsilon$$

we obtain:

$$C_{kj} + C_{kj'} = -2\frac{R}{r_{kj}}\,\frac{R}{r_{kj'}}\left(\frac{R}{r_{kj}} + \frac{R}{r_{kj'}}\right)\upsilon\,\tan\beta_j$$

the combined effect of P_j and $P_{j'}$ on Δ_{1k}.
Denoting the top of the prism-section by T_j, we have:

$$-\upsilon\,\tan\beta_j = \frac{h_j - h_{T_j}}{R} \;, \qquad r_{kT_j} > r_{kj}$$

A very rough approximation is then:

$$C_{kj} \simeq 2\left(\frac{R}{r_{kj}}\right)^3 \frac{h_j - h_{T_j}}{R} \;, \qquad r_{kT_j} > r_{kj}$$

This approximation is chosen to establish a link with the terrain or topographic correction from the literature. See [Moritz - Advanced Physical Geodesy - Wichmann, Abacus, 1980, section 48]. The formulas show much likeness, except that h_{T_j} has been replaced by the much more harmless h_k.

The Vening Meinesz integral formulas can now be obtained by calculating the partial derivatives

$$\left(\frac{\partial}{-\partial\varphi_k} + \frac{\partial}{-\partial\varphi_1}\right) \text{ and } \left(\frac{\partial}{-\cos\varphi_k\cdot\partial\lambda_k} + \frac{\partial}{-\cos\varphi_1\cdot\partial\lambda_1}\right)$$

of the Hotine and Stokes integral formula respectively, hence also of Δ_{1k} and consequently of $\left(C_{kj} - \frac{r_1}{r_k} C_{1j}\right)$.

Compute as an example

$$\frac{\partial}{-\partial \varphi_k} C_{kj} \underset{\text{say}}{=} D_{kj}$$

Then we have (again with $\sin(r_j, r_{kj}) \simeq 1$):

$$D_{kj} \simeq + \left(\frac{r_j}{r_{kj}}\right)^4 \frac{\partial \left(\frac{r_{kj}}{r_j}\right)^2}{\partial \cos(r_k, r_j)} \frac{\partial \cos(r_k, r_j)}{-\partial \varphi_k} \tan(r_j, n_j)_{r_{kj}}$$

$$= \left(\frac{r_j}{r_{kj}}\right)^4 \left(-2 \frac{r_k}{r_j}\right) \left(-\sin(r_k, r_j) \cos\alpha_{kj}\right) \tan(r_j, n_j)_{r_{kj}}$$

and again with $r_j \simeq r_k \simeq R, \quad \sin(r_k, r_j) \simeq \frac{r_{kj}}{R}$:

$$D_{kj} \simeq + 2 \left(\frac{R}{r_{kj}}\right)^3 \cos\alpha_{kj} \tan(r_j, n_j)_{r_{kj}}$$

Now do the same as was done with C_{kj} . Then:

$$D_{kj} + D_{kj'} \simeq 4 \frac{R}{r_{kj}} \frac{R}{r_{kj'}} \left[\left(\frac{R}{r_{kj}}\right)^2 + \frac{R}{r_{kj}} \frac{R}{r_{kj'}} + \left(\frac{R}{r_{kj'}}\right)^2\right] \cos\alpha_{kj} \upsilon \tan(r_j, n_j)_{r_{kj}}$$

In the same rough approximation which applied to C_{kj} one obtains:

$$D_{kj} \simeq - 6\left(\frac{R}{r_{kj}}\right)^4 \cos\alpha_{kj} \frac{h_j - h_{T_j}}{R}$$

a result which does not have a resembling counterpart in the literature.

Section 6

The author gratefully acknowledges his use of:
K.P. Schwarz
- Introduction to Inertial Surveying - Department of Geodesy, Delft, 1983, and discussions with Professor Schwarz.

Section 7 - 9

Use was made, among others, of:
- Reports of the Smithsonian Astrophysical Observatory (especially SAO Special Report 353, 1973);
- Reports of the Department of Geodetic Science, O.S.U. (especially the nos. 201, 1973, 284, 1978 and 294, 1979);
- Kaula's "Theory of Satellite Geodesy" (Blaisdell, 1966);
- Kovalevsky's "Mécanique céleste" in "Levallois and Kovalevsky - Géodésie Générale, Tome IV" (Eyrolles, 1971);
- Nagel's "Die Bezugssysteme der Satellitengeodäsie" (DGK, Reihe C, Nr. 223, 1976);
- Proceedings of the IAU Colloqium No. 26 (Torun, 1974) and No. 56 (Warsaw, 1980);
- Moritz's and Muellers's "Earth Rotation" (Ungar, 1987).

But it appears to me that most of the publications mentioned are more concerned with the theoretical side of the definition of coordinate systems than with the operational side.

In the sixties, George Veis has on my request at SAO executed computations concerning the introduction of an r_C -vector. The result proved to be detrimental to the satellite orbit, and his comment was not very flattering for my hypothesis. Presumably the reason was that he introduced a first degree term in the spherical harmonics expansion. After a long reflection I now propose the procedure presented here.

Section 10

Publications used are:

R. Rummel, O.L. Colombo
- Gravity Field Determination from Satellite Gradiometry - Bulletin Géodésique **59**, 1985.

R. Rummel
- Satellite Gradiometry, In: H. Sünkel, ed. - Lecture Notes in Earth Sciences **7**. Mathematical and Numerical Techniques in Physical Geodesy - Springer, 1986.

Section 11

The logarithmic notation is chosen. Provisionally, for all radial distances, also for points on S, the kernel letter s will be used. Then for Hotine's integral formula, with integration over S^*, the following is valid:

$$\frac{s_1}{s_k}\left[\Delta\left(\ln\frac{W_k}{W_1}\right) + \Delta\left(\ln\frac{s_k}{s_1}\right)\right] =$$

$$= \frac{1}{4\pi}\iint \frac{R}{s_k}\sum_{n=1}^{\infty}\frac{2n+1}{n+1}\left\{\left(\frac{R}{s_k}\right)^n Y_k'^{(n)} - \left(\frac{R}{s_1}\right)^n Y_1'^{(n)}\right\} Y_j'^{(n)} \, *$$

$$* \, \frac{s_1}{s_j}\left[\Delta\left(\ln\frac{g_j}{g_1}\right) + 2\Delta\left(\ln\frac{s_j}{s_1}\right)\right] d\Omega_j$$

Like in all parts of this publication, it is assumed that the rotational velocity of the earth is sufficiently known, so that $\Delta W = \Delta V$; besides, the difference between the radial direction and the normal direction is ignored.

A further assumption is that satellite orbits are nearly circular, so that:

$$s_1 \simeq s_j \simeq R$$

Then the Hotine's formula becomes:

$$\left[\Delta\left(\ln\frac{W_k}{W_1}\right) + \Delta\left(\ln\frac{s_k}{s_1}\right)\right] =$$

$$= \frac{1}{4\pi}\iint \sum_{n=1}^{\infty}\frac{2n+1}{n+1}\left\{\left(\frac{R}{s_k}\right)^n Y_k'^{(n)} - Y_1'^{(n)}\right\} Y_j'^{(n)} \, *$$

$$* \left[\Delta\left(\ln\frac{g_j}{g_1}\right) + 2\Delta\left(\ln\frac{s_j}{s_1}\right)\right] d\Omega_j \qquad \text{(a)}$$

Now the summation over n can be made to start at 2 (a warning for first degree terms!) if $\Delta\left(\ln\frac{s_k}{s_1}\right)$ is replaced by: $\left[\Delta\left(\ln\frac{s_k}{s_1}\right) - \left(B_k^{(1)} - B_1^{(1)}\right)\right]$.

In the same way as the so-called integral formula of Hotine is the direct solution of the integral equation (4.1.8) in [Baarda, 1979], the direct solution of the integral equation (5.3.4) in [Baarda, 1979] is, after re-writing as Hotine's integral formula:

$$\left[\Delta\left(\ln \frac{g_k}{g_1}\right) + 2\Delta\left(\ln \frac{s_k}{s_1}\right)\right] =$$

$$= \frac{1}{4\pi} \iint \sum_{n=1}^{\infty} \frac{2n+1}{n+2} \left\{\left(\frac{R}{s_k}\right)^n Y_k'^{(n)} - Y_1'^{(n)}\right\} Y_j'^{(n)} *$$

$$* \left[2\Delta\left(\ln \frac{\Gamma_j}{\Gamma_1}\right) + 6\Delta\left(\ln \frac{s_j}{s_1}\right)\right] d\Omega_j \qquad \text{(b)}$$

After changing indices, substitution of (b) into (a) gives:

$$\left[\Delta\left(\ln \frac{W_k}{W_1}\right) + \Delta\left(\ln \frac{s_k}{s_1}\right)\right] =$$

$$= \frac{1}{4\pi} \iint \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)(n+2)} \left\{\left(\frac{R}{s_k}\right)^n Y_k'^{(n)} - Y_1'^{(n)}\right\} Y_j'^{(n)} *$$

$$* \left[2\Delta\left(\ln \frac{\Gamma_j}{\Gamma_1}\right) + 6\Delta\left(\ln \frac{s_j}{s_1}\right)\right] d\Omega_j \qquad \text{(c)}$$

Finally we take (b) - 2(a):

$$\left[-2\Delta\left(\ln \frac{W_k}{W_1}\right) + \Delta\left(\ln \frac{g_k}{g_1}\right)\right] =$$

$$= \frac{1}{4\pi} \iint \sum_{n=2}^{\infty} \frac{(2n+1)(n-1)}{(n+1)(n+2)} \left\{\left(\frac{R}{s_k}\right)^n Y_k'^{(n)} - Y_1'^{(n)}\right\} Y_j'^{(n)} *$$

$$* \left[2\Delta\left(\ln \frac{\Gamma_j}{\Gamma_1}\right) + 6\Delta\left(\ln \frac{s_j}{s_1}\right)\right] d\Omega_j \qquad \text{(d)}$$

In these integral formulas W_1 , g_1 and $Y_1'^{(n)}$ are still to be eliminated because these relate to the datum point of the satellite orbit. Therefore we apply these formulas with $k \rightarrow \bar{1}$, $P_{\bar{1}}$ being the terrestrial datum point. Subtraction then results in the formulas sought, in which for points on S the kernel letter s can finally be replaced by r .