The Delft School of Geodesy

The development of a methodology for describing the quality of geodetic measurements, 1930 – 1980

Martien Molenaar

Translated by: Roel Nicolaï

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"Then I thought … of Bridgeman's question of … which co-ordinate system Einstein had used … The definition of coordinates is even worse than that of the Cadastre; it is completely ignored."

Baarda in a letter, dated 28 July 1997

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Foreword

In the autumn of 1967, I attended the first lectures of the geodesy curriculum of the Faculty of Geodesy of Delft University, together with a number of other freshmen. These lectures introduced us into the basic principles of this discipline, which was entirely new to us. One lecture series dealt with an introduction into Adjustment Theory. The meaning of that term was still entirely unknown to us. By way of familiarisation, we were asked to perform a simple exercise. A wooden plank, about one metre in length, was passed from person to person. Four small wood blocks had been attached to it at intervals of approximately 30 cm. Each wood block contained a rough hole, which had been filled with wax. The task was to stick a pin into the wax of each hole, such that it was located in the middle of each wood block. Each of us was asked to measure the distance between the successive pins with a ruler and add up these three distances to obtain the distance between the first and last block. And guess what, the results of the twenty students showed considerable differences.

That exercise was intended to make it clear to us that, although the concepts of 'distance' and 'length' are mathematically well defined, the application of these concepts to measurements in physical reality is not that simple. In fact, it turned out that the interpretation of the concept of 'length' was not straightforward and that it was not possible to assign an unambiguous value to a length. We were made to understand that in this situation the concept of 'true value' is meaningless. With real measurements, in the real world, one has to accept the fact that repeated measurements will yield different outcomes. That simple exercise demonstrated why Adjustment Theory, based on the least squares method, is important for geodesists.

In the later stages of the geodesy course, it became clear that at Delft University the foundations of this method were being subjected to renewed investigation. In the beginning of the twentieth century some fundamental thinking about the significance of mathematical models for the description of natural phenomena took place in various institutions. The exercise described above illustrated that challenge. A wider group of thinkers, including mathematicians, physicists and philosophers struggled with this problem and a suitable language to express its essence with clarity was lacking as yet. During the period from about 1930 to 1980, described in this monograph, the *Delft School of Geodesy* took shape, initially under the leadership of Jacob Menno Tienstra and then of Willem Baarda. Their struggle during these fifty years was strongly influenced by the challenge described in this paragraph.

This monograph is mainly based on the publications of the protagonists and their close fellow workers. Whilst that yields a somewhat monolithic approach, I hope it will provide the reader with some insight into the background of these developments and the manner in which the key ideas took shape. This monograph describes the development of their ideas up to the time of Willem Baarda's retirement.

Upon completion of the first draft, I sent this text to a number of friends and colleagues. Many replied, but I wish to mention a few of them in particular: Sieb Dijkstra, Robert Kroon, Jan de Kruif and Roel Nicolai. They supplied valuable additions and improvements on the text. Additionally, Hiddo Velsink provided some suggestions for historically-interesting references.

I especially thank Roel Nicolai, who acted as the final editor of the text and prepared most of the illustrations. He also translated the text for this English version. In several cases we found

that Dutch formulations and terminology could not be translated directly into English. We had to find new expressions that preserved the meaning of the original Dutch expressions. Roel proved to be very inventive in these cases. I sincerely thank him for his efforts to produce this present text. I also thank Richard Wylde, FRICS, who was kind enough the check Roel's English translation.

Martien Molenaar, December 2021

1 Introduction

During the twentieth century and particularly in its second half, a number of developments took place at the Faculty of Geodesy of Delft University of Technology that drew much attention, nationally, but definitely also internationally. These developments exerted an increasing influence on professional practice in geodesy and land surveying. These concerned the way in which reconnaissance surveys were conducted, measurement results processed and particularly the manner in which measurement quality was assessed. In the world of geodesy these developments are collectively referred to as the *Delft School*. Its most important pioneers were Jacob Menno Tienstra, his successor Willem Baarda and their co-workers. Their active contributions to this field span the period from 1930 to 1980 and that is roughly the period that will be discussed in this monograph.

The fascinating thing about this period is that problems, that were encountered in the work practice of the land surveyor and the geodesist, led to a thorough theoretical analysis of the work. That analysis yielded new concepts that were subsequently tested in practice, which led, in turn, to fine-tuning of the theoretical models. Therefore, any description of the developments of this period will have to be based on the interaction between theoretical developments and professional practice. Nevertheless, a split approach was chosen. This monograph discusses the development of theoretical framework in particular.

Two sub-periods are discussed in this monograph, focussed on the key players Tienstra and Baarda respectively. This format has been chosen for two reasons.

The Tienstra period was mainly aimed at the structuring of data processing methods: the so-called five *Standard Problems* of adjustment calculus. In that period the beginnings of an improved theoretical framework for the analysis of the quality of measurements became visible.

The Baarda period focusses in particular on the continuation of the development of that theoretical framework for quality analysis of geodetic measurements. The concept of *data quality* or *data accuracy* was refined by breaking it down into the separate concepts of *precision* and *reliability*. Additionally, Baarda applied new mathematical concepts to the analysis of the structure and quality of geodetic control networks.

The emergence of, firstly, so-called *total stations* (the combination of measurement of direction and distance in a single instrument) and secondly GPS from the 1980s onward, as well as the continually accelerating development of information technology and computing, have changed the daily practice of the geodesist fundamentally. The subsequent emergence of laser altitude measurement, radar interferometry and many other novel measurement techniques contributed to more changes in the work methods of the geodesist in an even shorter time frame. Additionally, the closing of the Faculty of Geodesy at the Delft University of Technology in 2003 and the termination of the geodesy curriculum at the Utrecht Polytechnic (….) have almost stopped the inflow of new Dutch geodesists in this professional field. Therefore, in the Netherlands, geodetic tasks are increasingly performed by professionals from other disciplines, who were not educated with the concepts of the *Delft School*. Because of this, this system of ideas is at risk of falling into oblivion, which is an important justification for writing this monograph. It is to be hoped that people do retain the awareness of what has been achieved in this phase of Dutch geodetic history. This contribution to geodesy had a global importance and impact.

The most important development phase of the ideas of the *Delft School* happened before those far-reaching technological changes in geodesy. For a proper understanding of this development

phase, a brief description of geodetic survey practice will be provided after this introduction. The period after that tipping point might be described in a later paper.

A large number of articles dedicated to various aspects of the theoretical developments of the *Delft School* have been published over the last decades, in part authored by the key players themselves. This monograph will make use of such publications; a selection of publications has been compiled. Taken together, these publications provide a good summary of the period described. This monograph provides the context in which these articles should be interpreted.

In the professional field, geodesy is often distinguished from land surveying. The development of the methods by the Delft School of Geodesy were of equal importance for both. These concepts will therefore be used interchangeably; use of one term implies the other. Furthermore, the terms 'observations' and 'measurements' will be used as synonyms.

2 Land survey practice in the twentieth century and the beginnings of the Delft School

A number of basic land surveying principles will be clarified in this chapter, because the ideas of the Delft School arose mainly from experiences in land survey practice. At a later stage these ideas were also applied to photogrammetry, but attention will be paid to that separately and later; only a brief outline will be provided in this monograph. For further details the reader is referred to the textbooks that were available at the time (Schermerhorn and Van Steenis 1964), (Alberda 1983), (HTW '56).

2.1 The primary control network of the *Rijksdriehoeksmeting* (RD-network).

The Netherlands is covered by a triangulation network that was constructed in the nineteenth and twentieth centuries. see *Figure 1*. The sides of the triangles vary between twenty and thirty kilometres. At the time, the theodolite was still the most important and most accurate survey instrument. Triangulation was therefore the mainstay of so-called 'higher-order measurements', that is, measurements with a, for that time, very high accuracy.

Figure 1: RD-network. The first-order geodetic control network, 1888-1904. See also Figure 13.

The coordinates of the corner points of the triangles were calculated in a national coordinate frame, of which the positive Y-axis pointed north and the positive X-axis east. This convention was also adopted in many other countries, but not all. Many countries adopted an alternative convention, as shown in *Figure 2*. The origin (0, 0) of the coordinate system was initially chosen to be the centre of the vertical bar of the weather vane on top of the spire of the *Onze Lieve Vrouwetoren* (Our Lady's tower) at Amersfoort. Later the origin was shifted such that all coordinates in The Netherlands have positive values and everywhere within the country Y>X.

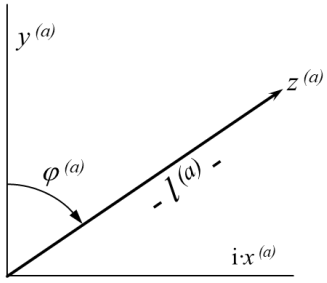
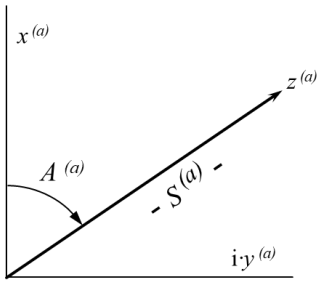
Dutch convention	Alternative convention
	
$z^{(a)} = y^{(a)} + i{\cdot}x^{(a)}$ $\varphi^{(a)}$ $l^{(a)}$	$z^{(a)} = x^{(a)} + i{\cdot}y^{(a)}$ $A^{(a)}$ $S^{(a)}$
$r_{jk} \equiv \varphi_{jk}^{(j)}$ $s_{jk} \equiv l_{jk}^{(j)}$	$r_{jk} \equiv A_{jk}^{(j)}$ $d_{jk} \equiv S_{jk}^{(j)}$

Figure 2: Definition of (X, Y) coordinates

The length of the sides of the triangles in the RD-network were calculated from a few baselines that were measured with high precision; see *Figure 3*. The length of the shortest side b was determined with high precision and by applying local triangulation the length of side $DP_k - DP_{k+1}$, one of the sides in the RD-network, could be calculated.

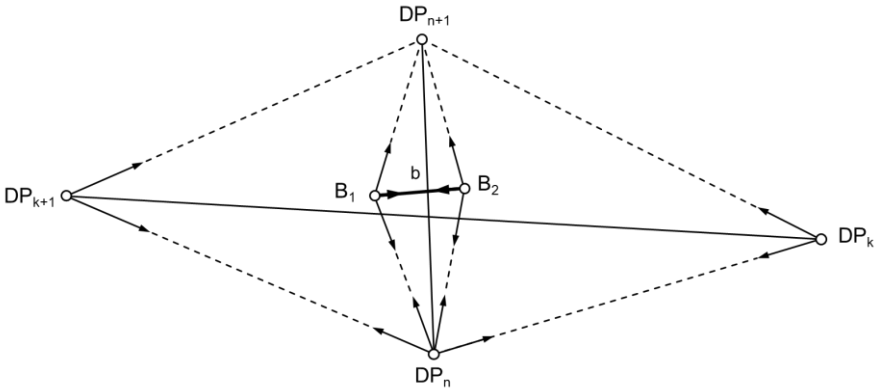


Figure 3: Measurement of the baseline of a triangulation network

The triangulation network of *Figure 1* was the primary or first-order network of the "Rijksdriehoeksmeting", the governmental triangulation agency. This was densified in several

steps by second and third-order measurements to networks with shorter sides. That resulted in a relatively dense field of control points, to which local measurements could be tied.

2.2 The theodolite

Because of the role of the theodolite, it is important to understand its construction principles, as that provides good insight into the nature of angular measurements. The core structure of the theodolite consists of a telescope, that can be rotated about a vertical and a horizontal axis. *Figure 4 (left)* shows an image of a Wild T2 theodolite, in which the design principles are readily recognised. *Figure 4 (right)* shows a schematic drawing of a theodolite. The theodolite is set up such that the vertical axis is aligned with the local direction of gravity. When the telescope is pointed successively at two distant targets, readings on the horizontal circle, which is perpendicular to the theodolite's vertical axis, can be taken, representing the directions to the targets. The difference between the values of those two directions is the angle subtended by the station and the two target points in the horizontal plane. A reading on the vertical circle, which is perpendicular to the horizontal axis of the instrument, can also be taken, representing the vertical direction to the target point. This reading is the zenith angle of the line between station and target. The elevation (or depression) angle can be derived from this, that is, the angle of the line between station and target with the horizontal plane.

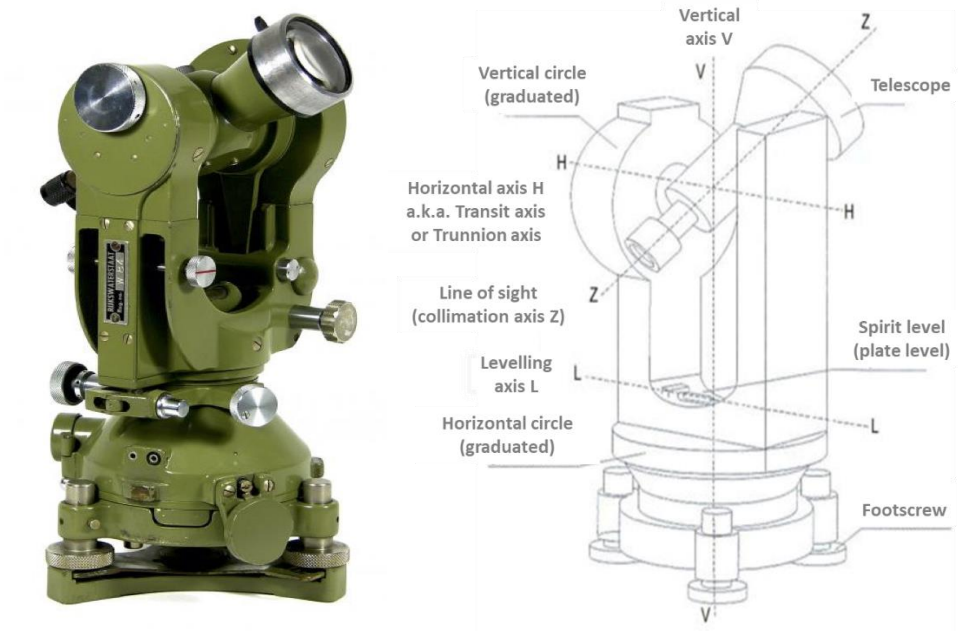


Figure 4: Wild T2 Universal Theodolite (left) and a schematic drawing of a theodolite (right)

The horizontal and vertical circles of theodolites used in The Netherlands and many other countries did not have a 360° graduation, but a graduation of 400 decimal grads, abbreviated as *gr*. A grad is also called *gon*. Thus, a right angle would be 100 gr. Every grad was subdivided to enable readings to a resolution of 10^{-4} gr = 1 dmgr (decimilligrad). With this unit a decimal system for angular measurements was realised. This approach was not universal; Anglo-Saxon (-oriented) countries preferred the 360° system.

2.3 Local densifications of fields of control points

In many cases geodetic control points were still too far apart to allow direct tying of local survey measurements to available control points. Hence, supplementary survey work had to be carried out to densify the control network further. For this task several measurement configurations were

in use. The first was the *traverse*; see *Figure 5*. This consisted of the establishment of a string of new points, called the turning points of the traverse, between two known control points.

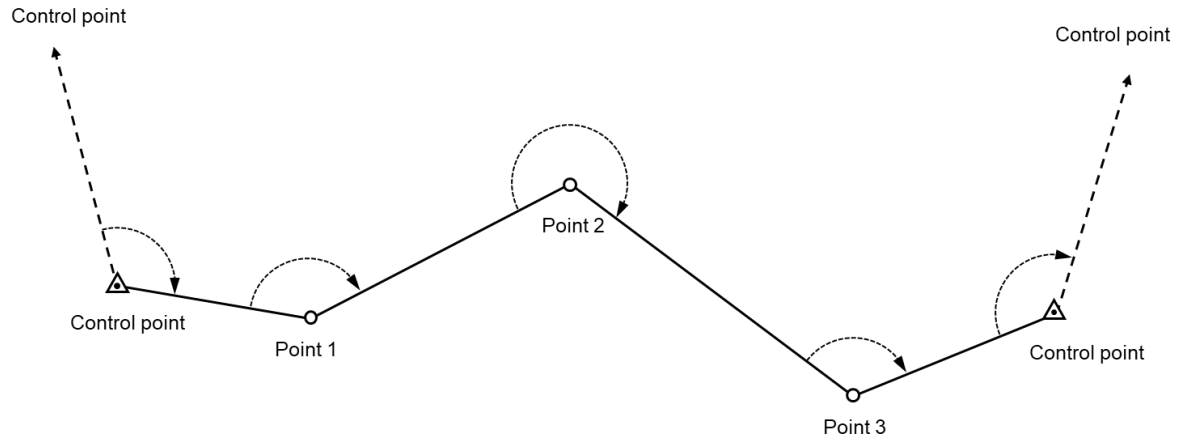


Figure 5: Measurement of a closed traverse

The angle, subtended at the first known point by another (distant) control point and the first new point of the traverse was measured by means of a theodolite. Thereafter, the theodolite was set up successively at each turning point of the traverse and the angle was measured between the forward target and the back target. The traverse was closed at the second known point by measuring the direction to another (distant) control point. This last step made it a *closed traverse*. When omitted, one spoke of an *open traverse*. Additionally, the distances between all points of the traverse were measured.

Hence this technique was based on the measurement of both angles and distances. Especially the accuracy of the distance measurement in lower-order surveys was limited and was conducted by means of measurement tape, optical distance-measurement equipment or by using a subtense bar, as shown in *Figure 6*.

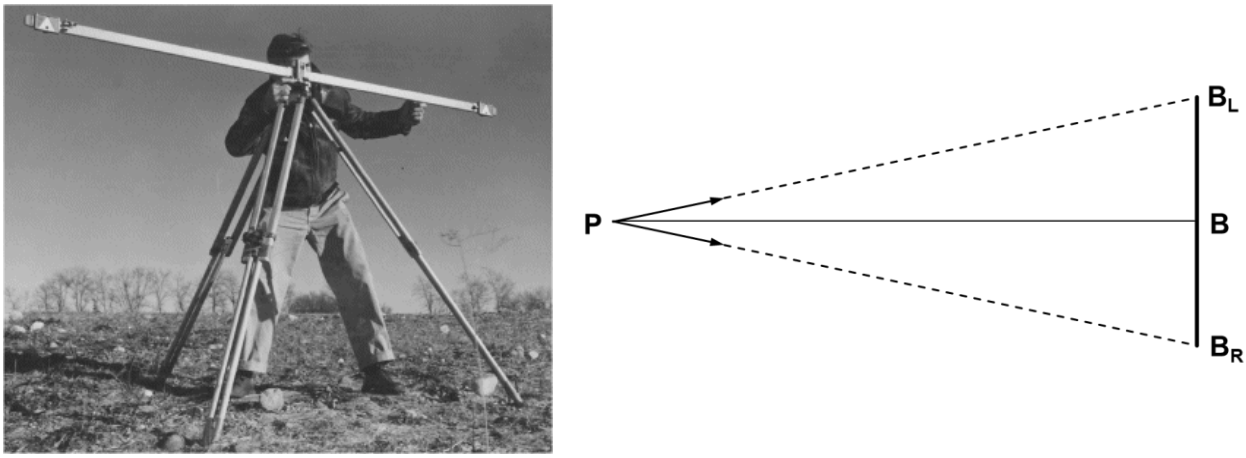


Figure 6: Subtense bar (left) and its principle of distance measurement (right)

Distance measurement with a subtense bar worked as follows: B_L-B_R is the subtense bar, which has a precisely determined length, usually 2 metres. It is set up such that the line between observer (P) and the middle of the bar (B) is the perpendicular bisector of the subtense bar. By measuring the angle subtended at P by the two end points of the subtense bar, the distance PB can be calculated.

Should there be no suitable control point in the area to serve as the start or end point of the traverse, the following solutions were available.

1. A baseline, consisting of two intervisible control points, was sought and a triangle was created with a new point as top. By measuring the two angles to the new point from each of the two baseline points the coordinates of the new point can be calculated. This technique is called *intersection*.
2. The theodolite was set up at a new point, from which three or more known points, evenly spread across the horizon, were visible and to which directions were measured. From these measurements the coordinates of the new point could be calculated. This technique is called *resection*; see *Figure 7*.

In the end these densification measurements yielded a group of control points, of which the coordinates were now known, which was dense enough to be used as tie points for survey measurements used for the mapping of local topography.

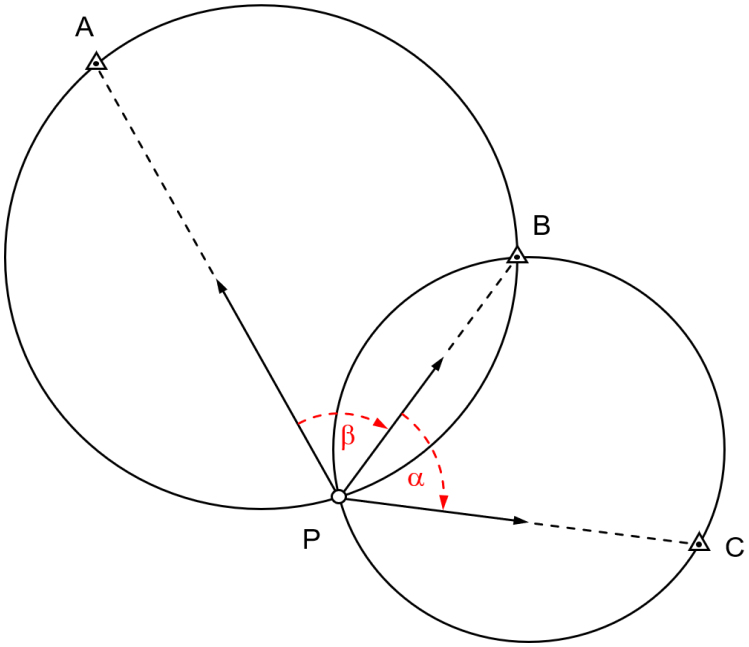


Figure 7: Resection from three control points

2.4 Research questions for the Delft School of Geodesy

Because measurement errors might occur, more measurements than were strictly necessary for the calculation of the coordinates of all points, so-called redundant measurements, were made. These enabled checks to be performed, which revealed that small contradictions occurred without any real measurement errors having been made. As early as the eighteenth century, this had led to the understanding that measurements have a stochastic nature and therefore behave like random variables, which show variations in their value when measured repeatedly. Based on this understanding it was no longer sufficient to calculate the coordinates of newly surveyed points; the small discrepancies that were caused by the stochastic nature of the measurements had to be resolved first. Furthermore, it was important to provide insight into the quality, or accuracy, of those coordinates.

These ideas generated the research programme of the *Delft School of Geodesy*, in which the following questions were addressed.

1. How does one resolve the contradictions, caused by the stochastic nature of the measurements, in a justifiable manner?
2. In what way does the stochastic nature of the measurements propagate into the calculated coordinates?
3. How can non-stochastic errors (gross errors) in the measurements be prevented?
4. How can the order of magnitude of any potentially undetected measurement errors and their impact on the calculated coordinates be estimated?
5. How should a survey plan be designed that ensures that the effects of 2, 3 and 4 are kept within acceptable limits?

In the Tienstra period work focussed mainly on the first two questions. Tienstra built upon advances in mathematical statistics, achieved in the nineteenth century. The *Method of Least Squares* had been developed during that century, partly based on experience with geodetic and astronomical measurements. Some well-known mathematicians had been involved, such as Legendre, Gauss and Laplace. See also (Stigler 1986) and (Teunissen 2000).

This period was completed in 1956, after Tienstra' death. In that year, his book (Tienstra 1956) was published posthumously. This book provides a schematic overview of five standard methods for processing measurements with redundant observations, the five so-called *Standard Problems*. It also addresses the question of how measurement uncertainties propagate into the calculated coordinates. These ideas are applied to the composition of the *HTW '56,* which stands for *Handleiding voor de Technische Werkzaamheden van het Kadaster* (*Manual for Technical Work of the Cadastre),* published in the same year. This manual was the successor to the 1938 version of the manual (HTW '38), also aimed at cadastral land surveyors and survey practice in general. The HTW contained guidelines for the reconnaissance, measurement and calculation of geometric structures in land surveying.

After Tienstra's death Baarda took over the baton. His focus would be mainly on questions 3 through 5. At around the early 1980s these questions had led to the completion of an attractive and consistent theory on the basis of which updated guidelines for survey practice were formulated. The advent of electronic distance measurement combined with angular measurement enabled novel designs of survey networks to be made. In conjunction with the rapidly developing information technology and computing, this provided a good basis for improving both the understanding of the theory and its application in survey practice.

At Baarda's retirement in 1982, he was offered a book in two volumes entitled *Forty Years of Thought* (LGR 1982). This book was a collection of articles written by colleagues, undergraduates and friends in honour of Baarda's contributions to geodesy, nationally, but also globally.

3 The Tienstra period

3.1 The first draft of a research programme

On October 26, 1931 Jacob Menno Tienstra accepted the position of Lecturer for the land surveying curriculum at what was then called the Agricultural University at Wageningen. The title of his inaugural address was *Usage of the method of least squares in land surveying*. The least squares method had been developed in the nineteenth century by some leading mathematicians in France and Germany. This story is exhaustively and absorbingly described in (Stigler 1986). The book shows how astronomy, but especially geodesy, played a stimulating role in the development of this method. A shorter and clear description is provided in (Teunissen 2000).

As described in Chapter 2, more observations were always taken in geodesy and land surveying than were strictly required for the calculation of the positions of new points. 'Redundant' observations were thus available to verify that no measurement errors were present. Because of the stochastic nature of the measurements, variations in the measurement results occurred, that always led to contradictions in the calculated results, even when no real measurement errors had been made. Particularly in scientific circles the least squares method was considered to be the purest way of reconciling these contradictions.

However, in survey practice it was a long way from being customary. The calculation processes for the solution of large systems of linear equations were too complex and time-consuming. The era of the computer had not yet begun. Calculations were executed by hand, with the aid of logarithmic tables and manual or electrical mechanical calculators, such as the Brunsviga and Friden respectively. Therefore ad-hoc methods were invented for the execution of the calculations needed to reconcile data contradictions.

In his inaugural address Tienstra showed that these practical solutions led to problems, because measurement errors and stochastic variations in the measurements were not taken into account correctly. That became even clearer when measurements were processed in successive stages for the stepwise expansion and densification of survey point sets. Improperly processed errors and contradictions in previous measurements then led to distortions that created problems when new measurements were being tied to earlier ones. He also stated that it was important to provide some indication of the quality of the calculated coordinates, to assess whether the quality of existing points was good enough for ties to additional measurements.

Tienstra used a metaphor in which a group of people began to occupy an uninhabited island. He explained how that island would be mapped. A land surveyor first established a triangulation network covering the island, which was densified subsequently by finer control networks. That was followed up by further densification by means of traverses and resections and then finally, a detailed survey of the topography.

This example thus showed the various work phases in survey practice. In the remainder of his lecture, he abandoned this example and addressed the aforementioned problems. After that he made a plea for the application of the least squares method, in which he discussed the following subjects:

- The processing of measurement results, whether done by ad-hoc methods or least squares, was aimed at calculating coordinates. It was not customary to provide information about the quality of these coordinates, but that ought to be done. Tienstra made a plea to additionally calculate the standard error ellipses (or error curves) of the coordinates (see Appendix 4).

- Because the activities of the surveyor or geodesist were divided up into several stages, a phased application of the least squares method, following the successive work steps was meaningful. The results of one step could then be used as input to the next, without the adjustment of the earlier step having to be repeated.
- Tienstra also argued in favour of a survey plan that would lead to circular error ellipses of the computed coordinates (see Appendix 4). As a result, the accuracy of the results is not sensitive to the direction from which a point is intersected. This places demands on the execution of the survey. Because he was mainly considering the surveying of triangulation networks, the accuracy of the angle measurement is important, as is the geometry of the triangulation network. This will be addressed in the next chapter.

3.2 Pursuing circular error ellipses

Tienstra indicated in his public lecture that he had found a way to generate point sets with circular error ellipses (he actually describes 'error curves', which are derived from error ellipses). He considered this the ideal result, as was common in geodetic survey practice of the time. This was followed by a publication with the Royal Academy of Sciences (KNAW), in which he explained his approach (Tienstra 1933).

In the introduction he stated:

> *When by measurement the position of a point in a plane has been determined, the accuracy of this determination is represented by the error-curve of the point according to the theory of the least squares. This curve is a foot-point curve of an ellipse. The determined point is the centre of this curve. In particular cases the error-curve is a circle. The point is then determined with the same accuracy in all directions. Generally, it will deserve recommendation to require equal accuracy in all directions when determining a point. This will, for example, be the case with an angle point of a net of triangles.*

Furthermore, he claimed that he would prove that:

> *... if the angles of a net of triangles without obtuse angles are measured with weights which are per triangle proportional to the cotangents of the angles, all the points of the net will have circular error-curves.*

The proof is provided through quite a few hefty mathematical derivations. Error ellipses will indeed be circles when the weights (see Appendix 1) of the measured angles are proportional to the cotangents of those angles. Tienstra demonstrated first that this is valid for the calculation of coordinates in a free network. This is a network that has not been tied to the established control network, hence a network with its own coordinate system. After that he showed that these results are retained when the free network is subjected to a similarity transformation, that is, when the coordinates are converted to an arbitrary other system. Mathematically speaking, that is correct, but it does invite a few comments:

1. The requirement is placed on the weights of the angles. If the survey is set up such, that the angles measured are uncorrelated, the weight coefficient (in that case the inverse of the weight) and therefore the variance of an angle is proportional to the tangent of that angle. However, the value of the tangent is infinite for an angle of 100 gr (90°) and negative for angles greater than 100 gr. That is impossible for the variance, which must always be positive. Tienstra does state that a triangulation network should not contain any obtuse angles, but the point is that a function is used for the calculation of the variance that is inadmissible on principle.

2. This approach only formulates requirements for the shape of the error ellipse of each point. The quality of the entire set of points, as it would be described later by the complete covariance matrix, is not considered. As a result, it would appear later that this form of network optimization does not work properly. The requirement to pursue circular error ellipses with a prespecified radius would turn out to be too strict. It cannot be met.
3. The previous two points were fundamental. But there were also practical objections against the specified requirement. This was due to the fact that the surveyor preferably measured several series of directions to the other points per station. Measurement of individual angles was considered impractical and inefficient. This drawback was compounded by the fact that each of the angles would have had to be measured with a different precision, depending on the value of its tangent or cotangent. The last point was not practicable anyway.

Apart from these comments, this study by Tienstra does indicate that criteria for the quality of geodetic measurements were being sought. Here the focus is mainly on the precision of point coordinates, as expressed by the standard (or 'one-sigma') error ellipses. The underlying idea is that optimal geodetic networks are obtained if they are measured in such a way that points with circular error ellipses are eventually generated. This idea will persist with many people well into the 1980s.

The idea of pursuing optimal geodetic networks would eventually be dropped in the Delft School tradition, because the requirement was considered to be too strict, as will be shown in section 3.3 below. The surveyor had to be free to adapt the structure of the network to be surveyed to the local terrain. Moreover, the objection described in point 3 above applied. Experience had shown that stable measurement procedures lead to good results. The land surveyor had to be able to work according to a fixed work routine, but the proposed method did not permit that. Therefore, it was considered better to aim for a precision that would not be worse than a certain lower limit: measurements had to be good enough. The search for a criterion theory for the precision of measurements would be continued with this new formulation in mind. This quest would lead to an approach in the early 1970s that appeared to be usable.

3.3 HTW '38

In 1932 Tienstra, together with geodesist Th.L. Kwisthout and land surveyor P.J. Hamelberg, was asked to work on the design of a manual for the technical activities for land surveyors of the Cadastre. This manual was published in 1938 and was therefore referred to as HTW '38.

This manual briefly described the structure of the coordinate system of the *Rijksdriehoeksmetin*g, the *RD-network*. This was followed by an explanation of the survey designs that were used to densify this network in several successive steps, to finally arrive at the level at which the set of points was densified sufficiently to allow ties to local topographical surveys. In addition to the survey design, attention was given to the calculation methods for processing measurement results and the calculation of the coordinates of newly established points. Particular attention was paid to the adjustment of contradictions (misclosures), resulting from the stochastic nature of the observations. In addition, it was explained how the accuracy (standard deviations, variances) of the measurements affect the calculated coordinates, through the so-called propagation laws, and how standard error ellipses for the newly established points are derived from those results. The reconnaissance requirements were now formulated less strictly than Tienstra had done (Tienstra 1933) for reasons that have already been given in paragraph 3.2 above. Or, as Baarda put it (Baarda 2002):

Allocation of a weight to an angular measurement, proportional to the cotangent of the angle, turned out to generate circular standard error ellipses of the corner points (the standard error ellipse is the 'image' of the covariance matrix of the two coordinates of the point; orthogonal projection of the standard error ellipse in a certain direction yields twice the standard deviation – the square root of the variance – in that direction). Circular standard error ellipses were the ideal in the geodetic world. However, an efficient and reliable measurement process precludes the measurement of angles with different weights. Selection of network points that ensures all angles are approximately sixty degrees provides the solution. Analogous reasoning holds for single point positioning and traversing. Hence, prior reconnaissance in situ of this measurement pattern – the so-called geometric framework – in an area to be surveyed is required to meet the objectives. As lead editor in an editing committee of three, Tienstra worded all this in the 'Manual for Technical Work of the Cadastre (HTW) of 1938 in a way that was unique for that time. The objective was formulated that the standard error ellipse of all points to be established had to lie inside a circle with a radius of 'd' cm. A value of d=3 was imposed on points of the Rijksdriehoeksmeting; for points in medium to small urban and rural areas, d=6; for areas where boundaries consisted of ditches or hedges d=12 (especially with a view to aerial mapping).

Hence, accuracy requirements were no longer formulated in terms of pursuing ideal point precision. They were now worded as follows (HTW '38):

… the square root of the mean of the squares of the major axes of the error ellipses, computed for a large-enough number of newly established points, shall be …

and then the specifications for types of terrain were given. Incidentally, these requirements differed slightly from what Baarda stated in the citation above.

Although the reconnaissance requirements were more relaxed now than in (Tienstra 1933), the limitation that only point error ellipses were considered remained. No consideration was given to covariances between points, from which, as we will see, relative standard error ellipses can be computed (see Appendix 4).

3.4 Revisiting the least squares method

3.4.1 Questions about the least squares method

In 1935 Tienstra switched to University of Technology at Delft, where he was initially appointed as associate professor and in 1939 as full professor. In his inaugural address he explained his struggle with the foundations of the method of least squares. From the beginning of the twentieth century until immediately after the Second World War, much thought was given to the relationship between mathematically formulated laws of nature and the stochastic nature of measurements, which was not yet well understood. In a broader context, this was about the meaning of mathematical models in relation to measurements that yield variable outcomes. In the Netherlands, the mathematicians G. Mannoury and L. J. E. Brouwer were the leading minds of this period. Their ideas were taught to a wider audience by David van Dantzig, a student of Mannoury and Brouwer.

Tienstra had contacts in that group, in particular with Van Dantzig, and was influenced by their ideas. His experience was that application of the least squares method yielded good results, whilst in the thinking of the time its foundations were considered questionable. His reflections concern two important aspects of current thinking about the relationship between the measurement process and scientific models:

1. Physical phenomena are described by means of characteristic quantities, between which relationships exist that are established in a mathematical model. Measurements are aimed at obtaining values for those quantities, thus quantifying the model.
2. Measurements are thus aimed at finding the values of those quantities, but due to their stochastic nature, variations occur in the measurement results, so that unambiguous values for those quantities can never be definitively established.

Regarding point 2, he indicates that justifying the method of least squares from the foundations of probability theory is problematic. This is because it is assumed that measurements comply with the *Central Limit Theorem*. That implies that repeated measurements will yield a spread of results, but that, if the number of repetitions is very large, this spread will approximate the *normal distribution*. Mathematically it is assumed that the normal distribution will turn out to be the limit of the spread function (probability density function) of a variable x, if the number n of measurements x_i would increase to infinity. The consequence is that for the mean X_n of each of the observations the following holds:

$X_n = \sum_{i=1}^{n} \frac{x_i}{n}$ with $E\{x\} = \lim_{n \to \infty} X_n = \mu_x$

in which μ_x is the *mathematical expectation* of the measurement results. Variable x is referred to as stochastic. The mathematical expectations of the measured quantities were also referred to as 'true values', that is, values for a model as referred to in item 1 above. The problem now is that in real measurement sequences hardly ever enough observations are made to establish the validity of these axioms. So, if the validity of these axioms is a condition for the applicability of the least squares method, there is a problem. But it is also questionable whether measurement processes can be specified that are stable enough to allow such large amounts of data to be collected.

Moreover, a mathematical model is always an idealisation of physical reality which one wishes to describe. Tienstra clarified that by means of an example in which the distance between two terrain points is measured. Let us elaborate on the example with three terrain points. One will have to understand that that these 'points' are idealisations of terrain objects. These objects are represented in the model by dimensionless points and that always implies some uncertainty about the link between the terrain object and the mathematical point. That means that it is not clearly defined which triangle is involved and that makes it problematic to specify the angles between the three points in a mathematical sense. Moreover, a theodolite that has its own physical characteristics is used, whilst the light rays involved propagate through the atmosphere from point to point and are subject to refraction that occurs during the measurement process. So, it is not certain either that the angles, even when they could be specified exactly, can also be measured in reality. Instead, something similar will be recorded.

Hence the link between physical phenomena and mathematical models is subject to inherent uncertainties and, additionally, the measurement process carries its own uncertainties. Measurement processes can therefore never supply definitive, true values. One is forced to work with estimates resulting from the measurements and the application of the least squares method. That means that the validity of the models as referred to in item 1 above can never be established with certainty; all one can attain are estimates. Even while those are frequently quite good, a method will have to be found to establish their reliability. Moreover, experience had shown that more accurate measurements, resulting from improvements of measurement techniques, often led to model adjustments.

After the Second World War, Tienstra therefore investigated the foundations of the method of least squares as it usually was applied in geodesy. His effort is nicely described in (Baarda 2002, pp. 133-134). He was mostly interested in gaining a thorough understanding of the reasons why

use of the method was justified and the true nature of the information it processed and made understandable. According to Baarda, in addition to being influenced by people such as Mannoury and Van Dantzig, Tienstra was also strongly influenced by the ideas of Von Mises, in particular by his book *Wahrscheinlichkeit, Statistik und Wahrheit* from 1936.

3.4.2 Foundations of observational calculus and the least squares method (Tienstra 1948)

The 1948 article (in English) begins immediately with strong criticism of the common terminology of the time:

> *In most classical textbooks on the calculus of observations in the first chapter we find a discussion on different kinds of errors: the true error, the systematic error and the accidental error. In connection with the true error, also a true value of an observation is introduced. Sometimes the true value is considered as a mysterious 'super value', that, may it not be measured by the imperfect methods of mankind, nevertheless must be considered to exist; sometimes it is defined as the limit to which tends the arithmetic mean of a number of observations when this number increases ad infinitum. It is clear that in both ways the conception is perfectly meaningless. Moreover it is in full contradiction with the theories in modern physics about the structure of matter. Therefore the true value and with it the true error has to be abandoned as belonging to a passed period of the history of science. … The theory can be built up simply, if we only begin with the beginning: the observations. The observations are a physical reality and in them is to be found the source of all our knowledge on measuring. It is nonsensical to introduce incontrollable highly mysterious and metaphysical conceptions on a domain of pure physics.*

Tienstra objected to the use of the Gauss-Laplace normal distribution, because its tails ran to infinity. In real measurement processes such large deviations did not occur. Moreover, as described above, he objected to the use of the concept of mathematical expectation. He also wanted to avoid the concept of probability as a mathematical concept. He wanted to avoid these, in his view, hypothetical concepts and develop his theory on a purely empirical foundation. This approach presumably arose from his contacts with people like Mannoury and Van Dantzig. As mentioned above, they were representatives of a group of scientists and philosophers who were retrying to get a grip on the empirical foundations of science in a broader sense.

Tienstra developed his ideas from this background. He made a very large number of observations of two mutually independent quantities x and y and, from those, built up a two-dimensional histogram. The values of x and y were plotted along two axes. He recorded the number of observations belonging to each pair of (x, y) values, that is, the relative frequency of occurrence of that pair of values. He then showed that the (x, y) pairs with the same relative frequencies formed approximately an ellipse of which the major and minor axes were parallel to the x- and y-axes. By means of a conversion to different length units along the x- and y-axes the ellipses can be converted to circles. Then it was explained how relative frequency distributions can be derived from this histogram for linear functions of x and y.

This idea can be extended for larger numbers of quantities x_i, with $i = 1, \ldots, n$ and linear functions of x_i. To this end, the propagation laws for the mean, the variance and the covariance were explained, or, instead of those, the propagation laws for weight coefficients (see Appendix 1), referred to as *cofactors* by Tienstra. He then introduced a n-dimensional space in which the coordinates of a point x corresponded to the n observations x_i. In that space points with the same relative frequency values generate hyper-ellipses or hyper-spheres instead of ellipses or circles.

This was followed by an explanation of least squares adjustment according to Standard Problem I; see Appendix 1. This was elucidated by a geometrical interpretation as shown in Appendix 3. The

conclusion of the story is that the method of least squares can be explained without invoking the normal Gauss-Laplace error distribution function and without invoking the formal concept of probability. He formulates this as follows:

> *To justify this we put for us once more the n-dimensional coordinate system … on which the representation of the n-dimensional frequency distribution of the observations had the property that points of equal frequency formed a system of concentric hyper-spheres and also a same system on which we represent the [Y-ruimte] and the 'observed point', both systems being placed before us in a same orientation. We now apply a translation on one of the systems relative to the other one, so that the observed point is placed on the hyper-sphere with the smallest radius and the centre of the hyper-spheres is in the [Y-ruimte]. Then this centre is the 'adjusted point' and as the observed point is now on the smallest hyper-sphere, its relative frequency is a maximum.*

In short: the point with the corrected observations as coordinates *X* is in the Y-space *[Y-ruimte].* The original point *x* now lies on a hyper-sphere with the point *X* as its centre and radius R. All points that satisfy the condition equations of the adjustment must be in Y-space. It now turns out that, of all points in Y-space, point *X* yields the smallest value for R. In essence, this result is actually the starting point of the adjustment: the search for the shortest projection of point *x* on the Y-space; see Appendix 3.

The formulation above is somewhat cumbersome; essentially, he explains the principle of what is referred to in literature as *maximum likelihood estimators*. But that concept was probably not known to him at the time. Of course, the train of thought developed by Tienstra is not quite independent of the approach developed in the textbooks, because, instead of adopting Gauss's normal distribution, he introduces a hypothesis that the measurements will generally behave in the way he discusses in his example. This is therefore a hypothetical induction step. In the textbooks this leads to the introduction of a mathematical error distribution and the concept of probability, which he wants to avoid. Baarda would later deal with this more pragmatically by following the philosophy of engaging and disengaging mathematical models.

At the end of his discourse Tienstra states that the functional model (he refers to it as the *algebraic model*) that applies to the measurements must be tested. At Baarda's suggestion he indicates that the function $v^T \cdot g_{xx}^{-1} \cdot v$ of the corrections to the measurements contains the information for this test. He will not work that out in detail; Baarda will do that later.

3.5 The five Standard Problems and adjustment in steps

3.5.1 An Extension of the Technique of the Method of Least Squares to Correlated Observations (Tienstra 1947)

In his 1947 publication, Tienstra first addressed the fact that applications of the least squares method usually rely on the input of uncorrelated observations. He demonstrated that this method could be generalized in a simple way, so that correlated observations could also be used as a starting point. At that time, this was an innovative insight, especially for geodesists. He then formulated four variants for the application of the least squares method. These variants make use of this generalization of the method.

The use of Ricci Calculus

To formulate these variants with their solutions, he used Ricci Calculus, a complicated index notation that defines additions and products of several types of variables in the following style:

$y^r = B^r{}_s x^s$

See the explanation in annex 7, pp. 302-303 in (Tienstra 1947). This notation is used in tensor calculus. Geodesists have come to know that through differential geometry, a special application of tensor calculus is used in geometric calculations on curved surfaces such as the ellipsoid. This notation is very useful for formula systems in which many different types of quantities occur, but this flexibility is at the expense of the readability of the developed formulas. This made Tienstra's theoretical developments difficult to fathom for many. See also the remarks on this in (Alberda 2005). Tienstra probably opted for this notation because he was not yet sufficiently familiar with matrix calculus.

From uncorrelated to correlated quantities

Tienstra now re-derived the propagation laws for mathematical expectations and covariances of functions *y* of the original observations *x* with the aid of Ricci Calculus. See also annex 7, pp. 304-305 in (Tienstra 1947) and Appendix 1. The values of the variances and covariances are often not exactly known. Therefore, so-called weight coefficients are used instead. Tienstra also refers to these as cofactors; see Appendix 1. When starting with the assumption of uncorrelated quantities *x*, it will appear that quantities, derived by applying the propagation laws, will be correlated. In other words, if $g_{x_i x_j} = 0$ for all x_i and x_j ($i \neq j$), then for two functions of *x*: *y* and *z*, it will be found that $g_{yz} = g_{zy} \neq 0$.

The least squares method therefore appears to be applicable whether or not the entered observations are correlated. When *v* are the corrections to the observations to be calculated, and the least squares requirement is to minimize the expression (see Appendix 1):

$$v^T \cdot g_{xx}^{-1} \cdot v$$

then it is irrelevant for the solution algorithm whether g_{xx}^{-1} and hence g_{xx} are or are not diagonal matrices. If both are diagonal matrices, the quantities are uncorrelated; if not, they are correlated. With this remark Tienstra made the method more generally applicable than was customary in geodetic practice, because, in general, uncorrelated observations were assumed. In addition, observations with equal weights were usually assumed, which implies that all observed quantities would have been measured with equal accuracy.

Five Standard Problems

After these introductory considerations (Tienstra 1947) took a fresh look at the structure of adjustment problems in geodesy. Four basic structures were distinguished in that publication and a fifth was added later. An overview of these five Standard Problems is given in Appendices 1 and 2. In this chapter the notation used in the appendices is used.

The first formalism, *Standard Problem I*, was based on the fact that redundant observations were included in the survey design as check measurements. These were used to formulate condition equations, such as 'the sum of the angles of a triangle equals 200 gr (180°)', or, 'the sum of all angles measured at a single station equals 400 gr (360°)', etc. These conditions concerned functions of the observations, the values of which were known in advance. These functions can always be written such that the value of the function equals zero. If the observations are designated by *x*, as in Appendix 1, and the conditions are formulated as follows:

$$t = U \cdot x + U_0$$

then the expected result will be $E\{t\} = 0$. However, as a rule, $t \neq 0$ due to the stochastic nature of the observations. These contradictions will have to be resolved, which is done by applying the least squares algorithm of Appendix 1.

In practice the formulation and processing of condition equations is not always as straightforward as described above. This led to the formulation of *Standard Problem II*, an alternative algorithm in

which the corrected observations were expressed as a function of the unknown quantities, or unknowns, leading to the *observation equations*. Hence, the corrected directions, angles or distances – let these be X – were expressed as functions of the coordinates Y. If these relationships have the simple, linear form:

$X = A \cdot Y + A_0$

the solution can be found by applying the algorithm of Appendix 2. Usually, this relationship is not linear. Then an approximate solution will be found for the functional relationships $X = F(...Y...)$. Under the assumption that the residual quantities ΔX and ΔY are small, this relationship will be linearised, so:

$\Delta X = A \cdot \Delta Y + A_0$

Standard Problems I and II are the key algorithms for formulating the other three Standard Problems, which are described in Appendix 2. It is noted here that the fifth formalism was not given in (Tienstra 1947), but in his later book (Tienstra 1956); see below. It was therefore probably not recognized as a new variant until later.

Solution to Standard Problem III and adjustment in steps

Evaluating *Standard Problem III* deserves more attention here, because it led to the insight that the method of least squares allows problems to be solved in several steps. Here, the starting equations consist of two groups (see Appendix 2):

$X = A \cdot Y + A_0$ and $U \cdot Y + U_0 = 0$

The first group consists of observation equations, as per Standard Problem II. The solution to the first step can thus be found with the algorithm from Appendix 2. This yields interim results. The set of equations of the second step specifies condition equations with the unknowns. Dependencies will therefore exist between the unknowns. Hence too many unknowns have now been introduced. The interim results for the unknowns will be written as Y' which has the matrix of weight coefficients $g_{Y'Y'}$.

The second group of equations therefore consists of condition equations as per Standard Problem I, but now formulated in terms of the unknowns Y'. In (Tienstra 1947) it was demonstrated that the conditions can also be formulated for functions of the original observations. That implies that the solution algorithm of Appendix 1 can be applied. The input to the algorithm is the interim result Y' instead of the observations x, and, furthermore, the weight coefficients matrix $g_{Y'Y'}$ is used instead of g_{xx}.

This yields the least squares estimator Y for the unknowns. The corrected observations are then computed from:

$X = A \cdot Y + A_0$

Tienstra demonstrated that a solution obtained in multiple steps will yield the same results as a single-step solution. Solutions for *Standard Problems IV* and *V* are readily derived through such a division into steps.

For the case in which the adjustment is based on the formalism of *Standard Problem I*, the adjustment in steps proceeds as follows. When the observations are entered into the condition equations, the followed misclosures are obtained:

$t = U \cdot x + U_0$

These equations can be split into two groups:

$t_1 = U_1 \cdot x + U_{01}$ and $t_2 = U_2 \cdot x + U_{02}$

When the first group is solved by applying the Appendix 1 algorithm, the interim result is X' with $g_{X'X'}$. When entered into the second group of equations, this yields:

$t'_2 = U_2 \cdot X' + U_{02}$

Entering $g_{X'X'}$ into the Appendix 1 algorithm instead of g_{xx} will yield the adjusted observations X as the final result. Again, the same result would be obtained from a single-step adjustment. The condition equations therefore can be adjusted groupwise. Baarda would make use of this property later in the development of testing methods for the detection of gross errors in observations.

Tienstra also demonstrated that a two-step solution could be split further into an arbitrary number of steps. The maximum number of steps depended on the number of redundant observations and therefore the number of conditions. Moreover, he showed that this approach is valid for all least squares problems. The method thus proved to be generally applicable.

Tienstra refers to the option to split adjustment problems into several steps as "the principal property of the method of least squares". This was of immense importance to land surveying and geodesy, because surveys were often executed in several successive steps. The processing results of an earlier step could now be entered into the next processing step after the addition of new measurements; that is, as survey networks were extended in a piecemeal fashion. Moreover, Baarda in particular would later distinguish between the adjustment of newly surveyed networks in isolation and their tying to known control points as a second step.

Note: the process of adjustment in steps, as described above, is defined for the situation in geodesy and land surveying in which the surveying of control networks is done piecemeal over a relatively long period of time. That implies a rather static approach. However, the algorithm is consistent with the Kalman filter, which was formulated for dynamic measurement processes.

3.5.2 The book

The contents of (Tienstra 1947) and (Tienstra 1948) was incorporated into the syllabus of a course that Tienstra presented at the Mathematical Centre in Amsterdam in 1949. The lecture notes *Observational calculus, parts I and II* were published in 1951, the year of Tienstra's death. These lecture notes probably have a considerable overlap with the earlier mentioned syllabus. *Standard Problem V* was also added to part II of the lecture notes; see Appendix 2.

This course therefore provided a fairly complete overview of the work with which he shaped observational calculus over many years. This approach would henceforth be followed in the further development of the philosophy of the Delft School, up to the 1980s. After his death in 1951, the lecture notes were converted into a book, published by Elsevier in English in 1956 with the title *The Theory of the Adjustment of Normally Distributed Observations.* This book has had a large impact in international geodetic circles. References to this book in scientific publications and later textbooks were made until long after its publication.

The book mentions that its publication was facilitated by his friends. After Tienstra's death Baarda took over the baton. It is clear that he played a major role in its publication. He would continue where Tienstra had to leave off.

3.6 Some concluding remarks

3.6.1 The use of complex numbers

The publication (Tienstra 1933) was discussed in Section 3.2. In addition to the main theme of this publication, it is interesting to see how Tienstra used complex numbers. He did this for

defining circular standard error ellipses and for performing a similarity transformation on the coordinates. Geodesists were familiar with complex numbers because these were used in mathematical formulations of map projections. In one of the derivations (Tienstra 1933, bottom of p. 664), he arrived at the following formula, somewhat paraphrased here:

$ln\ \xi_{jik} = A_{jik} + i \cdot a_{jik}$

A_{jik} is the natural logarithm of length ratio of two sides of a triangle, *ij* and *ik*, and a_{jik} is the angle between these two sides. In this formula the structure of Π-variates, which Baarda would introduce later, may be recognized; see section 4.4 below.

Baarda wrote in (Baarda 2002) that Tienstra also used complex numbers in an article about traverses. I have been unable to locate that article. Presumably it concerns a report that was discussed during conferences of the "Nederlandse Landmeetkundige Federatie" (Netherlands Land Surveying Federation) in 1937-39 (Tienstra 1937). The HTW '38 also used complex numbers for performing similarity transformations. In chapter 15 of HTW '38 (pp. 68-69) the transformation formulas are described with complex numbers, as in (Tienstra 1933). This approach is clearly recognisable as a precursor of the use of complex numbers in the *S-transformation* which would be formulated later by Baarda.

3.6.2 The development of a theory without good computational facilities

As explained in Section 3.1, computation techniques were still based on manual methods in the time Tienstra began his research. Calculations were executed by hand, with the aid of logarithmic tables and manual or electrical mechanical calculators, such as the Brunsviga and Friden respectively. Therefore, practical ad-hoc methods were invented for the execution of calculations for reconciling, or at least controlling, data contradictions. The computational processes for solving large systems of linear equations were felt to be too complex and time-consuming. The era of the computer was still far away.

It was therefore so admirable that Tienstra had the insight that the use of all those approximation methods would lead to tensions in geodetic networks, which would eventually become very disturbing. He understood that geodesy needed a coherent scientific theory that allowed computational methods to be developed that dealt with the stochastic nature of measurements in a justifiable manner.

His approach was in line with developments in mathematical statistics of his time, but he was very reluctant to apply the mathematical models of the nineteenth century without further ado. He attempted to work from a purely empirical basis without (too many) a priori assumptions, as was described in Section 3.4. Widespread application of his ideas in survey practice was delayed until the advent of modern computers in the 1960s and 1970s. Later Baarda would provide a great impulse to this with the establishment of the Geodetic Computing Centre (LGR). But Tienstra's work did form the scientific basis on which Baarda would continue to build.

3.6.3 Few references to source literature

Few traces exist that might shed light on influences on Tienstra's thinking in addition to his having been strongly influenced by the circle around Van Dantzig and Mannoury. Baarda mentions the book by Von Mises and additionally the philosophers A. N. Whitehead and H. Reichenbach in (Baarda 2002).

A handful of references to literature exist in (Tienstra 1956). His remaining publications contain few or no references to literature. Yet many important publications had appeared before his time. For example, Gauss had already identified the importance for geodesists of adjustment based on condition equations (Gauss 1828) and Legendre had formulated the adjustment method based on

observation equations (Legendre 1805). Additionally, the principle of adjustment with correlated observations had been explained already by (Aitken 1936). And in (Helmert 1907) a summary treatment was given of the use of the least squares method in geodesy, among other disciplines. No references to these or other sources are found in Tienstra's publications. Apparently, this was not customary in the past and journal editors might not have asked for them yet.

4 The Baarda period

4.1 Practical experience as a land surveyor

Willem Baarda, born on July 20, 1917 at Leeuwarden, was appointed lecturer in *Land Surveying, Spirit Levelling and Geodesy* at the Civil Engineering faculty of Delft University of Technology (TU Delft) on November 1, 1946. Before that, he had held various positions as land surveyor at the Cadastre from 1940 onwards (Hoek 1982), (Alberda 2005). During the time he had been appointed associate surveyor at Zwolle, his task consisted largely of establishing a geometric framework for the recently reclaimed Noordoostpolder. As described by Van der Hoek, the experience he gained during that period was to have a major impact on his later theoretical work.

Baarda was appointed Land Surveyor with the Cadastre on January 1st, 1942 and transferred two months later to the Zwolle Office of the Service for Exceptional Land Surveys. This service had been set up shortly before to provide re-surveys and other cadastral innovations. Baarda arrived as an 'associate' land surveyor, which means that he was deployable throughout the wide area covered by the Zwolle Office. A major part of a land surveyor's work at the Service for Exceptional Land Surveys consisted of the establishment of geodetic frameworks in accordance with the guidelines in the Manual for Technical Activities of the Cadastre, the "HTW 1938". In short, the method of plane point positioning of that time boiled down to choosing resection stations, connecting these to traverses and designing a network of survey lines within the framework of traverses that allowed topographical surveys to be tied to. This procedure was entirely acceptable even to those with a more than superficial professional interest. But not so to Baarda! He soon arrived at the conclusion that the method was 'not quite right', that is, that it led to contradictions.

One of the things that were 'not quite right' was the incorporation of the known coordinates of RD control points in the calculations from the outset, as a result of which the checking of one's own measurements was obscured by errors in those coordinates, among other things. The fact that following various paths through the adjusted traverse network led to different coordinates for the same point led Baarda to the conclusion that it was actually not clear what the meaning and the value of the coordinates were! Yet another aspect that drew Baarda's scientific attention were the, at first sight inexplicable jumps in the observation sequences of direction measurement.

The Noordoostpolder had been reclaimed during 1942 and was developed piece by piece using the limited means available during the war. The establishment of the cadastral administration of this wide expanse of mud flats and reed beds was one of the Zwolle Office's tasks. Baarda completed a very substantial part of this work. He used the opportunity to conduct research into various aspects of error theory. He also conducted measurements of refraction, as a result of which he formulated a theory for the calculation of regular density differences. The Head of the Zwolle Office, mr. H. Vermeulen, pursued a wise policy and allowed Baarda complete freedom in doing this work. Baarda never forgot the opportunity he had been offered. For years he kept in touch with his former boss until his death.

While living in digs at Zwolle, Baarda began reconstructing the theory he had been taught by Tienstra. He had to work from memory because he had not brought the relevant lecture notes and travelling had become impossible due to the war circumstances (it was in the year 1944).

Building on the experience that misclosure vectors of traverses usually pointed in the direction of the traverse he designed adjustment method no. II. In the same lodgings Baarda also found the

principle that would later lead to the entire theory of S-referencing.[1] The year 1944 was probably his most fertile period of contemplation. All his later scientific work consisted of working out the ideas of that period. As he used to say himself: the execution of a programme of action.

It is clear from the text above that in that period Baarda realised it was desirable to design the measurements such that the new framework could be tested in isolation to verify that no errors were present in the new measurements. Only after that, in a second step, would the new survey be tied to existing control points, to check whether these were undisturbed. With this he had essentially formulated the basic ideas of his later theory of *S-referencing*.

Around 1950, inspired by these findings, he made an analysis of the various methods for tying new survey networks to established control points used in land surveying. He compared the results of these methods with the results obtained by the application of least squares adjustment to ties based on similarity transformations. The results of this analysis were published later as an extended explanation of the treatment of these subjects in the HTW '56; see (Baarda 1956). This was the first step of a process that would ultimately lead him to the formulation of the theory of *S-referencing*. Additionally, it became clear to him that the approximation of the precision of established control points and the desired precision of the end results, as described in the HTW '38, were too simple and too stringent. This insight led him later to the formulation of his criterion matrices and the replacement matrices derived from them.

At that time, he was frequently in contact with his teacher, Tienstra; these contacts will have played an important role in his appointment as lecturer at the TU Delft. As a lecturer he worked closely with professor Tienstra and was introduced to the latter's approach to his theoretical work. Hence he was the ideal successor to Tienstra and took over Tienstra's role in the preparation of the HTW '56. He also had a major part in the posthumous publication of the book (Tienstra 1956), which summarised Tienstra's work. Baarda would adopt Tienstra's approach of adjustment calculus as a starting point for his further work, which was aimed at the development of a theoretical framework for the assessment of the quality of geodetic measurements.

4.2 David van Dantzig's influence

As discussed above, Tienstra was influenced by the ideas of Mannoury and particularly those of David van Dantzig. He introduced Baarda to those ideas. During the first half of the twentieth century, Van Dantzig worked at the Mathematical Centre in Amsterdam and at the TU Delft. Particularly at the Mathematical Centre he was involved in the revolutionary development of mathematics of that time, as described in (Albers 2000, a, b and c. A central theme in that development was the search for the role that mathematics began to play ever more emphatically in application areas such as physics, economics, biology and technology. It became increasingly evident that those disciplines always assumed measurements of which the values showed a certain spread. Hence the results of those measurements could not automatically be used as input

[1] Baarda introduced the concepts of *S-referencing*, *S-base* and *S-system* for the first time in his public lecture of 1947. He proposed to connect a newly measured network to only two control points, to check how the shape of the network fitted to the shape defined by the configuration of the whole set of control points. He likened this method with carpenters suspending a door or window to two hinges. If the door or window is not strong enough it will deform and will not fit into its frame. Dutch carpenters call such a deformation 'schranking'. His lecture of 1947 was in Dutch. Later in his publication of 1973 he realised that this term was an unfortunate choice. For non-Dutch readers he explained his symbol '*S*' as being an abbreviation of 'similarity'.

into self-consistent mathematical models. This created a problem that was not understood straight away. It invited philosophical questions about the role and applicability of mathematics and mathematical statistics, which had been developed recently.

Baarda attended the statistics course that Van Dantzig gave at the TU Delft in 1946-47. This course treated the following themes:

a) the meaning of the stochastic nature of observations (measurements);

b) the role of mathematical models in the processing of these observations;

c) engaging and disengaging these models in the processing of measurement results;

d) the occurrence of contradictions in those results, due to:
   - the stochastic nature of the observations, or:
   - deficiencies in the mathematical model, or:
   - (gross) measurement errors.

Presumably he also attended the course that Van Dantzig gave in the context of the *Studium Generale* at the TU Delft, a series of public academic lectures (Dantzig 1945-47). A summary of those ideas can be found in (Dantzig 1947). These ideas exerted a strong influence on Baarda during the remainder of his active life. Notably the items c) and d) above play an important role in Baarda's work. Engaging and disengaging mathematical models occurs at two levels:

1. The normal distribution is adopted as the probability density function of the observations, when supported by the behaviour of the observations, evidenced by the spread in the values of a large number of repeated observations. Regarding to this aspect, Baarda is more pragmatic and states that experience in land surveying provides enough support for usage of this model as a standard, unless the observations really suggest otherwise. In this pragmatic approach he thus uses the *Central Limit Theorem*. As discussed earlier, in that case the arithmetic mean of a the values of a stochastic variable *x* will stabilize close to the mathematical expectation *E{x}* of that variable, when the number of repeated measurements *n* is large enough:

   $$X_n = \sum_{i=1}^{n} \frac{x_i}{n} \rightarrow E\{x\} = \lim_{n\to\infty} X_n = \mu_x$$

2. At the second level, this plays a role in the definition of a functional (mathematical) model for the observations. This is a model of the section of physical reality that is desired to be quantified through measurements. It is formulated by expressing mathematical relationships between the observation quantities (in the form of conditions that should be satisfied by the observations) and between the observation quantities and the parameters that need to be determined, for example the coordinates of new points.

Baarda was strongly influenced by this way of thinking. He was therefore able to take a less dogmatic stance than Tienstra in the rejection of the axioms that were described as the foundations of the least squares method in professional literature.

In the posthumous publication of (Tienstra 1956) the influence of this approach is evident in the introduction to chapter 9.

THE CONTROL OF THE MODEL AND NUMERICAL DETERMINATION OF THE MODULUS; MEAN (SQUARE) ERROR.

9.1 Introduction. *– It is obvious that an observer will be interested in knowing how well his observations fit the assumed mathematical model. If the model is given in the form used in Standard Problem I, the contradictions in the relationships apparently provide this information.*

However, the difficulty is that these contradictions depend not only on the fluctuating character of the observations, but also on the degree of perfection with which the assumed model fits the actual conditions.

In the existing terminology the contradictions resulting from the observations are called 'accidental errors'; those resulting from the model are called 'systematic errors'. Systematic errors are considered to be traceable to the influences of temperature, the observation model employed, etc. When these have been sufficiently removed, the remaining errors are considered to be accidental. This notion of 'error' is the central point in the classical development. The concept of "true value" is also used: there is a true value of each quantity to be measured, and a true value minus a measured value is a 'true error'. The classical theory also uses an "apparent error". By this is meant a reference value minus an observation. The reference value is the adjusted value found by the application of least squares. The apparent error is therefore the opposite of the 'correction' applied to the observations in the theory as developed in this book.

Criticism of the classical development is based on the following points.

1. *It is wrong to base the theory on the concept of 'error' as starting point. We may only speak of an error after we have defined a reference value. This reference value can be nothing but a function of the observations. The observation, therefore, is primary and should logically be the point of departure in developing the theory.*
2. *The concepts of 'true value' and 'true error' are completely fictitious and physical absurdities. There is, for instance, no true length of a measuring rod.*
3. *It is certainly improper to speak of an 'accidental error' as a defined quantity, as there is no end to eliminating systematic errors.*
4. *The term 'systematic error' is at best misleading. In the classical theory 'error' is always connected with observation. The fact that systematic errors indicate imperfections in the mathematical model is more or less concealed by the misleading name.*

None of these doubtful concepts were necessary to the development of the theory presented here. We are only concerned with the observations, the mathematical model, and the method of least squares for adjusting the observations in such a manner that they fit the assumed model. The model may be more or less complete, the observations may be made carefully or roughly; these two factors determine the size of the contradictions when the observations are substituted in the model. This is the true character of the problem.

A remark on page 305 in (Tienstra 1948) and one in the chapter "FILOSOOF" (Philosopher) in (Baarda 2002) lead us to suspect that this chapter was added at Baarda's initiative, initially to the syllabus (Tienstra 1951) and later to the book (Tienstra 1956). This introduction discusses the same subjects as mentioned in (Tienstra 1948), but they are expanded here to the starting points of the testing theory that Baarda would develop later. In this chapter Baarda puts words into action by demonstrating that Tienstra's histograms are described well by the Gauss-Laplace probability density function (the normal distribution). Later this allowed him to use testing methods based in this distribution. that had been developed in mathematical statistics.

From this point on Baarda would formulate the functional model for the expectation values of the observation variates and the unknown parameters. The original observations do not comply with the model on account of their stochastic nature. The least squares method yields estimators that *do* comply with that model. This philosophy of engaging and disengaging models allowed him to opt for a pragmatic approach. This necessitates the use of effective testing methods to evaluate whether the modelling choices made were justified. The misclosures of the condition equations

contain the information on which these tests must be based. Baarda would devote a large part of his work to the formulation of such a testing method.

4.3 The HTW '56

In 1950 Tienstra and the Cadastral Land Surveyors 1st Class D. de Groot and F. Harkink were asked to compile a new HTW. After Tienstra died in 1951, Baarda took over his role. A few aspects of this new HTW are noteworthy.

Chapter II discussed the surveying of new point fields and their tying to the RD network. Section II.2 discussed straight away the reconnaissance criteria for new surveys. Baarda's experience, gained from his work in the Noordoostpolder, are evident from the fact that criteria for absolute point accuracy were de-emphasized. The new criteria were formulated in terms of the required *relative* point accuracy, that is, the accuracy of one point relative to another point. The required accuracy of the coordinate differences of two points was specified as a circle with a radius that was a function of the distance between these points. The required absolute point accuracy was specified as a circle with a radius that is a function of the distance to the closest existing control point. These circles were interpreted as accuracy criteria; calculated standard error ellipses had to lie within them. Baarda discovered later that this approach would also lead to problems and developed a *criterion matrix* in his 1973 publication.

Section II.3 began with a treatise on the essence of coordinates. Here too Van Dantzig's influence could be discerned, particularly in the first paragraph. Whether an average practical land surveyor would have been able to understand this is a valid question.

Section IV.10 discussed the tying of densification networks (triangulation networks) to the RD network. The similarity and affine transformations were described, both the direct (on the minimum number of tie points) and the overdetermined (on more tie points than required) versions. The overdetermined similarity transformation was explained in section 1.1a (pages 106 and 107) with the remark that it is advisable to tie to only two points initially, to evaluate whether any significant deformations exist between the two systems. This remark clearly foreshadowed adjustment and testing in steps, as executed by applying the later methodology of *S-systems*.

The overdetermined non-linear similarity transformation was described in Section I.2 (pages 108 and 109) by means of complex numbers, as Tienstra did in the HTW '38. The example described the tying of two sets of points by means of a second-degree polynomial. The RD coordinates of the newly surveyed points were computed between the tie points using so-called Lagrange interpolation. When simplifying interpolation to a polynomial of degree one (that is, a regular similarity transformation), the formulas of Baarda's later *S-transformation* can be easily derived from the interpolation formulas. As will be shown later he indeed developed his ideas while the concept of *S-transformations* was germinating in his head. He had been working with this idea in mind from the 1940s. It is not entirely clear why he omitted to mention this explicitly in his publications.

Chapter X discussed the testing of measurement and calculation results. For the explanation of symbols used in this paragraph, see Appendix I. For the condition equations $t = u \cdot x + u_0$, the expectation values of the misclosures t are zero: $E\{t\} = 0$. Therefore, it is to be expected that any measurement errors or disturbed tie points will result in values for t that differ significantly from zero. In chapter X, a testing method was introduced that was aligned rigorously with recent developments in mathematical statistics, as Baarda had become acquainted with through Van Dantzig. For his work he made extensive use of earlier editions of (Cramer 1957). The approach in that book is aligned with the comments that had also been made in chapter 9 of (Tienstra 1956).

For the corrected observations $X = x + v$ the following equation held: $u \cdot X = 0$, which confirmed that they satisfied the requirements. For the least squares corrections v, the following equation was valid: $v^T \cdot g_{xx}^{-1} \cdot v = t^T \cdot g_{tt}^{-1} \cdot t$ and variable σ_t^2 was computed as: $t^T \cdot g_{tt}^{-1} \cdot t/b = \sigma_t^2$.

Long-term experience led to the choice of a value for the variance factor σ^2, such that the matrix $\sigma^2 \cdot g_{xx} = \sigma_{xx}$ correctly represents the precision of the observation variates x. The existence of b condition equations led to the variable σ_t^2/σ^2 having a Fisher distribution with degrees of freedom b and ∞ (Cramer 1957 and Hogg and Craig 1970). Hence, $\sigma_t^2/\sigma_0^2 = \underline{F}_{b,\infty}$. This led to the introduction of a rigorous statistical testing method. Additionally, the concept of *Power of the Test* was explained. That indicated the magnitude of a bias in the observations that might be detected with a probability of β, which was commonly fixed at 80%. With these concepts Baarda would later develop the so-called *B-method* of testing and introduce the concept of *Marginally Detectable Bias* (*boundary value*).

4.4 The birth of Π-variates[2]

4.4.1 The introduction of the variable $\Delta\Lambda_{ik}$, calculated from coordinates

Baarda had been occupied for some time by the subject matter addressed in chapter X of the HTW '56. In 1957 he published an article in the journal *Kadaster en Landmeetkunde*, in which he elaborated on this subject (Baarda, 1957). This elaboration was formulated entirely in terms of complex numbers, as Tienstra had done before him in (Tienstra, 1933) and the HTW '38, and Baarda had done himself in the HTW '56. He considered the transformation of coordinates *(x, y)* in a given coordinate system to coordinates *(x', y')* in another system with only small differences in the values of the coordinates, such that *(x', y') = (x + Δx, y + Δy).* See *Figure 8*.

$$\left.\begin{aligned} x' &= x + \Delta x; \quad \Phi' = \Phi + \Delta\Phi; \\ y' &= y + \Delta y; \quad L' = L + \Delta L. \end{aligned}\right\} \quad (3)$$

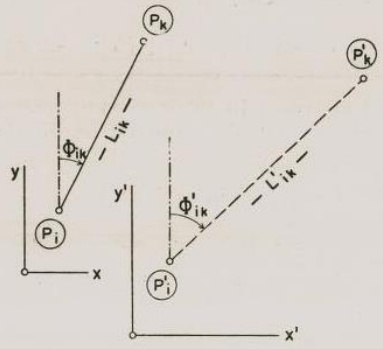


FIG 1

Hoeken A verkrijgen we uit het verschil van twee argumenten, lengteverhoudingen V uit het quotient van twee afstanden.
Ter bereiking van meerdere symmetrie zullen we de (natuurlijke) logaritme ln V van V verder beschouwen. Dit betekent uitbreiding en wijziging van (3) tot:

$$\left.\begin{aligned} x' &= x + \Delta x; \quad \Phi' = \Phi + \Delta\Phi; \quad A' = A + \Delta A; \\ y' &= y + \Delta y; \ln L' = \ln L + \Delta \ln L; \ln V' = \ln V + \Delta \ln V. \end{aligned}\right\} \quad (4)$$

Figure 8: Extract from (Baarda, 1957)

Baarda's formulation, shown in Dutch in *Figure 8*, is noteworthy:

[2] Baarda used the term "Π-quantities".

We obtain the angle A by subtracting two azimuths (Φ) and the length ratio V by dividing two lengths. To achieve additional symmetry, we will henceforth consider the (natural) logarithm ln V of V. This implies extension and modification of (3) to:
[followed by two lines of formulas]

A bit further down (not shown in *Figure 8*) the usage of complex numbers was introduced: $z = y + i \cdot x$ and $z' = y' + i \cdot x'$ and, based on the paragraph above: $z' = z + \Delta z = (y + \Delta y) + i \cdot (x + \Delta x)$.

The coordinate differences of two points *i* and *k* could then be expressed as:
$z_{ik} = z_k - z_i$ with azimuth Φ_{ik} and length L_{ik} and
$z'_{ik} = z'_k - z'_i$ with azimuth Φ'_{ik} and length L'_{ik}.

The variable $\Delta\Lambda_{ik}$ was then defined as: $\Delta\Lambda_{ik} = ln\ (z'_{ik} / z_{ik}) = ln\ (z'_{ik}) - ln\ (z_{ik})$

Considering that $\Delta\Phi_{ik} = \Phi'_{ik} - \Phi_{ik}$ and $\Delta ln\ L_{ik} = L'_{ik} - L_{ik}$, the following relationship held: $\Delta\Lambda_{ik} = \Delta ln\ L_{ik} + i \cdot \Delta\Phi_{ik}$

Later, these variables were extended to cover the relationships between three points *j*, *i* and *k*, with the length ratio $V_{jik} = z_{ik} / z_{ij}$ and the angle $A_{jik} = \Phi_{ik} - \Phi_{ij}$. This led to the following relationship.

$ln\ \Delta V_{jik} + i \cdot \Delta A_{jik} = \Delta\Lambda_{ik} - \Delta\Lambda_{ij}$

In this article Baarda described the tie between two coordinate systems with small differences in coordinate values of the same point. He did not yet arrive at the following interpretation. $\Lambda_{ik} = ln\ z_{ik}$ and from there $ln\ V_{jik} + i \cdot A_{jik} = \Lambda_{ik} - \Lambda_{ij}$.

Tienstra had adopted this approach earlier, be it with different symbols, in (Tienstra 1933). Baarda referred to that in the introductory chapter of (Baarda 1973) and in (Baarda 2002) and he also mentioned an article about traverse calculations in which Tienstra used complex numbers. In (Baarda 1977) he stated that, around 1958, he had completed formulating the theory of the description of plane point positioning entirely in terms of complex-number algebra. Yet it was not until the second half of the 1960s that he published on this. His lecture notes about plane point positioning were published from 1966 and his publications with the Netherlands Committee for Geodesy about this subject also appeared around 1966. See below.

4.4.2 Arrival of the Π-variates (Baarda 1962)

Whilst, in Baarda's opinion, the HTW '56 was a step forward compared to the HTW '38, the following questions remained unanswered.

a) How should the quality (precision) of the coordinates of the tie points in densification surveys be described in a satisfactory manner?
b) What requirements should be formulated for the precision of newly surveyed stations?
c) How should one test for measurement errors in densification networks?
d) How should tie points be tested for possible disturbances?
e) How can these two sources of error be separated?

In the meantime, Baarda had joined Special Study Group 1:14 (SSG 1:14) of the International Association of Geodesy (IAG). In this group he met a number of colleagues who were studying the same subjects, but their approaches were all different. He tabled Van Dantzig's ideas there and worked these out in detail in a number of reports that were later published by Netherlands Committee for Geodesy. These may be found now at www.ncgeo.nl under menu item *Publications*.

Van Dantzig had pointed out that the specification of a functional model is of vital importance for the processing of observation data. That implied that this model ought to be specified in terms of estimable quantities that are generated by the measurement process. In other words, the Central Limit Theorem had to be applicable. Therefore, the model should not be based on non-repeatable

observation variates, nor should it dependent on incidental choices. Only when these conditions are met will it be possible to investigate, on the basis of the processed measurement results, whether:

a) the measurements have been taken correctly;
b) the functional model has been well chosen (that is, it aligns with the measurements).

The position of stations

These ideas are worked out in a report for IAG-SSG 1:14 (Baarda 1962) for the positioning of stations in a network. The argument used is that a coordinate system must be chosen whenever coordinates are computed. This concerns the origin (two coordinates), the orientation of the Y-axis (and, consequently, the X-axis) and a scale parameter to relate the unit of measure of the measured distances to the metric of the coordinate system. That amounts to four parameters that must be chosen. One might alternatively choose the coordinates of any two stations (also four parameters). Those points then serve as the computational base for the calculation of the coordinates of the remaining stations. Baarda therefore concluded that coordinates do not meet the requirement of being estimable quantities. In the report for SSG 1:14 he elaborated on this; see Appendix 5, where we find that:

$\Lambda_{ik} = ln\ z_{ik}$

From that, it follows that:

$\Pi_{jik} = ln\ (z_{ik} / z_{ij}) = ln\ z_{ik} - ln\ z_{ij} = \Lambda_{ik} - \Lambda_{ij} = ln\ V_{jik} + i\ A_{jik}$

The definition of this Π-variate is identical to the expression used in (Tienstra, 1933); see Section 3.6 above. When two stations are chosen as the computational base for calculating the coordinates of the remaining stations in the network, the precision σ_{XY} of the new stations turns out to depend only on the Π-variates that were used in their calculation. Hence, the precision of the coordinates is a direct function of the precision of these Π-variates. When a different computational base is chosen, a different precision σ_{XY} will result for the coordinates of the remaining stations. This demonstrates again that Baarda was already thinking in terms of the concept of *S-systems*.

An analysis of the measurement process

The idea of the Π-variates was expanded, and its results are described in chapter 4 of (Baarda, 1967). An analysis of the measurement process in land surveying demonstrated the following points.

- Direction measurement by means of a theodolite yielded quantities that did not meet the requirements of the Central Limit Theorem, because each time a theodolite was used at a new station, or each time its horizontal circle was rotated, the reference (zero) direction for direction measurement changed. However, measuring the angle α_{jik} in station *i,* subtended to two other points *j* and *k,* would eliminate this problem, because the same reference direction would be used for both directions r_{ij} and r_{ik}, because $\alpha_{jik} = r_{ik} - r_{ij}$.
- Distance measurement was conducted with equipment that had its own unit of measure of length, which might vary over time. Hence, the length of, for example, the side *ij* of a triangle, equals $l_{ij} = s \cdot d_{ij}$, where *s* is an unknown scale factor for length and d_{ij} is the distance, measured by (or with) the instrument. Because *s* might change over time, distance d_{ij} did not satisfy the requirements of the Central Limit Theorem either. However, when an additional distance d_{ik} between stations *i* and *k* was measured, the following relationship held: $v_{jik} = l_{ik} / l_{ij} = d_{ik} / d_{ij}$.

By adopting angles as estimable quantities, the relevance of the instrument's reference direction, hence, its relationship to north, is eliminated from the measurement process. Because length ratios (distance ratios) are also estimable quantities, the relevance of the recording of absolute

time is also eliminated. For electronic distance measurement, when measuring multiple distances from one station, local time measurement to determine the travel time of the electromagnetic signal suffices. For length ratios only the ratios of the travel times matter.

The use of natural logarithms of length ratios, $ln\ v_{jik}$, in the measurement process yields Π-variates, which can now be formulated in terms of measurement results: $\Pi_{jik} = ln\ v_{jik} + i \cdot \alpha_{jik}$. That demonstrates that the analysis of coordinate systems and measurement results can be based on the same Π-variates. It also follows from this analysis that that the tying of two coordinate systems, but also the tying of new survey measurements to an existing system, must be performed by means of a similarity (or: conformal) transformation. Such a transformation does not distort the Π-variates. Essentially, this agrees well with the comparison of tying methods in (Baarda, 1957), in which a preference for this method was also expressed.

Remark 1: In the above paragraphs α_{jik} and v_{jik} were introduced as estimable quantities but do require a footnote. The angle α_{jik} is computed as the difference between two readings on the horizontal circle of a theodolite, whilst its horizontal circle is considered to be parallel to the horizontal plane, hence at right angles with the direction of the local vertical. Whether that is true depends on how well the vertical axis of the theodolite is aligned with the local gravity direction (see Chapter 2). It is commonly assumed that the deviations in the alignment of a theodolite are so small, that α_{jik} can indeed be considered to represent a horizontal angle. The ratio v_{jik} is computed from two distances, measured from the set-up station *i* to the target points *j* and *k*. In general, these points will have different heights, but the height difference will be small in The Netherlands. The zenith angles from the set-up station to the target points can be determined by taking readings of the vertical circle of the theodolite, which allows slope distances to be projected to the horizontal plane. Because of the small height differences the effect of this computational step on the accuracy of the projected distances will be small, so that one may assume that this accuracy is the same as that of the slope distances. With these two assumptions the quantities α_{jik} and v_{jik} can be considered to be estimable. This leads to the concept of *plane survey networks*.

Remark 2: In aligning the vertical axis of a theodolite with the local vertical, the additional assumption is made that the gravity directions of several set-up stations are parallel. This assumption is valid if the survey network covers an area that is sufficiently small. For larger networks this is not the case. Those should be treated as *spatial survey networks*. See Section 4.9.

4.5 Traverse circuits

In the period around 1960, when Baarda was formulating his concept of Π-variates, the first electronic distance measuring instruments became available commercially. The first versions were still somewhat unstable, as a result of which the length unit of the instruments would vary over time. Additionally, corrections for temperature and pressure had to be applied and these corrections had to be updated several times during the day. These issues led to the idea of deriving distance ratios from the recorded distance values at each station. That was analogous to the angles computed from the recorded directions and aligned neatly with the parallelism of angles and length ratios in Π-variates. However, over the years the instruments became increasingly stable and accurate. Soon, distances of several kilometres could be measured with good accuracy. This resulted in more freedom for the land surveyor in choosing survey configurations in the field.

Until then high-accuracy densification networks depended on surveying triangulation networks and/or triangulation chains. That required a theodolite to be set up in at least two of the three stations of each triangle, preferably all three, from which angular measurements would be carried out. This imposed severe constraints on reconnaissance and led to an intensive measurement process. The new instruments allowed the configuration of each triangle to be determined by

surveying the complete Π-variate in only one of the three stations, hence one angle and one length ratio.[3]

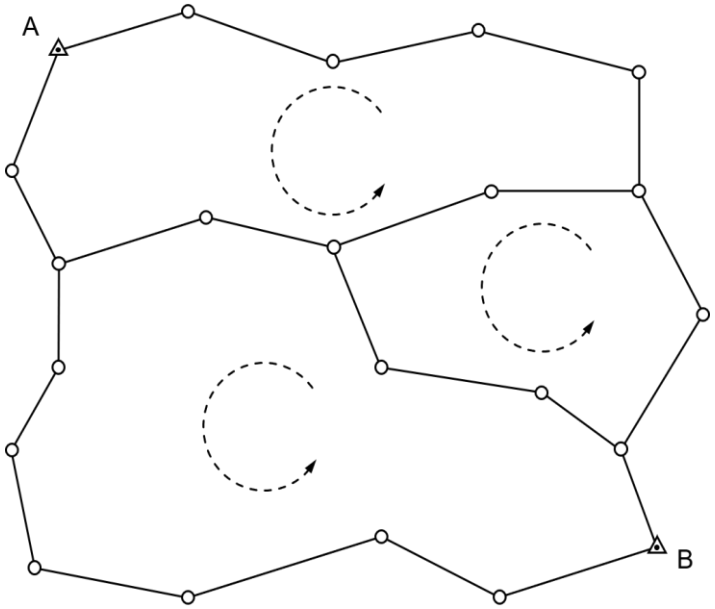


Figure 9: Three traverse circuits and their ties to stations A and B (four known coordinates)

Around 1961–62 this already led to the insight that it was better to build up survey networks from closed polygons, or traverse circuits, as shown in *Figure 9*. Personal reminiscences in (Kruif 2019) indicate that first experiences with this survey approach were gained by students during their field work in the summers of 1961 and 1962. This proved that the quality of these traverse circuits, surveyed with the then modern instruments, was the same or even better than that of the triangulation networks of the past. Moreover, not only were fewer stations required, because it was no longer necessary to measure complete triangles, but it was also possible to have a less dense distribution of stations, so that fewer open sightlines were required. That resulted in faster and cheaper surveys, which meant a considerable gain for practical surveying.

This method was elaborated on in a series of voluminous lecture notes (Baarda 1969). These discussed a number of standard designs, for which condition equations were formulated to facilitate adjustment according to the *First Standard Problem*. These were described in terms of complex numbers. Baarda preferred this formulation for analytical reasons. It provided good insight in the possibilities of finding errors in the measurements or in the coordinates of the tie points. Quee would revert to this later in his research into the structure of three-dimensional networks. He supplied a number of reasons why survey analysis based on the First Standard Problem leads to a better understanding of the functional model, because of the explicit elimination of unknown parameters. These parameters might remain hidden when using the formulation of the *Second Standard Problem* (Quee 1983).

However, from a practical perspective, it was much more convenient to use the *Second Standard Problem* (Appendix 2) for the processing of the observations, because the observation equations have a relatively simple structure in which each observation is expressed as a function of the

[3] A 'distance' is a measured quantity; a 'length' its mathematical abstraction. In practical terms the length ratio is identical to the distance ratio, because both distances share the same scale factor, which relates the distance to the corresponding length. See page 27, *An analysis of the measurement process.*

unknown coordinates. Consequently, the software did not need to be adapted each time to the surveyed network structure, which did have to be done in the case of the formulation of condition equations. Use of the *First Standard Problem* additionally required linear independence of the condition equations.

Soon distance measurement equipment became so good, that practical experience indicated that the scale factor of an instrument tended to remain constant during the surveying of the entire network, so that only a single scale factor for each instrument used had to be established. The implication of this was that the length of each traverse leg did not have to be measured from both stations. The length of each traverse leg only needed to be measured once, which speeded up the surveying process. The consequence was that not all Π-variates in the closed traverse networks were measured. The formulation of condition equations became less systematic as a result, while the formulation of the observation equations for the Second Standard Problem could be adapted in a straightforward manner.

Remark 1. The approach for adjusting measurement results, described above, demonstrated that the formulation of the *First* and *Second Standard Problems* sufficed in land survey practice. As a result, the remaining three standard problems disappeared from the lecture notes. According to (Alberda 2005), it was around 1967 that Baarda introduced the term *Adjustment Theory* instead of the term *Observations Calculus,* used by Tienstra.

Remark 2. Baarda had taken over the use of Ricci notation from Tienstra. In the meantime, he had become acquainted with matrix calculus (Hoek 1982) and adapted his notation accordingly. That resulted in a hybrid notation form, a combination of Ricci and matrix notation (indices and parentheses), which made his publications rather inaccessible for the non-expert reader. This turned out to be a considerable barrier to the uptake of his ideas. See also the remark on this in (Alberda 2005).

4.6 The B-method of testing

Chapter X of the HTW '56 gave the first impetus to the development of a method to check the results of a calculation for the possible occurrence of errors. This was based on the quadratic sum of the misclosures of the conditions:

$t^T \cdot g_{tt}^{-1} \cdot t/b = \sigma_t^2$

This quantity yields a Fisher-distributed variable $\sigma_t^2/\sigma_0^2 \sim \underline{F}_{b,\infty}$ in which σ_0^2 is an assumed variance factor, based on experience. The assumption is that: $E\{\sigma_t^2\} = \sigma_0^2$, or, formulated differently: $E\{\sigma_t^2 / \sigma_0^2\} = 1$. When true, σ_t^2 is a good estimator of the variance factor σ_0^2. All information required for the testing of adjustment results is captured in the misclosures t of the condition equations.

Earlier in Chapter X of the HTW '56 it was shown that $v^T \cdot g_{xx}^{-1} \cdot v = t^T \cdot g_{tt}^{-1} \cdot t$. This means that the corrections (or: residuals) v contain the same information as the misclosures t.[4] Therefore, when the adjustment was done according to the *Second Standard Problem*, testing can also be performed on the basis of the least squares corrections (or residuals) *v.* That also holds for *Standard Problems III, IV* and *V*.

[4] Strictly speaking the residuals are *minus* the corrections. So, when the corrections are designated by *v*, the residuals would be -*v*. In Dutch geodetic literature the preference is to speak of *corrections*; in English geodetic literature *residuals* appears to be preferred.

These findings formed the starting point for the development of a method for finding errors in the original observation data. This method could be formulated entirely in terms of mathematical statistics, which allowed the ad hoc methods, common in land surveying, to be abandoned. As it turned out, this new approach also made it possible to estimate the magnitude of errors that could be detected with it. This gave an idea of the *reliability* of the measurements.

4.6.1 Introduction of the concept of "Boundary Value" (Baarda and Alberda 1962)

The first elaboration on these ideas was published in (Baarda and Alberda 1962). This article was largely a brief recapitulation of (Tienstra 1956). It began with an extensive introduction, which explained how, in accordance with Van Dantzig's line of thought, mathematical-statistical models were linked to the measurement process in land surveying. It asserted that the normal distribution (the 'Gauss bell curve') provides a good description of the stochastic nature of the observations. Then it was demonstrated that the results, the least squares-corrected observations X, are so-called *Best Linear Unbiased Estimators (BLUEs)* of the expectations $E\{x\}$, that were sought to be determined. It was also reiterated that an adjustment might alternatively be executed in multiple steps and that this would yield the same results as adjustment in a single step.

The introduction of ∇-variates

Next, in chapter 5, hypothesis testing was discussed. The *null hypothesis* H_0 is the assumption that the observations and the known coordinates are unbiased, hence $E\{x\} = \mu_x$. Therefore $E\{t\} = 0$, where t are the misclosures of the condition equations; hence $E\{t\} = 0$. An *alternative hypothesis* H_a assumes that one or more observations are biased. Under an alternative hypothesis the following holds:

H_{a_i}: $E\{x_i\} = \mu_{xi} + \nabla x_i$ and $E\{t\} = 0 + \nabla_i t = \nabla_i t$

This group of alternative hypotheses, in which only the i-th observation is assumed to contain a bias, is, as we shall see below, the basis for the testing of individual observations and the calculation of the effects of undiscovered biases on the computed coordinates. Baarda termed this group the *Conventional Alternative Hypotheses*.

The power of the test and the concept of Boundary Values

In the presence of b redundant observations, b mutually independent condition equations can be formulated. The number b is also referred to as the number of degrees of freedom. The effect of a bias ∇x on the value of the above-mentioned estimator of the variance of the misclosure is:

$E\{\sigma_t^2/\sigma_0^2\} = 1 + \lambda/b$, in which $\lambda \geq 0$.

In the HTW '56 it had already been demonstrated that, under H_0, the variable σ_t^2/σ_0^2 is Fisher-distributed; hence $\sigma_t^2/\sigma_0^2 \sim F_{b,\infty}$. This creates the option of testing whether σ_t^2/σ_0^2 does not deviate from unity by too much, or, in other words, whether σ_t^2 does not deviate from σ_0^2 by too much. That might be an indication of the presence of biases in one or more observations. Testing proceeds as follows.

1. Choose a value for α and compute the value $F_{1-\alpha;b,\infty} = C$. Parameter α is called the *significance level of the test* and indicates the probability that the test will incorrectly lead to rejection of the null hypothesis.
2. When there are no biases present, that is, under H_0: $P(\underline{\sigma}_t^2/\sigma_0^2 > C \mid H_0) < \alpha$. That is, the probability that $\sigma_t^2/\sigma_0^2 > C$ is smaller than the chosen value of α.
3. In practice common values for the significance level are $\alpha = 0.05$ and $\alpha = 0.001$.

When H_0 is not valid, but instead some alternative hypothesis H_a is, a bias in one of the observations is assumed. In that case $E\{\sigma_t^2/\sigma^2\} = 1 + \lambda/b$. This results in a shift of the Fisher distribution. The question is how large this shift needs to be for the bias to be noticed with a

probability greater than β, that is, how large λ needs to be such that $P(\sigma_t^2/\sigma_0^2 > C \mid H_{a_i}) > \beta$. Parameter β is known as the *power of the test*. In practice a common value for β is 0.80. In other words, there is a probability of 80% or greater that a bias is detected. When that bias generates the value λ_β, the combination (λ_β , β) defines the reliability of the test.

The above description concerns subject matter that can be found in textbooks on mathematical statistics. Baarda referred to an earlier version of (Cramer 1957). In (Baarda and Alberda1962) this information was interpreted in the context of the processing of geodetic measurements. Furthermore, it was supplemented at two important points:

1. For every alternative hypothesis H_{a_i}, that is, for every observation x_i for which a test is being executed, one can calculate the value $\nabla_\beta x_i$ that leads to the value of λ_β , such that $P(\sigma_t^2/\sigma_0^2 > C \mid H_{a_i}) = \beta$. For this value $\nabla_\beta x_i$ the concept of *boundary value*[5] was introduced.

2. Additionally, it was explained how this test works for an adjustment in steps. Three steps were proposed:
   a. calculation of the arithmetic mean of repeated measurements (angles and distance ratios), per station: the station adjustment.
   b. The adjustment of the newly surveyed network, independent of any tie points. This enables new measurements to be tested on their own for possible gross measurements errors (biases).
   c. The tying of the new network to known control points. This allows the coordinates of these points to be tested for potential disturbances.

   Evidently the number of steps can be increased for each new expansion of the new survey network, that is, each densification phase of a given section of the control network can be individually tied to the expansions that have been surveyed before.

Notably the separation of the steps b. and c. is an important step towards the concept of *S-systems*, which was not long in coming.

<u>The reliability of a network (Baarda, 1967)</u>

The above-mentioned ideas were further elaborated on in (Baarda 1967) and (Baarda 1968). Firstly, it was explained in (Baarda 1967), which was originally a paper presented at a conference at Brussels in 1966, that adjustment according to the *First* and *Second Standard Problems* yields identical results. That meant that, whilst Baarda preferred to use the formalism of the *First Standard Problem* for his analyses, its results are also valid for the outcome of the *Second Standard Problem*. He then explained how the Fisher test works for an adjustment in steps and what the implications are for the choice of the critical region of the test (that is, for the choice of α). He proceeded to show what the effect of a measurement error is on the results of the adjustment and the Fisher test if that error is not detected. Additionally, he showed what its impact on quantities correlated with the measurements would be. He introduced the latter as X^R-variates (derived quantities). Essentially, this concerns the following two types of quantities.

1. The coordinates of control points that are not tie points but are correlated with the coordinates of the tie points.

2. The survey results that were adjusted in an earlier phase and are correlated with the input data of a later survey phase.

[5] The concept of *boundary value* is referred to in some geodetic literature as *Marginally Detectable Bias (MDB)*.

Several types of ∇-*variates* were introduced, but the term *boundary value* is not used.

Furthermore, he discussed the situation in which the original covariance matrix is unavailable, and a substitute matrix is used instead. This subject will be encountered later.

In addition, an analysis followed of the requirements of this approach for the formulation of a mathematical functional model for observation variates. Based on that, Baarda concluded that the geodetic survey process yields distance ratios and angles as estimable quantities. These should therefore be central in the formulation of condition equations and observation equations for adjustment according to the First and Second Standard Problem respectively. We were also able to see this in the last paragraph of Section 4.3 above.

4.6.2 "Data Snooping" and the B-method of testing (Baarda 1968)

In 1965 Baarda became President of IAG's *Special Study Group 1.14*. In that quality he prepared an IAG meeting in 1967 with a very extensive report that was eventually published as (Baarda 1968). Although he presented this report as chairman of the study group, its contents appear to be mainly his own work. Nowhere it appears that others contributed, although reference is made to the work of others. This publication begins with a geometric interpretation of adjustment according the *First Standard Problem*; see Appendices 1 and 3. This is therefore an adjustment with condition equations.

$t = U \cdot x$, in which $E\{t\} = 0$.

In this equation x is a vector with m elements in m-dimensional space. Let this be named the *x-space*; See Figure 10. Each of the m observation variates in a geodetic network therefore specifies a dimension of this space and has its own axis in the m-dimensional axis frame. Because there are b conditions ($b << m$), vector t lies in a b-dimensional subspace. Let this be named the *t-space*. The least squares corrections v calculated for the observations x then span a vector with m elements, such that $v = B \cdot t$ (see Appendix 1). This is in agreement with the assertion made earlier that t and v contain the same information.

Baarda now introduces the group of *conventional alternative hypotheses,* which means that an error is assumed in only one of the observation variates x_i successively. This aligns with the findings in (Baarda and Alberda 1962) mentioned earlier. This means that, in the x-space, an incorrect value is assumed along the axis corresponding with x_i . *Figure 10* shows the effect of this assumed error on the misclosure vector t. *Figure 10* shows a two-dimensional t-space, in which the projection of the x_i-axis on this space has been drawn. The direction of that projected axis is determined by the i-th column of matrix U.

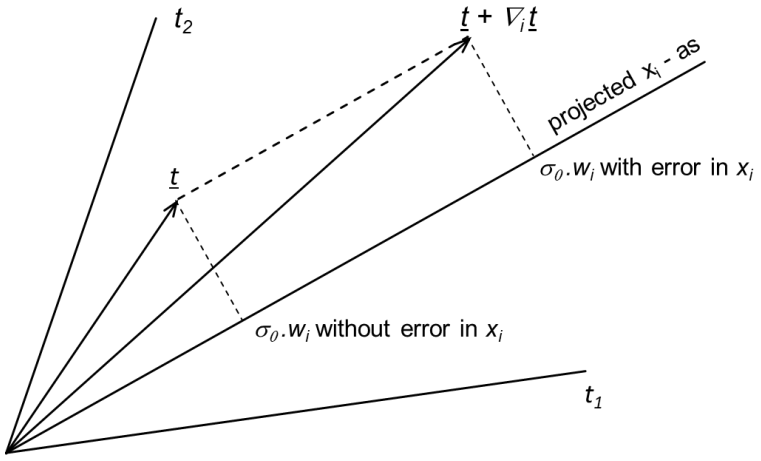


Figure 10: The geometric representation of the specifying test variate w_i

An error in observation x_i yields the misclosure vector $t + \nabla_i t$. The shift $\nabla_i t$ is parallel to the projected x_i-axis. When t is projected orthogonally on the projected x_i-axis, variate $\sigma_0.\underline{w}_i$ is produced (σ_0 is used as a scale factor here). This illustrates that the effect of the error in observation x_i corresponds entirely with the difference between the values of $\sigma_0.w_i$ *with* the error and *without* the error. This *w*-variate is the starting point of the testing of each individual observation. That procedure is called *Data Snooping*; see Appendix 6.

The proposed method entails that after the adjustment the first test to be executed is:

$\sigma_t^2/\sigma_0^2 < F_{1-\alpha;\, b,\, \infty}$

in which b is the number of condition equations. A value for α needs to chosen that is commensurate with b. If this test leads to rejection, which is usually the case, *Data Snooping* is required. Baarda named this the *B-method of testing*. The usual question at the time was where this name came from. (Alberda, 2005) answered this question in its discussion of (Baarda, 1968):

> *… As usual Baarda had studied the international literature extensively. He provides some quotations from a book by the well-known American statistician R.L. Ackoff, among other things about multi-dimensional tests according to a method Ackoff named the 'A-method'. With a satisfied smile Baarda said: "In that case we shall call our method of testing the B-method!"*

Remark. For the test $w_i^2 < F_{1-\alpha_0;\, 1,\, \infty}$, described in Appendix 3, it can be demonstrated that it is the *most powerful test* (Cramer 1957) – in (Hogg and Craig 1970) and other textbooks this is called the *best test* – for the testing of H_0 (no errors in the observation x_i) against H_{a_i} (error in observation x_i). That means no more sensitive test can be found. Baarda does not use this term, even though it would be an essential argument in proposing this procedure as a protocol for data processing in survey practice. He must have known that it is an optimal test. He was familiar with an earlier edition of (Cramer 1957) and according to Alberda he had familiarised himself extensively with literature on this subject. I suspect that he became aware of the fact that this is a *best test* through the geometric interpretation of the adjustment problem, as shown in *Figure 10*. That is apparent from the illustrations in his publications and from the way he discussed this subject matter during lectures. During those he explained the algorithms with illustrations as shown in Appendix 6.

4.6.3 Internal and external reliability of networks

In addition to the proposed testing procedure the new concept of *reliability of geodetic networks* was now introduced (Baarda 1968, Chapter 11). With every w_i-test the associated variate ∇x_i was now calculated. Their values collectively describe the *internal reliability* of a network, which is the order of magnitude of the errors that might be detected in the network. Additionally, the effects on the coordinates of all newly surveyed points (∇X, ∇Y) for every ∇x_i are calculated. Those values are collectively referred to as the *external reliability* of the network. This quantifies the potential biases in the calculated coordinates if the tests failed to identify the error(s). This is important when using this data for the expansion of the network in densification surveys. In (Baarda 1968) he describes the ∇-values as the lower bound of the error-detecting capability of the tests, but he uses neither the term *Marginally Detectable Bias* nor the term *boundary value*.

Data Snooping and the associated reliability analyses are elaborated on in (Baarda 1968) for an adjustment in multiple steps. That was definitely important for survey practice, but to understand the concepts an explanation for an adjustment in a single step suffices.

4.6.4 Testing the functional model

Baarda formulated the *Data Snooping* procedure as the test of a specific group of alternative hypotheses, for each of which, a shift in the expectation value of one of the observations was

assumed. However, his starting point was a more generalized formula that also allowed other alternative hypotheses to be tested. In (Baarda 1968) he introduced a C-vector, indicating to what extent the various observations are involved for any alternative hypothesis. See Appendix 6. With Data Snooping this C-vector carries the value 1 (unity) only for the element that corresponds with the observation that is to be tested; all other elements have a value of zero. In that way all observations in turn can be tested.

Other alternative hypotheses, hence other tests against the functional model are represented by different specifications of this C-vector. What Baarda mainly had in mind was the testing for disturbances of the coordinates of tie points. This had to be done after new measurements had been adjusted in the first step, that is, as a free network, hence before tying the free network to existing control points. The tying to existing control was then done in the second adjustment step. Because the measurements had already been tested in the first step, any problems showing up in the second step would be attributable to errors in the coordinates of the tie points. Individual points might possibly be affected, but alternatively the whole group of tie points, or a subset of those, might be distorted. His method offered the possibility of a focussed search for errors through the formulation of relevant alternative hypotheses.

4.7 S-transformations, criterion matrices and substitute matrices

4.7.1 S-systems

Section 4.1 describes how Baarda was commissioned to create a geometric framework in the newly reclaimed Noordoostpolder and how the experience he gained provided the impetus for his later work. In (Baarda 2002) the following passage can be found:

> *During the war years a close pupil of Tienstra tried to put his ideas into practice. However, the surveyed regions turned out to be larger than Tienstra had envisioned, so that the theory had to be refined. Moreover, the pattern of the detected coordinate differences in the tie points of traverses deviated from the expected pattern according to the above objective, which further complicated the checking for measurement errors. Eventually the solution was found by changing the definition of the concept of 'coordinate'. That did not change the numerical values, but the stochastic influence (according to probability theory) of absolute position, length scale and orientation on the geometric framework in a region was eliminated; in short: stochastically, only shape was relevant. It would take years for these so-called 'S-referenced' coordinates to be truly understood. The HTW '56 was still in two minds about that.*

This is a somewhat curious formulation because that pupil was Baarda himself: a strange kind of modesty? Essentially the above quotation shows that the concept of *S-systems* was conceived there and then. In the first chapter of (Baarda 1973) he briefly describes these early roots of his later work.

In 1946 he was appointed lecturer in *Land Surveying, Spirit Levelling and Geodesy* at the faculty of Civil Engineering of TU Delft.

He elaborated on his experiences in the Noordoostpolder in his public lecture (Baarda, 1947). He discussed two main themes: the definition of coordinate systems and the assessment of the quality of calculated coordinates. He explained that land surveyors actually determine length ratios and angles between triplets of stations and that coordinates cannot be calculated until the coordinates of one point, as well as the orientation of one of the sides plus a scale factor for the distances measured, are adopted. However, it is easier to choose the coordinates of two points as the basis for the calculation and to enter them as error-free quantities. He preferred the latter option for practical reasons and named the selected pair of points an *S-system*. Hence, a network is 'suspended' from those two points. He names that the *S-referencing* of a network. In the remainder

of the discourse, he explains that the computed standard error ellipses of the points, the relative error ellipses between pairs of points *and* the S-referencing of the network describe the quality of the coordinates. Thus, he laid the foundation of the theory of *S-transformations and criterion matrices.* In 1966 he described his ideas in an internal publication (Baarda 1966). It was not until 1973 that that the theory was really published (Baarda 1973).

The quotation above mentions that the HTW '56 (chapter II) already contained the first hints of a development in this direction. Baarda discusses in (Baarda 1960) the meaning of the calculated (co)variances of coordinates and shows how the *S-referencing* of coordinates in one *S-system* can be transformed to another *S-system*. He wrote the formulas with complex numbers. These show a strong relationship with the formulas encountered in the earlier-mentioned publications of Tienstra and of Baarda himself. This concerns two internal publications. For that matter, this principle is also described in textbooks such as (Schermerhorn and Van Steenis 1964, chapters III and XIV). These describe how standard error ellipses of newly surveyed points should be calculated relative to a computational base, just like for an intersection. That implies that their value for the representation of point accuracy by means of these ellipses must always be interpreted as being relative to such a base. Furthermore, just before the publication of (Baarda 1973) a paper had been published in the USA by Allan Pope (Pope 1971), which also discusses the same subject matter. It is unclear whether Baarda was aware of that; he does not refer to it.

4.7.2 Distinction between precision and reliability

(Baarda 1968) is the first real publication in which Baarda commented briefly on the fact that error ellipses depend on the computational base (*S-base*) of a network; see *Figure 11*. For that figure a covariance matrix of the coordinates was calculated twice from the adjusted observations. The first calculation was based on the base points 1 and 3; the second on points 69 and 71. It is easy to see that this leads to different point standard error ellipses and relative error ellipses for the remaining points and hence to different covariance matrices for these points. However, he did not yet use the term *S-base* here.

From this point on he divided the concept of *accuracy* into the concepts of *precision* and *reliability*. The precision of a survey network is expressed by the covariance matrix of the coordinates of these points; reliability by the boundary values of the adjusted observations. In an example he demonstrated that the internal reliability of a network, as expressed in the boundary values of the adjusted observations, was not directly related to the precision of the calculated coordinates, visualised by point error ellipses and relative error ellipses. That is because the internal reliability does not depend on the chosen *S-base*. But the external reliability does depend on the *S-base*. This distinction between precision and reliability was new in geodesy at the time.

4.7.3 S-transformations

When wishing to merge two adjacent survey networks into one, it will generally be found that these networks have been computed in two different *S-systems*. One then needs to identify common points and select two of those as the *S-base* of the merged network. That means that the points of each network need to be recalculated from their respective original *S-base* to this new common *S-base*. These ideas were elaborated on in (Baarda 1966) and, later, the concept of *S-transformations* is published (Baarda 1973).

He formulated the *S-transformation* in terms of complex numbers. In the text below, the notation introduced in Section 4.4.1 shall be used. The position of an arbitrary point *i* is designated by the complex number z_i. The concept of *S-systems* was developed mainly to describe the stochastic nature of the calculated coordinates. That is, the accuracy of the coordinates must be expressed relative to an *S-base*, as shown in *Figure 11*. The reasoning is as follows:

It will be assumed that the two adjacent networks are described in the same coordinate system, for example the Dutch RD system. Let one network having been calculated relative to base points p and q (their S-base). The coordinates of point i can then be designated by complex number $z_i^{p,q}$.The coordinates of some point j in the other network, calculated relative to points u and v, the S-base of the second network, can be designated analogously by $z_j^{u,v}$. When both networks have been calculated in the same coordinate system, $z_i^{p,q} = z_i^{u,v}$ for a point i common to both networks. The *numerical values* of the coordinates of common points are the same and do not need to be transformed. However, there might be small differences Δz between these values and the expectation values of the coordinates. It is for that reason that the stochastic characteristics of the coordinates. expressed by their covariance matrices, differ for the two S-systems, as shown in *Figure 11*.
Expressed in symbols: $E\{ z_i^{p,q}\} \neq E\{ z_i^{u,v}\}$.

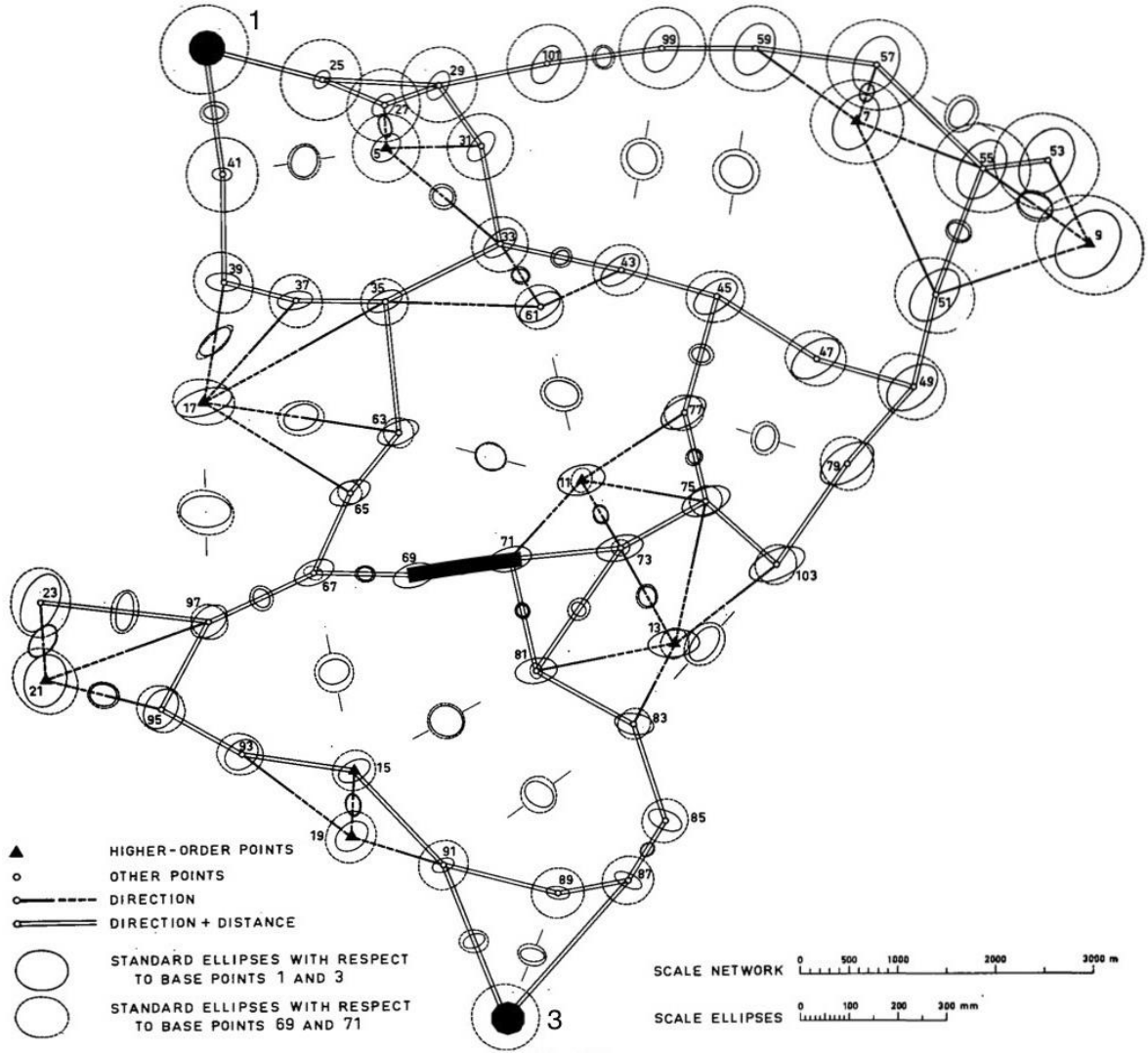


Figure 11: Error ellipse dependency on S-base (Baarda, 1968, p. 90)

When two common points r and s are chosen as the new *S-base* for the merging of the two networks, the old (p, q) system and the old (u, v) system need to be transformed to the new (r, s) system. In the configuration chosen here, this concerns the transformation of the relatively small difference quantities Δz. Baarda derived the formulas for this transformation from first-degree Lagrange polynomials, which he had also used in (Baarda 1957). In short, the transformation from an old system to a new (r, s) system takes the form: $\Delta z_i^{r,s} = S^{r,s} \cdot \Delta z_i^{p,q}$

The indices of transformation *S* refer only to the new *S-system*, because it does not matter what the old system was, provided it satisfies the description above. Baarda therefore often used an index *a* to refer to the old system when it was not specified. Hence, it is not necessary to know exactly what the *S-base* of the old system was, to be able to transform to a new *S-base* that *is* specified. This is relevant for the reworking of the covariance matrix of existing survey networks to a new *S-system*. Let the coordinates be defined in an unknown *S-system*, designated by $\Delta z_i{}^a$. Their covariance matrix will then be $\sigma_{zz}{}^a$. When these coordinates are transformed to some (*r,s*) system, the formula, analogous the one above is:

$$\Delta z_i{}^{r,s} = S^{r.s} \cdot \Delta z_i{}^a$$

the laws of propagation of covariances dictates that:

$$\sigma_{zz}{}^{r,s} = S^{r.s} \cdot \sigma_{zz}{}^a \cdot S^{r.s\,\mathrm{T}}$$

in which matrix $S^{r.s\,\mathrm{T}}$ is the transpose of matrix $S^{r.s}$.

This then yields the covariance matrix of the coordinates in the new (*r, s*) system.

4.7.4 Criterion matrices

The history

To gain insight in the origin of the theory of criterion matrices and the role that S-systems play in it, it might be helpful to summarise some of the findings from the foregoing.

We saw that (Tienstra 1931) and (Tienstra 1933) specified that circular point error ellipses were considered ideal, because these imply no directional preference regarding the precision of surveyed points. In other words, when these points are used to tie to new surveys, it does not matter from which direction they are surveyed. In the same vein, it was also desired that the size of these error ellipses be independent of the location of these points, so that their location is not a factor for precision considerations when they are used in the expansion or densification of survey networks. However, it soon became clear that it was impossible to achieve circular error ellipses in practice.

Reconnaissance requirements were therefore formulated differently in the HTW ’38; see Section 3.3. above. Ideal precision was no longer pursued. Instead, a criterion was formulated for the mean of the squared major axes of the standard error ellipses for moderately large networks. That mean value should not exceed a certain threshold and was specified for three terrain types. Baarda's experiences in his time as a practicing surveyor in the early 1940s led to the following three findings.

1. It turns out to be practically impossible to carry out a survey such that the requirement for the mean square of the major axes of the point standard ellipses is met. It is better to formulate a criterion as a lower limit for the desired precision.
2. In formulating such a criterion, the focus should not be exclusively on the precision of the surveyed points. The relative precision of pairs of points is at least as important. This is relevant because new survey will always be tied to multiple control points. The relative precision of these points is therefore more important than their ‘absolute’ precision, expressed in the point error ellipses.
3. In point 2 the word ‘absolute’ has been placed in quotes because it had become clear to Baarda that the precision of the points in a network is always calculated relative to a computational base for which the coordinates are adopted. This is therefore an *S-base*, of which the coordinates are not stochastic.

The HTW ’38 already took item 1 into account in its formulation of requirements for reconnaissance. In addition, item 2 formed the basis for the formulation of these requirements in the HTW ’56. Item 3 was not yet taken into consideration. If the reconnaissance requirements are based only upon point error ellipses and relative error ellipses, it is unclear how an *S-base* can be factored in. That is probably why the third item played no role at the time. However, later Baarda repeatedly stated that he was unhappy about the formulation of the reconnaissance rules in the HTW ’56. It was a half-hearted approach that demanded improvement. In (Baarda 1973), he published a consistent and more satisfying approach for the first time.

The search for a criterion matrix

The improved insights into the correct application of the least squares method since the 1940s gave a better understanding of how, as well as estimating the unknown parameters, the covariance matrix of these parameters could be calculated. In the application of this method to land surveying, the coordinates of the surveyed points constituted the unknown parameters. This meant that their full covariance matrix could also be calculated. From the 1960s onwards, better computation facilities quickly became available, due to the rapid development of computer technology. That enabled these extensive calculations to be carried out.

This new situation offered the option to implement the concept of *S-systems* rigorously. It was now also possible to calculate the covariance matrix of the new coordinates relative to a specified *S-base* for larger networks, as shown in Figure 11. The question now was how to formulate a criterion for the precision of those coordinates. That criterion should be valid for the entire covariance matrix of the coordinates of all points and not just for the covariances of coordinates of individual points and/or the covariances of coordinate differences of point pairs. The question therefore came down to how to formulate a criterion for this matrix. The search was for the structure of a *criterion matrix*, which will be designated by the letter *H*.

Of course, the point error ellipses and the relative error ellipses still can be derived from the calculated covariance matrix. It was therefore reasonable to require that it should be possible to derive circles for points, and pairs of points, from a criterion matrix *H*, which could then serve as upper bounds for those standard error ellipses. However, that would be an incomplete criterion. The complete criterion ought to apply to the entire covariance matrix. Moreover, in an *S-system* the magnitude and the orientation of the point error ellipses depends on the location of those points relative to the points of the *S-base*, as shown in *Figure 11*. Furthermore, the points of the *S-base* are treated as non-stochastic, so that the circles of those points, derived from the criterion matrix would necessarily have a radius of zero. These two requirements make it clear that *H* needs to be calculated in the same *S-system* as the covariance matrix of the coordinates. The search is therefore for a matrix $H^{r,s}$.

In (Baarda 1973) a solution to this problem was published for the first time. Baarda had developed this in close cooperation with his co-worker ir. Jouke Alberda and had described this earlier in an internal report (Baarda 1966). The chapters of (Baarda 1973) that eventually led to the formulation of the criterion matrix show some fierce juggling of equations, but, in short, the reasoning is approximately as follows.

a. The covariance matrix of the coordinates in an *S-system* is a direct function of the covariance matrix of the shape elements, that is, the Π-variates, of the network. Therefore, it should be possible to derive a criterion for the covariance matrix from a criterion for these shape elements.

b. The old requirement of circular error ellipses for points and point pairs of which the size is independent of their location in the network must be translated to a requirement for the shape elements. The criterion for the precision of these shape elements should be

independent of their location in the network. Hence the criterion should depend only on the shape and the size of the triangle spanned by the three points of the shape element.

c. In a triangle with points 1, 2 and 3, the elements of the criterion matrix, corresponding with the coordinates of the points, are defined as follows (the symbol *h* is used here instead of σ; additionally, $i \neq j$ and *i, j = 1, 2, 3*).

$h_{x_i x_i} = h_{y_i y_i} = d^2$ and $h_{x_i y_i} = h_{y_i x_i} = 0$ and
$h_{x_i x_j} = h_{y_i y_j} = d^2 - d_{ij}^2$ and $h_{x_i y_j} = h_{y_i x_j} = h_{x_j y_i} = h_{y_j x_i} = 0$

It must be pointed out that symbol *d* does not designate distance, as earlier in this report. Here, d^2 designates a constant used in the criterion matrix.

If $d_{ij}^2 = f(l_{ij})$, that is, if d_{ij}^2 is a function of the length between points *i* and *j*, the precision of the shape elements calculated from this triangle is only a function of the length of its sides and a chosen constant d^2. For the variates α_i (angle) and $ln\ v_i$ (natural logarithm of length ratio), both calculated for point *i*, the following relationships hold: $h_{\alpha_i, \alpha_i} = h_{ln\ v_i, ln\ v_i}$ and $h_{\alpha_i, ln\ v_i} = 0$.

On Alberda's advice, Baarda chose the function: $d_{ij}^2 = c.\ l_{ij}$. Other options were available but were constrained by the requirement that the matrix thus generated had to be positive definite.

	Δy_i	Δy_j	Δy_r	Δy_s	Δx_i	Δx_j	Δx_r	Δx_s
Δy_i	d^2	$(d^2 - d_{ij}^2)$	$(d^2 - d_{ir}^2)$	$(d^2 - d_{is}^2)$	0			
Δy_j	$(d^2 - d_{ji}^2)$	d^2	$(d^2 - d_{jr}^2)$	$(d^2 - d_{js}^2)$				
Δy_r	$(d^2 - d_{ri}^2)$	$(d^2 - d_{rj}^2)$	d^2	$(d^2 - d_{rs}^2)$				
Δy_s	$(d^2 - d_{si}^2)$	$(d^2 - d_{sj}^2)$	$(d^2 - d_{sr}^2)$	d^2				
Δx_i	0				Same partial matrix as top left			
Δx_j								
Δx_r								
Δx_s								

Figure 12: The structure of criterion matrix H^a

Based on this line of thought a solution could be found by generating a H^a matrix in an unspecified *S-system*; see *Figure 12*. That matrix will satisfy requirements *b* and *c*. This matrix can be transformed to matrix $H^{r,s}$ on *S-base* (*r, s*) through an *S-transformation*. If the covariance matrix $\sigma^{r,s}$ of the coordinates has also been computed relative to this base, the standard error ellipses for points and relative standard ellipses for pairs of points can be calculated. Those can be compared with the corresponding criterion circles that are computed from $H^{r,s}$. Visual comparison will then reveal whether the standard error ellipses lie within the criterion circles. Therefore Baarda proposed to execute the testing of matrix $\sigma^{r,s}$ against criterion matrix $H^{r,s}$ by solving the *eigenvalue* problem, as formulated in text books on linear algebra and matrix calculus.

$$|\ \sigma^{r,s} - \lambda\ H^{r,s}| = 0$$

Hence, all information contained in both matrices is used instead of only the calculated standard error ellipses. The eigenvalues λ do not depend on the chosen S-base.

To solution must satisfy the requirement that the smallest eigenvalue $\lambda_{min} \geq 0$, which is dictated by the requirement that covariance matrices, hence also the criterion matrix, have to be positive definite (see the textbooks). But it is additionally required that the largest eigenvalue $\lambda_{max} \leq 1$.

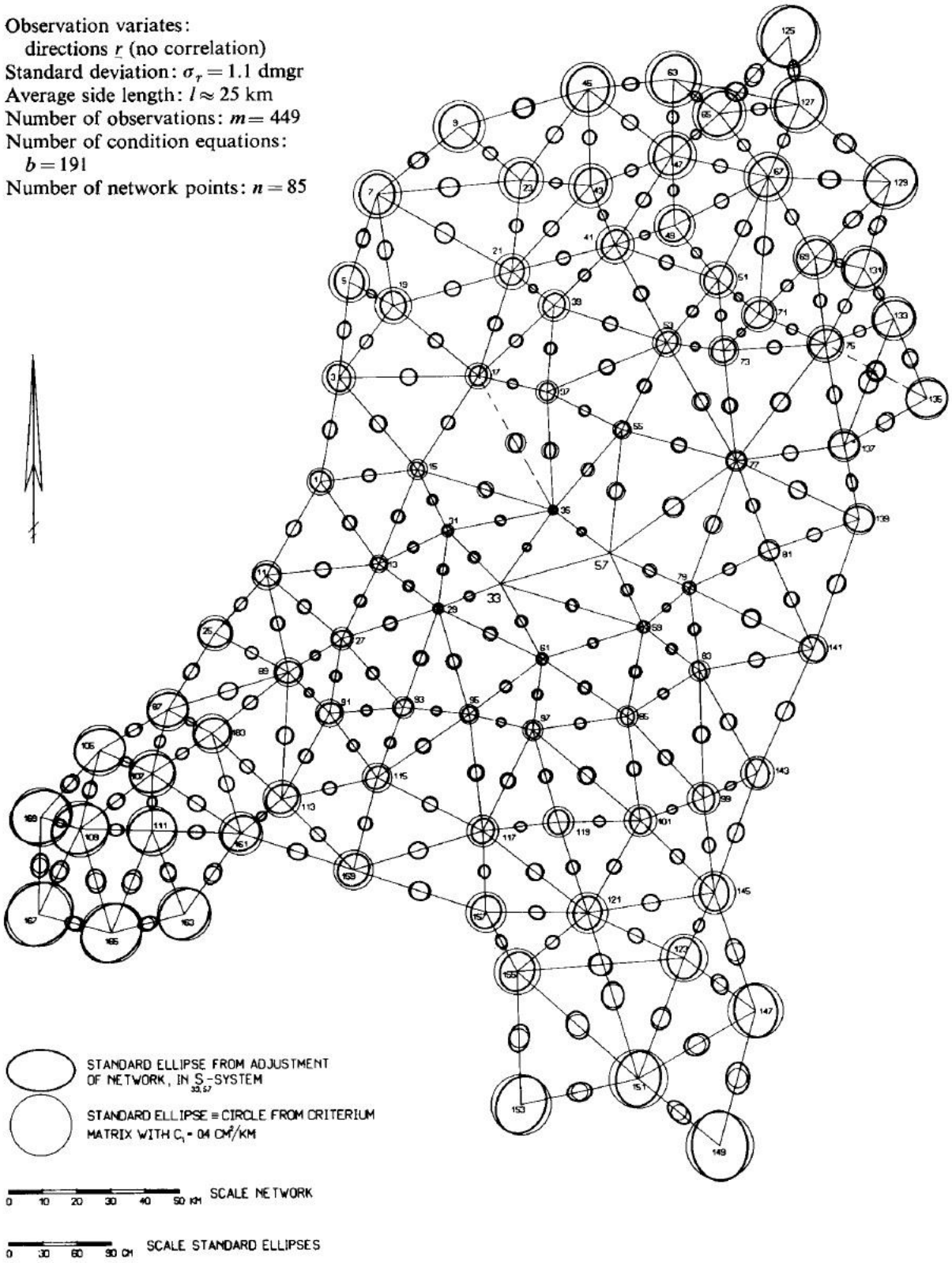


Figure 13: Standard error ellipses of the Dutch RD network, compared with criterion circles. The S-base is (33, 57).

This may be visualised as follows. The geometric equivalent of the covariance matrix for n points is a $2n$-dimensional hyper-ellipsoid and that of the criterion matrix is a $2n$-dimensional hyper-sphere. When the largest eigenvalue $\lambda_{max} < 1$, the hyper-ellipsoid lies entirely within the hyper-sphere and when $\lambda_{max} = 1$, the hyper-ellipsoid is tangent to the hyper-sphere. In both cases the precision of network is within the specified requirements. When some eigenvalues are greater than unity, the hyper-ellipsoid will protrude from the hyper-sphere, indicating weak parts in the network. From the associated eigenvectors it can be deduced which parts of the network are out of spec.

Point standard error ellipses and relative standard error ellipses can be derived from the covariance matrix. Analogously, the criterion matrix yields point criterion circles and relative criterion circles. When the maximum eigenvalue $\lambda_{max} \leq 1$, all ellipses will either lie within the circles or will just be tangent to them. *Figure 13* shows the comparison of the point standard error ellipses and relative standard error ellipses of the Dutch RD network with a criterion matrix. The largest part of the network satisfies the criterion, except in the lower right corner. Here the ellipses just protrude from the criterion circles in several places. However, were an *S-base* 1-5 chosen (Berkheide – Schoorl, along the west coast), poor quality would be concluded for the *entire* network. The results of the visual comparison thus depend on the chosen *S-base*. The specification of the criterion can be tailored to the precision requirements imposed on the network by choosing an appropriate value for parameter c in the above-mentioned function $d_{ij}^2 = c \cdot l_{ij}$ in addition to the value that is chosen for the parameter d^2.

In fact, parameter c determines the part of the criterion that relates to the correlation between points and is determined by the structure of the network. To a large extent parameter d^2 determines the quality of the identification of points. These two parameters replace the parameters in chapter II of the HTW '56.

4.7.5 Substitute matrices

The structure of the criterion matrix described above could also be used in a different way. In the practice of that time, it was customary to omit the coordinates of existing control points from the adjustment when tying new measurements to these points. These points were used to tie to, but their coordinates did not participate in the adjustment. In other words: they received no corrections. They were even considered non-stochastic.

There was a good pragmatic reason not to correct them because the coordinates of these control points were recorded on maps and in printed tables, which made it impractical to correct them each time they were used in a new survey. Even later, when this data was available in digital files, it remained impractical to continually change them. However, that was not a valid reason to consider them non-stochastic. Originally, their coordinates had been calculated from survey measurements and therefore they ought to have a covariance matrix. But this matrix was never stored, because technically that could not easily be done. The history of the origin of those coordinates was not recorded in data files either, so reconstructing the matrix was impossible.

Initially this was not seen as problematic because it was assumed at the time that higher-order densification surveys were so much better than later (and lower-order) network densifications that the control point coordinates resulting from the former could be considered error free. However, due to advances in survey technology this argument was rendered obsolete. All new densification surveys had a quality comparable with the earlier higher-order surveys. Moreover, the approach of adjustment in steps offered the possibility to calculate and test new survey measurements on their own first. After this first phase the new survey network could be tied to the existing control points, which would be tested in the next step. This led to the realisation that the coordinates of existing control points had to be considered stochastic and had to be tested.

The unavailability of their original covariance matrix could now be remedied by introducing a substitute matrix that described the approximate precision of those coordinates. The structure of the criterion matrix could be used for this, with the values of the parameters c and d^2 chosen such, that a reasonable approximation of the precision of the existing control points was achieved. This substitute matrix could then be referenced to an *S-base*, common with the new survey network. The adjustment would be executed in two runs, the first run rigorously, which would include the coordinates of the tie points in the adjustment, so that their coordinates could be tested in accordance with the method described in Section 4.6. After removing any control points rejected in the test, a second adjustment run would be executed, in which the coordinates of the remaining tie points were left uncorrected. But the substitute matrix was used in the calculation of the precision of the new points, by applying the laws of propagation of covariances, be it in a somewhat amended form. The latter was referred to as a *pseudo least squares adjustment*.

4.8 The Geodetic Computing Centre (LGR)

4.8.1 Beginnings

In the 1950s Baarda had the idea of establishing a centre, in which theoretical research could be conducted, linked to the development of computing technology; see (Alberda 2005). Electronic computing technology was on the rise, and it was clear that the ideas being developed then led to the need for large computing capacity. With the support of the Board of Trustees of the TU Delft he established the *Geodetic Computing Centre* (Dutch: *Laboratorium voor Geodetische Rekentechniek*) in 1958. Its first employees were ir. B.G.K Krijger, ir. F. Meerdink and ir. J.E. Alberda.

One of the ambitions of LGR was to participate in large international projects. The first opportunity was the adjustment of the European levelling network in the middle of the 1950s. That consisted of levelling chains covering the whole of Western Europe. Its data was processed by Institut Géographique National (IGN) in Paris, the Technische Hochschule (TH) in Munich and LGR. This was a good opportunity for the Delft group to try out the methodology of adjustment in steps, as well as the ideas regarding the testing of large survey networks. Moreover, LGR was the only group that used electronic computation facilities. These facilities concerned TNO's ZEBRA-computer, which had been designed and built by PTT's laboratory.[6] LGR was the only group that had computed the full covariance matrix of the heights. With this project LGR established its name both at home and abroad.

Around 1960 the decision was made internationally to compute a new pan-European adjustment. The intention was to improve the ties between the national triangulation networks in order to define an improved European ellipsoid. In the meantime, electronic distance measurement equipment had become commercially available, initially the Geodimeter and later the Tellurometer. These allowed tests to be carried out with the configuration of traverse circuits, which would be processed by LGR.

[6] TNO (Technisch Natuurkundig Onderzoek) is an independent research organisation that conducts research on a contract basis for and in cooperation with government agencies, commercial companies, and educational and other institutes. PTT (Post, Telegraaf and Telefoon) was the government-owned company responsible for post, telegraph, and telephone communications, as well as some radio communications services in the Netherlands.

In 1967 a length base was measured by means of invar wires at the Afsluitdijk.[7] This new base was part of a dedicated survey network 'Afsluitdijk', which had to be tied to the RD-network. It was in those days that the connection with the German triangulation network was being recalculated. The computations associated with these activities were so extensive that they had to be executed on computers.

4.8.2 The development of computational facilities at the TU Delft

In the same period the mathematician Professor R. Timman established the Computing Centre of the TU Delft. He became the chairman of a committee that had to consider the purchase of a computer for the university. Baarda became the secretary of this committee. Initially the question of whether the university should purchase an analogue, or a digital computer was debated. Based on the committee's recommendations a digital computer was selected, the Telefunken TR4.

Many debates took place in the committee about the consequences of the purchase of such a computer for the disciplines that would be making use of it. Soon it became clear that these would be radical. Their impact would be greater than the mere fact that computations would be faster. Baarda states the following about this (Baarda 1977).

> *In that period there were consultations with the mathematician R. Timman concerning the establishment of a computer centre for the Delft University of Technology. We both held the view that independent programming of existing methods was the easiest but also the least efficient way to introduce automation. It was better to consider existing methods as variants of a basic pattern which should be as simple as possible and concentrate on the programming of this pattern. In addition, it was required to limit the volume of the output by the choice of suitable parameters. In other words: programming should be preceded by model research.*

Obviously, this dovetailed neatly with his research interests. In addition to the more theoretical influence by Van Dantzig, which pushed him towards fundamental model research, this constituted a second, supplementary impulse in that direction. The research described in the preceding sections originated from a twofold stimulus to the extent that these were required in addition to Baarda's own curiosity. Perhaps it is better to speak of a twofold justification.

4.8.3 Software development

Eventually the TR4 was put to service by the TU Delft in 1963, with an Algol-60 compiler. That enabled the real programming work of LGR to begin. Jan de Kruif was one of the first undergraduates who wrote a computer programme in Algol-60 for LGR as part of his master's thesis. After he graduated in 1964, he became an LGR employee. He began writing software procedures needed for the execution of adjustments, such as multiplication and inversion of matrices. It was important to pursue efficiency in input and output and in the storage of intermediate results, because, at that time, computers still had limited memory capacity and the input and output of data was not very fast. That is why the software was initially only usable for processing networks of a limited size. Furthermore, the software was aimed at working with sparse matrices, matrices of which most elements have a value of zero. Separate procedures were developed for such matrices.

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[7] See https://en.wikipedia.org/wiki/Afsluitdijk.
Invar is nickel-iron alloy with a very low coefficient of thermal expansion. In geodesy it was used for the calibration of length bases of triangulation networks, but it was also used in land surveying in subtense bars (see Section 2.3).

In 1968 development started for the programming of adjustment according to the *First* and *Second Standard Problems*. As was mentioned in remark 1 of Section 4.5, the insight that these two forms sufficed for the processing of land survey data had been gained. The *First Standard Problem* required formulation of condition equations for traverse circuits. The *Second Standard Problem* required formulation of the observation equations. Due to the consistence of the structure of traverse circuits, both the condition equations and the observation equations acquired a fixed form, which lent itself well to being programmed. The measured angles and length ratios (directions and measured distances), as well as the approximate values of the station coordinates involved in the survey, had to be entered into the calculation. Here the advantage of standardisation, as Baarda intended it in the quotation shown above, became apparent. The software that was developed became known as SCAN. Later Jan de Kruif's successors, led by Johan Kok and Henk de Heus, developed a new version called SCAN II.

4.8.4 The computation of land survey networks

In 1968, consultations took place between Baarda and associates with A.L.J. Claassen of the Dutch Cadastre (Kruif 2019). The Cadastre was in the process of preparing many reallotments, for which geodetic frameworks had to be surveyed to create ground control points for (aerial) photogrammetric mapping. The Cadastre was prepared to test the principle of traverse circuits in practice and to have several of their employees undergo additional training. Using their software, LGR would adjust the survey data of those traverse circuits. The methods for the testing of observations and the testing of the precision of the calculated coordinates against a criterion matrix, as developed by Baarda, had to be implemented in the software.

Around 1969 the software was expanded to the application of the *w-test* for the detection of errors in traverse circuits. After an adjustment a test on the entire network was executed first with $\sigma_t^2/\sigma_0^2 < F_{1-\alpha;\, b,\, \infty}$; see Section 4.6. This proved to be unsatisfactory because it (almost) always led to rejection of the adjustment result for the entire network. Experience thus led to the rule that the *w*-variates for all observations always needed to be tested, that is, *Data Snooping* would always have to be used. That allowed gross errors to be located faster, so that remeasuring of the whole network could be avoided. This software addition also allowed the boundary values for all observations (internal reliability) to be computed together with their effects on the boundary values of the calculated coordinates (external reliability).

Around 1970 programming efforts for setting up criterion matrices and substitute matrices and executing *S-transformations* began. Software was also developed for the comparison of the covariance matrix of coordinates with a criterion matrix through calculation of the eigenvalues. Suitable parameter values for the testing of the precision of coordinates were found through test calculations. The first results for simulated survey networks were published in (Baarda 1973).

Suitable parameter values were also sought for substitute matrices for the purpose of tying new survey measurements to existing control in a second adjustment phase. That really enabled adjustments to be carried out in two steps. In the first step, traverse circuit networks were adjusted as free networks and in the second step this adjusted network would be tied to existing control points. The substitute matrix was used in that second step instead of the real, but usually unknown, covariance matrix of the coordinates of existing control points. That enabled the coordinates of existing control to be tested for possible disturbances. After that test, the second adjustment step was usually repeated as a pseudo-adjustment, because correcting the coordinates of existing control points was too cumbersome. The substitute matrix was also used to calculate the final precision of newly surveyed points.

4.8.5 Reconnaissance of land survey networks

With this software it was possible to make an important improvement in the efficiency of survey work. It was now feasible to design a survey network while being in the field. The availability of approximated coordinates enabled the calculation of internal and external reliability, as well as the covariance matrix of the coordinates of the planned network points. Usually, these approximated coordinates could be scaled from a topographical map. Thus, on the basis of such a test computation, the reliability of the network could be assessed, and the covariance matrix compared with a criterion matrix. From then on it was possible to assess the quality of the planned network in advance and, where necessary, to amend its design before actual survey work would begin. This resulted in significant time savings and cost reduction in the execution of land survey work. In (LGR-staff 1982) the LGR staff of that time give a nice overview of the state of development of ideas of the *Delft School* in the early 1980s.

4.9 Expansion to three dimensions

With the developments described above, the methodology for plane point positioning and the assessment of its quality was largely completed. These methods were practicable enough for use in established land survey practice. Parallel to this development, since the early 1960s, the question arose of how to escape from plane surveying and create truly spatial, that is, three-dimensional, networks. Traditionally, a two-pronged approach had been followed with separate horizontal (x, y) and vertical (H or height) positioning. For surveys covering larger areas, the curvature of the earth's surface had to be considered. The common and approximate solution was to project the horizontal positioning on the surface of a sphere or an ellipsoid. From a theoretical perspective this was judged to be inadequate and therefore research was conducted into the development of truly three-dimensional methods.

Baarda's objections to the ellipsoidal approximation arose from his following Van Dantzig's ideas. The starting point was that the measurement process determined what the estimable quantities were, based on which the functional model would be developed. Transformations of this model should leave these quantities unchanged. In other words, the estimable quantities must be invariant under these transformations. Since angles and length ratios are the estimable quantities in geodesy, this concerns similarity transformations. These entail scale changes, rotations and shifts in x- and y-directions. The shape elements on the surface of an ellipsoid are only invariant with rotations about the minor (polar) axis, not about any other axis. Although the distorting effects would only be noticeable in very large networks, Baarda rejected the ellipsoid model on principle.

What he was seeking was an approach for spatial networks along the same lines as that he had used for plane networks. The starting point remained that angles and length ratios are the estimable quantities in the measurement process, but in this case not projected on the horizontal plane. So, this concerned truly spatial quantities. In the plane, an elegant approach was possible based on complex numbers, because complex number algebra allowed angles and length ratios to be described well. In scientific literature, in the early 1960s, he found that *quaternions* provided an extension of complex numbers to four dimensions, which, thanks to a convenient interpretation, could also be used for the description of three-dimensional spatial shape quantities; see Appendix 8. Now, the following three questions had to be answered.

a) How should an *S-base* in three-dimensional space be defined?

b) What is the form of a criterion matrix in three-dimensional space?

c) Wat is the structure of three-dimensional survey networks?

These questions are related to the formulation of the mathematical model. The procedures for *Data Snooping* and the calculation of internal and external reliability followed directly from the adjustment algorithm and thus did not require reformulation for three-dimensional applications.

4.9.1 Spatial S-systems and criterion matrices

Naturally, with the transition from two to three dimensions, Baarda immediately came up with the question of how coordinates ought to be defined and how they were linked to the measurement process. The estimable quantities in three-dimensional applications are also angles and length ratios, but these are now defined in a triplet of points in three-dimensional space and hence not projected on a horizontal plane.

Seven parameters are required to be chosen to define a three-dimensional coordinate system: three coordinates for the origin, two directional components that will fix the direction of one of the three axes in space, a rotation about that axis to fix the spatial direction of the other two axes, and one scale factor. By choosing the coordinates of two network points, let those be P_r and P_s, six parameters are chosen. In effect, this defines the coordinates of the origin and the scale factor (for the traverse side *r, s*). This fixes the entire coordinate system apart from one more rotation. That requires a third point P_t, but its role is different from that of the first two points. It does not participate fully, because point P_t is needed to fix only one parameter and not three, as the other two points do. By choosing this third point, the rotation about traverse side *r, s* is chosen, thus fixing the spatial direction of the (unit) vector orthogonal to the plane spanned by points P_r, P_s and P_t. To visualise this, see the illustration with formulas (4.2.1) and (4.2.2) in Appendix 8 and replace the indices *i, j, k* by *r, s, t* respectively.

This definition of an *S-base* for 3D-coordinates no longer has the simplicity of that for 2D-coordinates. Added to this is the difficulty that quaternion multiplication is not commutative (see Appendix 8). Those facts make the analysis of 3D-networks considerably more complicated than that of plane networks.

Notes show that Baarda probably developed his ideas for 3D *S-systems* as early as the 1960s or early 1970s and also derived the associated *S-transformations*, but he never published them. I first defined an *S-base* myself in my master's thesis (Molenaar 1972) for the analysis of the structure of Doppler satellite measurements. Later I elaborated on this definition with the associated 3D *S-transformation* for my doctoral dissertation (Molenaar 1981). I ran into this issue in the 1970s when I wanted to apply the concept of criterion matrices to the results of adjusted photogrammetric blocks. A 3D criterion matrix, which had to be transformed to an *S-base*, was also developed for this. I discussed this extensively with Baarda during my PhD research. He wanted to adopt the principle of spherical point error ellipses and relative error ellipses. My objection to that was twofold.

a) This would only be valid for a criterion matrix that had not been *S-referenced*. As soon as that is transformed to an *S-base*, the error ellipsoid for base point P_t would be flattened to a circle in the plane spanned by base points P_r, P_s and P_t. Consequently, the remaining error ellipses would have a flattened shape.

b) The experiences in aerial photogrammetry as well as in geodesy led to the insight that the accuracy of height determination is generally worse than that of horizontal point positioning.

Baarda never really admitted I might be right, but he did let me elaborate on the criterion matrix in my own way. I did that by elaborating on the horizontal criterion for spherical coordinates, so that it would 'bend along' with the earth's surface for varying horizontal positions. Likewise, the criterion for height would bend along with the earth's surface. This applied to survey

measurements covering large areas. For smaller-sized networks this matrix defaulted into the structure of the criterion matrix for the plane (Molenaar 1981). Test calculations demonstrated that for area of 500x500 km^2, the differences between the (plane) matrix from (Baarda 1973) and this new matrix were negligible. That was enough reason for me to cease refining the model and stop looking for a matrix that would bend along with the surface of an ellipsoid.

4.9.2 Spatial networks

Besides the problem of the operational definition of coordinates, that is, of 3D *S-systems*, the question of how Baarda's approach to networks in the complex plane could be extended to three dimensions was still unanswered. What was the structure of spatial traverse circuits? Baarda had already explored at an early stage how quaternions could be used as analogues of complex numbers for the calculation of spatial networks. The results of this work are unfortunately only recorded in personal notes. The first study accessible to others was Erik Vermaat's master's thesis (Vermaat 1970). In this thesis he investigated the structure of quaternion rotations. This was important preparatory work for the analysis of the structure of networks.

The first to take up that task was Herman Quee. He developed software for calculating the condition equations of spatial traverse circuits with LGR as part of his master's thesis in 1971. He formulated these equations in terms of quaternion relationships. Their structure was comparable with those that were given in (Baarda 1969) for traverse circuits in the complex plane. The numerical results indicated dependencies between these conditions, but, at the time, that had not yet been demonstrated analytically. In his PhD research he elaborated on this and published that in (Quee 1983). It should be noted that this took place at a time before the advent of GPS and other new measurement techniques. He distinguished two network types.

a) Networks with traverse circuits, covering larger areas and surveyed as higher-order geodetic control networks. The height differences in these networks are relatively small, compared to length of the traverse legs. The survey stations are relatively far apart and the direction of gravity follows the earth's surface. As a result, it can no longer be assumed that the directions of the theodolite's vertical axis in multiple stations are approximately parallel. Astronomical latitude and longitude will then have to be determined to establish the change in the direction of gravity. Additionally astronomical north will have to be measured for the orientation of the horizontal direction measurements. This data is therefore part of the formulated condition equations, which makes the functional model considerably more complex than for plane closed traverses.

b) Local networks as surveyed for civil engineering and cadastral work. In such networks the stations are sufficiently close for the gravity vectors in all points to be considered parallel. That implies that all stations share a common horizontal plane. That eliminates the need for astronomical observations. For such a survey the height differences are generally larger relative to the length of the traverse legs.

This study practically completed the development of a functional model for 3D terrestrial networks based on the use of quaternions. The methodology, which Baarda had developed for applications in the plane, had thus been extended to applications in 3D space. Spatial *S-systems*, spatial *S-transformations*, spatial criterion matrices and the structure of spatial networks of traverse circuits had now been defined. Application in practice, aimed at establishing whether this approach was practicable, was now awaited.

4.9.3 Photogrammetry

In parallel with the above developments Baarda also attempted to find a description of photogrammetry using quaternions. He made a design for that himself and discussed it with

photogrammetry experts such as Schermerhorn, Roelofs and Van der Weele, as well as with their co-workers. However, they responded with little enthusiasm. At the time this took place photogrammetry worked mainly with instrumental solutions for mapping from aerial photography and, where calculations were needed, analogue methods were preferred.

The rise of computer technology led to the development of mathematical models, but these were based on entirely different principles than those used in land surveying and geodesy. As shown above, models in land surveying were mainly developed from the principle of angle and distance measurements. In the photogrammetry two approaches existed.

a) Aerial photographs were processed in pairs with photogrammetric instruments that formed stereographic models. 3D-coordinates of terrain points were measured in these models. Then blocks were formed from groups of contiguous models, in which the models were converted to a common coordinate system through similarity transformations and then subjected to a so-called *block adjustment by independent models.* This approach thus focussed on transformation of coordinate systems instead of adjustment based on shape elements.

b) The second approach consisted of measuring terrain point images in the overlap zone of aerial photographs. These points were subsequently linked to the physical terrain points by applying the projective transformation formulas to their measured coordinates. If the terrain coordinates of a sufficiently large number of terrain points were known, the terrain coordinates of all other points measured on the photographs could be calculated by applying a so-called *bundle block adjustment*.

The method of working in photogrammetry was therefore so different from that in land surveying, that Baarda was unable to find an opening for the use of his approach. Personal notes that I saw from him also showed that his approach was wrong. His assumptions regarding the relationship between point triplets on the aerial photograph and the terrain were incorrect. He determined the Q-variates[8] based on the shape elements of the point triplets on the photograph and linked those to the corresponding Q-variates in the terrain. The trouble was that the relationship between photograph and terrain is described by a projective transformation and not by a similarity transformation. This approach therefore failed.

In the 1970s, during my first period at ITC[9], Baarda asked me to investigate this. This question occupied me for some time and led to the research on the spatial S-transformation and criterion matrix. What Baarda was looking for was an alternative formulation of the bundle block adjustment mentioned in item b) above. His fundamental assumption was that shape elements between terrain points played a key role in that. However, that was incorrect. If aerial photographs are interpreted as bundles of direction measurements from the camera focal point to points in the terrain, the analogy with theodolite measurements is obvious. That is the approach I followed. In formulating the functional model by means of quaternions I ran into the issue that only direction measurements were available and no distance measurements of length ratios. Consequently, all relationships between quaternions defaulted to their vectorial components and the model might just as well be described by vector algebra (Molenaar 1982).

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[8] The Q-variate is the 3D analogy of a Π-variate in the plane. See Appendix 8.

[9] The International Training Centre for Aerial Survey (ITC) was established at Delft in 1950 to provide training to survey department members of development countries. Later, it relocated to Twente University and is now known as the Faculty of Geo-Information Science and Earth Observation.

Remark: The formulation in (Molenaar 1982) was new in photogrammetry. It turned out that only three and not five unknown parameters, as was customary, had to be resolved for determining the relative orientation of two overlapping aerial photographs. Moreover, no accurate approximate values were required for these unknown parameters to start the solution algorithm. Furthermore, this approach offered the option of formulating the block adjustment in accordance with the *First Standard Problem* (actually, Tienstra's Standard Problem IV), which enabled adjustment in steps. This algorithm was similar to that of the Kalman filter, as was mentioned at the end of Section 3.5.1 above.

5 Epilogue

The end of an era

The completion of the theoretical extension from two to three dimensions approximately coincided with Baarda's retirement. This ended the period that is discussed in this monograph. However, this last part of the methodology has not really been tested in practice. Moreover, there was some doubt about the feasibility of this approach to an adjustment in two steps of three-dimensional networks. In particular, networks of type a), described in Section 4.9.2, would probably not be stable enough to be processed as free networks in step one of the adjustment. Experiences in photogrammetry led to similar findings.

The rise of GPS in particular has given enormous impetus to the changing role of geodesy and geodesists. However, another, be it rather delayed, conclusion of covered period came with an updated version of the HTW in 1996. Its subject matter clearly showed the dawn of a new period. Geodesists were being confronted with different questions, while GPS made it possible for precise positioning for most applications to be executed by professionals from other disciplines. Earth observation really took flight and modern computer science made a definitive breakthrough, with major consequences for the work of the geodesist. All this provides a good reason to describe the developments after the early 1980s in a separate monograph.

Publications and external influences

It is quite remarkable that this successful research tradition, covering about fifty years, has led to a relatively small number of publications, of which only a handful was published in internationally acknowledged scientific journals. Nevertheless, the results of this work have been taken up in a major way in geodesy and its associated disciplines. Tienstra's book and Baarda's publications in the 'Yellow Series' of the Netherlands Centre of Geodesy and Geo-Information Science (NCG)[10], have had a considerable impact. Equally remarkable is the fact that by far the largest number of publications are attributed to the two main actors, Tienstra and Baarda. Baarda wrote one article together with Alberda (Baarda and Alberda 1962) and Alberda contributed to several of Baarda's publications, but ultimately these were published under Baarda's name alone.

In the fifty years described in this monograph a consistent methodology, highly relevant for survey practice, took shape. It is therefore surprising that no textbook has ever been published, providing an accessible and coherent introduction of this work with examples of how the methods can be used in practice.

In the current monograph I adopted a somewhat monolithic approach in my reconstruction of the history of the development of the Delft School methodology, doing so mainly based on the publications of the protagonists and those that were involved directly. It is unclear from those publications how others might have influenced Tienstra's and Baarda's work. Naturally an exception must be made for Van Dantzig and his circle. Baarda did refer to the standard literature about mathematical statistics. Cramer has been mentioned earlier and Fisher's work was clearly addressed. However, any influence exerted by fellow geodesists remains unclear. He did refer to their work but did not show how he incorporated that in the development of his own ideas. That would be worth investigating more closely. To get an idea of the thoughts formulated elsewhere in the geodetic community, especially around the 1980s, see (Mierlo 1982), in which Jan van Mierlo provides an overview of international developments in mathematical geodesy.

[10] See https://www.ncgeo.nl/index.php/en/publicatiesgb

PhD students

Another remarkable point is that from the generally well-qualified co-workers no one was ever awarded a doctorate. Baarda himself had not been awarded one either. On this subject there was considerable restraint, just as in the publication of work results. Not until towards the end of his career did he accept four PhD students and even that he did with some hesitation. I remember I had regular meetings with Baarda regarding the research I conducted with the aim of applying his ideas in photogrammetry at ITC in the 1970s. One of those meeting was attended by my friend and then colleague Cor van der Hout. Baarda let slip that the material I showed was actually worthy of a doctoral dissertation, at which point Van der Hout jumped in and said, "I'm a witness, Baarda!" After the work had progressed a few years, another meeting took place in which Diederik van Daalen participated. At the end of the meeting Van Daalen remarked that more than enough material was now available to draw the research project to a conclusion. Baarda consented and a date was agreed for the PhD award process, during which I had to publicly defend the project, to be awarded a doctorate, subject to the defence being successful.

Of the four PhD projects Baarda supervised towards the end of his career, Herman Quee's and mine (Molenaar 1981 and Quee 1983) contributed clearly to the conclusion of the research period described in this monograph. Frits Brouwer's research concerned the application of the methodology developed to a related discipline: astronomy, specifically Very Long Baseline Interferometry (Brouwer 1985). Finally, Peter Teunissen brought this research into a new phase with his doctoral dissertation (Teunissen 1985). He would continue the work after Baarda and give it new impetus.

Finally ...

The term *Delft School* is customary among geodesists, particularly those with a background from TU Delft. Diederik van Daalen, as a non-geodesist and hence a relative outsider, devoted an essay (in Dutch) to this subject in a quintennial festive book of the society of geodetic students *Landmeetkundig Genootschap Snellius* (Daalen 1985).

In summary, we may conclude this monograph describes a development in geodesy that was of great importance for the discipline, both nationally and abroad. The appreciation for this is also apparent from the many honorary titles awarded to Baarda. One of the most important was an honorary doctorate of the University of Stuttgart, awarded to him in 1982.

What has been shown in this monograph is that this line of research was initiated and elaborated on by the personal commitment of two great men, Tienstra and Baarda. Their intuition and power of work have been decisive for the success of the *Delft School*. Tienstra died too early, so that he had to hand over his tasks prematurely to Baarda, who proved himself a more than worthy successor. At his retirement in 1982 the question therefore was whether he was going stop. Evidently that was not the case.

The question of the connection between measurement systems and the definition of coordinate systems continued to occupy him. An essay about these questions had been published by NCG in 1979, just before his retirement and in 1995, he was 78 years old then, a sequel was published. These publications were difficult to understand but they did demonstrate that he persisted into old age. That also became apparent in 1997, at his eightieth birthday. On that occasion his four former PhD students and the chairman of the then Netherlands Society of Geodesy (NVG) were asked to contribute to a special issue of the trade magazine *NGT-Geodesia*. They were thanked for this by a personal letter from Baarda (Baarda 1997). In those letters he briefly responded to the contributions, after which he wrote that he was still wrestling with the concept of coordinates in modern geodesy. This shows that this question occupied him all his life, into old age, as the following sentence from this letter shows:

Then I thought … of Bridgeman's question of … which coordinate system Einstein had used … The definition of coordinates is even worse than that of the Cadastre; it is completely ignored.

Appendices

Appendix 1: Standard Problem I

Least squares adjustment with condition equations

Let x be a vector of n observation variates with covariance matrix $\sigma_{xx} = \sigma^2 \cdot g_{xx}$, in which σ^2 is a scale factor (variance factor).

- Matrix g_{xx} is then called the matrix of weight coefficients of x.
- Matrix g_{xx}^{-1} is the inverse van g_{xx} and is called the weight matrix of x.
- Diagonal elements of this matrix are the weights of the observation variates x.

For linear functions $y = B \cdot x$ the propagation laws of probability calculus hold:

- for the mathematical expectation values: $E\{y\} = B \cdot E\{x\}$
- and for the matrix of weight coefficients: $g_{yy} = B^T \cdot g_{xx} \cdot B$ and hence also $\sigma_{yy} = B^T \cdot \sigma_{xx} \cdot B$.

If $t = U \cdot x + U_0$ designates b (linear) functions of the observation variates $(b<n)$, $E\{t\} = U \cdot E\{x\} + U_0$. If $E\{t\} = 0$ these functions are termed *condition equations*. These are conditions that should be satisfied by the observation variates. If, for a certain set of measurements: $t=U \cdot x + U_0 \neq 0$, vector t is termed *misclosure vector* and corrections need to be applied to the measurements, such that: $X = x + v$ and $U \cdot X=0$. Many solutions for v exist, but in this case the so-called least squares solution is sought, for which the following equation is valid.

$\underline{v}^T \cdot g_{xx}^{-1} \cdot \underline{v}$ = minimal!

The *propagation laws of weight coefficients* yield the matrix of weight coefficients of the misclosure vector t:

$g_{tt} = U^T \cdot g_{xx} \cdot U$; hence also: $\sigma_{tt} = \sigma^2 \cdot g_{tt}$

The solution for v can be found in textbooks:

$\underline{v} = g_{xx} \cdot U^T \cdot g_{tt}^{-1} \cdot \underline{t} = B \cdot \underline{t}$

where: $g_{vv} = B \cdot g_{tt} \cdot B^T = g_{xx} \cdot U^T \cdot g_{tt}^{-1} \cdot U \cdot g_{xx}$

and the matrix of weight coefficients of the corrected observations X equals: $G_{XX} = g_{xx} - g_{vv}$.

Remark: With Tienstra's terminology g_{xx} might be termed the matrix of cofactors of x. This term is also used in tensor calculus.

Appendix 2: Standard Problems II, III, IV and V

Standard Problem II: Least squares adjustment with observation equations.

In this form the *n* corrected observations X are expressed as a function of the *n-b* unknowns *Y.*

$X = A \cdot Y + A_0$, where $X = x + v$; hence: $A \cdot Y + A_0 = x + v$

In this case too the corrections *v* must satisfy: $\underline{v}^T \cdot g_{xx}^{-1} \cdot \underline{v}$ = minimal!

When *N* is defined as: $N = A^T \cdot g_{xx}^{-1} \cdot A$; then: $Y = N^{-1} \cdot A^T \cdot g_{xx}^{-1} \cdot (x - A_0)$, where $g_{YY} = N^{-1}$.

Usually the starting point is a non-linear function $X = F(Y)$. The values of Y are now sought. By choosing good approximate values y_0, a solution between approximate values can be found as follows.

$x_0 = F(y_0)$

Corrections to the approximate values x_0 and y_0 need to be found in order to compute the corrected values $X = x_0 + \Delta X$ and $Y = y_0 + \Delta Y$ that do satisfy the start equations $X = F(Y)$. When ΔX and ΔY are relatively small, a linear relation can be found from this that has the structure of Standard Problem II:

$\Delta X = A \cdot \Delta Y + A_0$

where *A* is the Jacobian matrix of function *F*. As a result of the introduction of x_0 the quantity $\Delta x = x - x_0$ will take over the role of the original observations *x*. The least squares solution is then found as follows:

$\Delta Y = N^{-1} \cdot A^T \cdot g_{xx}^{-1} \cdot \Delta x$, with *N* defined as above.

Tienstra formulated another three variants of the standard problems. Only the start equations are given here. The reader is referred to relevant textbooks to find the solutions.

Standard Problem III:

Using the same symbols, the start equations consist of two groups:

$X = A \cdot Y + A_0$ and $U \cdot Y + U_0 = 0$

The first group thus consists of observation equations as in Standard Problem II. The second group consists of condition equations, but applied to the unknown parameters *Y*.

Standard Problem IV:

Using the same symbols, the start equations are:

$A \cdot Y = U \cdot X + U_0$; or $A \cdot Y - U \cdot X - U_0 = 0$

In these equations the observations are not expressed as functions of the unknowns, but (linear) functions of the observations are expressed as functions of the unknowns. This approach is often adopted for more complex problems, for example in photogrammetry.

Standard Problem IV:

The start equations are in this case a combination of the equations given for Standard Problems III and IV.

$A \cdot Y + A_0 = U_1 \cdot X$ and $U_2 \cdot Y + U_0 = 0$

Appendix 3: The geometric interpretation of the standard problems

Let n observations x_i (i = 1 … n) be the scalar components of the n-dimensional observation vector x. These can be represented geometrically in a space ('x-space') with n axes. Two complementary subspaces can be defined in x-space, viz.:

- A b-dimensional t-space. This is the subspace of the condition equations; vector t lies entirely in t-space;
- An a-dimensional Y-space. This is the subspace of the unknown parameters Y, which are to be estimated.

The two subspaces are complementary, that is, $a+b=n$; the dimensions of two subspaces added together yields the dimension of x-space.

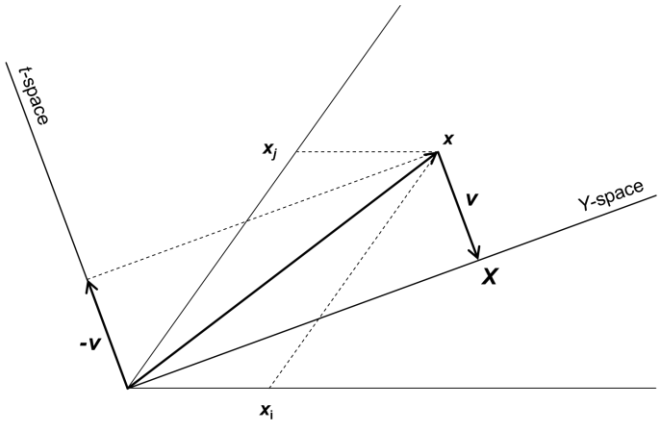


Figure A3: Geometrical representation of the X-space, Y-space and t-space

In Appendix 1 and 2 it was shown that in the case of least squares estimators: $X = A \cdot Y$, which implies that vector X lies entirely in Y-space. Additionally $U \cdot X = 0$.

When a measurement process is repeated, the results for x will vary and that means the results for Y and X will vary. However, for <u>all</u> results the following relation holds: $U \cdot A \cdot Y = 0$. From that follows: $U \cdot A = 0$. Vector X is therefore orthogonal to t-space; hence Y-space and t-space are orthogonal. These relations are graphically represented in *Figure A.3*.

x_i and x_j are the parallel projections of vector x (the observation vector) on the respective axes of x-space. Vector v lies entirely in t-space and is therefore orthogonal to Y-space. The vector of corrections v projects vector x therefore also orthogonally to Y-space and that is exactly what least squares estimators do. The (weighted) sum of the squares of the corrections $v^T \cdot g_{xx}^{-1} \cdot v$ is the length of the correction vector and that is minimal when it is orthogonal to Y-space.

In x-space, as it is drawn in *Figure A.3*, the axes are not orthogonal. Therefore, the length of a vector X cannot be calculated by adding the sum of the squares of the scalar components X_i, that is, by applying Pythagoras's Theorem. Its length must be calculated from the product $X^T \cdot g_{xx}^{-1} \cdot X$. In this product g_{xx} is called the metric tensor; this compensates for the effect of the non-orthogonality of the axes and for the effect of potentially different units of measure along different axes. In our application that is the matrix of weight coefficients. In t-space the length of the t-vector is computed analogously as $t^T \cdot g_{tt}^{-1} \cdot t$.

Appendix 4: Standard error ellipses and error curves

Let x be a normally-distributed random variable with expected value $E\{x\} = \mu$ and variance $E\{(x - \mu)^2\} = \sigma^2$. The probability density function is then represented by Figure A.4.

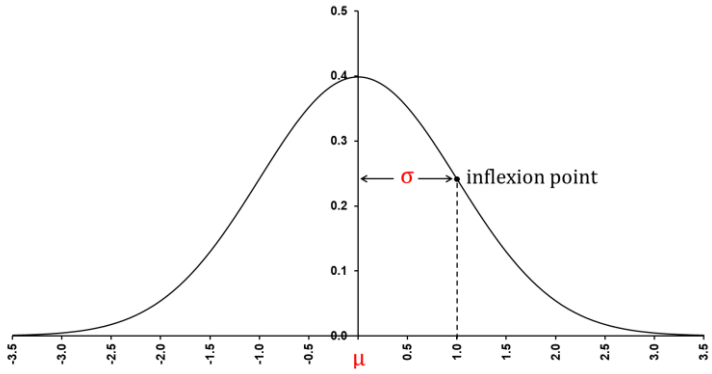


Figure A.4: Normal distribution

Figure A.4 demonstrates that σ determines the shape of the curve, which is why this parameter is used as a measure to express the spread of x. The following relations are valid (see mathematical statistics textbooks, e.g. (Cramer 1957) and (Hogg and Craig 1970)).

$$E\{(x - \mu)^2 / \sigma^2) = 1$$

This holds for the spread of a single normally-distributed variable. In the case of n normally-distributed variables, expressed jointly as vector x, the expected value is:

$$E\{x\} = \mu$$

where μ is now also a vector.

Instead of the (scalar) variance used above, the n observation variates now jointly have a $n \times n$ covariance matrix σ_{xx} . This matrix takes over the role of the variance σ^2. That yields:

$$E\{(x - \mu)^T \cdot \sigma_{xx}^{-1} \cdot (x - \mu)\} = 1$$

Analogous to the above case, σ_{xx} determines the shape of the n-dimensional distribution. If two variates are involved and $x - \mu = x'$ is substituted, then, the formula:

$$x'^T \cdot \sigma_{xx}^{-1} \cdot x' = 1$$

describes an ellipse, of which the shape is determined by σ_{xx} . In geodetic surveying point standard error ellipses are calculated from the (X, Y) coordinates of surveyed points using the above formula. From the covariance matrix of the *coordinate differences* between two points relative standard error ellipses may be computed analogously. The latter ellipses reflected the relative accuracy of any two points.

Appendix 5: The definition of Π-variates

Extract from (Baarda 1962).

If we consider a rectangular x, y coordinate system with equal units of measurement on the axes, the only transformation admissible within Euclidean geometry is the general similarity transformation.

The arbitrary choice of a coordinate system can therefore be expressed by the introduction of a coordinate system which is not uniquely determined but can be subjected to similarity transformations.

If we introduce the complex quantity

$$z = x+\mathrm{i}y \qquad (3.1)$$

this means that the following z' system is equally admissible:

$$z' = \gamma z+\delta \qquad (3.2)$$

in which γ and δ are complex transformation constants.

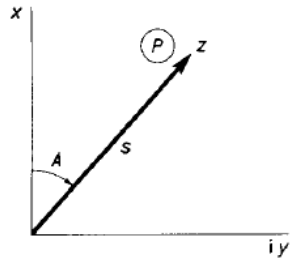


Fig. 3-1

Fig. 3-2

For the theory to be developed, the essential quantities are those that are invariant with respect to the transformation (3.2.)

Consider the points P_i, P_j, P_k, P_l and define relative coordinates or coordinate differences as follows:

$$x_{ij} = x_j - x_i$$
$$y_{ij} = y_j - y_i$$

and consequently

$$z_{ij} = x_{ij}+\mathrm{i}y_{ij} = z_j - z_i \qquad (3.3)$$

For point coordinates we have, by (3.2):

$$\left.\begin{aligned} z'_i &= \gamma z_i+\delta \\ z'_j &= \gamma z_j+\delta \\ z'_k &= \gamma z_k+\delta \\ z'_l &= \gamma z_l+\delta \end{aligned}\right\} \qquad (3.4)$$

and for relative coordinates, by (3.3):

$$\left.\begin{aligned} z'_{ij} &= \gamma z_{ij} \\ z'_{kl} &= \gamma z_{kl} \end{aligned}\right\} \qquad (3.5)$$

The introduction of relative coordinates results in quantities that are invariant with respect to changes in δ, but not with respect to changes in γ.

Quantities invariant with respect to changes in both δ *and* γ follow from (3.5):

$$\frac{z'_{kl}}{z'_{ij}} = \frac{z_{kl}}{z_{ij}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.6)$$

Let

$$\frac{z_{kl}}{z_{ij}} = e^{(\Lambda_{kl}-\Lambda_{ij})} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.7a)$$

with (ln denotes the natural logarithm):

$$\left.\begin{aligned} \Lambda_{ij} &= \ln z_{ij} = \ln s_{ij} + \mathrm{i}A_{ij} \\ \Lambda_{kl} &= \ln z_{kl} = \ln s_{kl} + \mathrm{i}A_{kl} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \quad (3.7b)$$

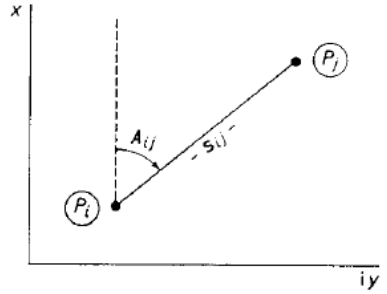


Fig. 3-3

Separating the real and imaginary parts results in:

$$\left.\begin{aligned} \mathrm{R}\{\Lambda_{kl}-\Lambda_{ij}\} &= \ln s_{kl} - \ln s_{ij} = \ln \frac{s_{kl}}{s_{ij}} \\ \mathrm{I}\{\Lambda_{kl}-\Lambda_{ij}\} &= A_{kl}-A_{ij} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (3.7c)$$

It will turn out to be more important to consider (3.7) for three points instead of four, and we will consequently introduce a new symbol for this case.

For the points P_i, P_j and P_i, P_k, (3.7) results in the quantity:

$$e^{\Pi_{jik}} = e^{(\Lambda_{ik}-\Lambda_{ij})} = \frac{z_{ik}}{z_{ij}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.8a)$$

with:

$$\Pi_{jik} = \ln \frac{z_{ik}}{z_{ij}} = \ln \frac{s_{ik}}{s_{ij}} + \mathrm{i}(A_{ik}-A_{ij}) \quad \ldots \ldots \ldots \ldots \quad (3.8b)$$

If we put

$$\left.\begin{aligned} v_{jik} &= \frac{s_{ik}}{s_{ij}} \\ \alpha_{jik} &= A_{ik}-A_{ij} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.8c)$$

we obtain:

$$\boxed{\Pi_{jik} = \ln v_{jik} + \mathrm{i}\,\alpha_{jik}} \quad \ldots \ldots \ldots \ldots \ldots \quad (3.8d)$$

In the complex plane we can represent Π_{jik} in the same way as α_{jik} in the real plane, i.e. *with* the direction of rotation. The positive direction is chosen as in Fig. 3-1 and Fig. 3-3, i.e. clockwise. See Fig. 3-4.

Appendix 6: Data snooping

See (Baarda 1968) p. 28, formula (4.3).

Let C_p be a vector that indicates to what extent observations are part of an alternative hypothesis H_{ap}. *Data snooping* is the procedure in which a series of alternative hypotheses H_{ai} are tested. In each of these alternative hypotheses only one observation is assumed to contain an error. For each of these hypotheses the *i*-th element of the associated C_i vector will contain the value 1; all other elements will be zero. Data snooping proceeds as follows.

For each value of $i = 1 \ldots n$, calculate:

$N_i = C_i^T \cdot g_{xx}^{-1} \cdot g_{vv} \cdot g_{xx}^{-1} \cdot C_i$

N_i is therefore the *i*-th diagonal element of the following matrix: $N = (g_{xx}^{-1} \cdot g_{vv} \cdot g_{xx}^{-1})$

The test variate for observation x_i is then:

$w_i = C_i^T \cdot g_{xx}^{-1} \cdot g_{vv} \cdot g_{xx}^{-1} \cdot (x - A_0)/\sigma\sqrt{N_i}$

All observations are tested individually and successively for the occurrence of possible errors by applying the criterion:

$w_i^2 < F_{1-\alpha_0;\, 1,\, \infty}$

A fixed value for α_0 is used these tests, commonly α_0 =0.001.

Appendix 7: Ricci calculus

The accessibility of Tienstra's publications and later also Baarda's is difficult for most readers because of the index notation both authors used. Geodesists might be somewhat familiar with Ricci notation from their exposure to differential geometry. The effects of earth curvature will be felt when geodetic survey measurements start cover medium-large areas. When high accuracy is pursued and when large areas are covered, the fact that the polar axis of the earth is shorter than its equatorial diameter must also to be taken into account. In that case the shape of the earth is approximated by an ellipsoid. That means that a north-south cross section of the earth will have the shape of an ellipse with the equatorial diameter as major axis and the polar axis as minor axis. By allowing the ellipse to rotate about its polar axis a three-dimensional ellipsoid is obtained.

Euclidean geometry does not suffice for geometric calculations on such a surface; Riemann geometry is required for curved surfaces. Calculations on curved surfaces are performed by means of differential geometric calculus, an important application of tensor analysis. It is here that Ricci notation was used. This approach had become more widely known in the early twentieth century because Einstein applied it in the formulation of the Theory of Relativity.

Tienstra probably used the Ricci notation because he was not yet well versed in linear algebra and matrix calculus; see the remark made on this in (Hoek 1982). The Ricci notation has a great expressive power and makes it possible to distinguish many types of quantities through the combination of basic symbols and indices it uses, whereby many types of spaces could be distinguished in the geometric interpretation of formulas (Tienstra 1947).

With the new variables X^α, the linear transformation will be given by

$$\begin{aligned} X^1 &= a_1^1 x^1 + a_2^1 x^2 + \ldots + a_n^1 x^n \\ X^2 &= a_1^2 x^1 + a_2^2 x^2 + \ldots + a_n^2 x^n \\ &\vdots \\ X^n &= a_1^n x^1 + a_2^n x^2 + \ldots + a_n^n x^n \end{aligned}$$

These formulae can be written down in an abridged way

$$X^\beta = \sum_{\alpha=1}^{n} a_\alpha^\beta x^\alpha \qquad (\beta \rightarrow 1, \ldots, n)$$

It is a proposal of *Einstein*, to suppress the sumsign therefore to write simply

$$X^\beta = a_\alpha^\beta x^\alpha \ (\alpha, \beta \rightarrow 1, \ldots n) \qquad (3)$$

and to make once for all the agreement, that whenever a letter enters in a product as upper and lower index, the sum over this index has to be taken. Moreover it is convenient in order to avoid the constant repeating of the significance of the used indices as in (3), to agree beforehand on the letters that will be used for denoting a certain set of numbers. In our case we state that the indices

$$\alpha, \beta, \gamma, \varepsilon \text{ and eventually } \alpha', \beta', \gamma', \varepsilon' \qquad (4).$$

will run from 1 to n. It is to be remarked that between the indices α, β, ... and 1, 2, 3..... n there is quite a difference. The former represent each of the numbers 1 to n, as is agreed upon; the latter are fixed and represent only the number for which they stand.

We may now write (3) as

$$X^\beta = a^\beta_\alpha x^\alpha \tag{5}$$

and this expression has the meaning of a set of n linear equations in the n quantities x^α. In accordance with the agreement about the indices

$$X^\alpha = a^\alpha_\gamma x^\gamma$$

is identical with (3) and therefore it is not essential which letter is used for denoting a certain index, if it is only one of the set (4). It is the *place* of the index that counts. An upper index is also called a contravariant-, and a lower one a covariant index.

As stated in the introduction, the X^α introduced in (5) may be considered as a set of observations themselves and in this paragraph, we will examine accuracy and correlation of these quantities.

We suppose that the determinant on the a^β_α is not zero; in that case the equations can be solved to the x^α. Let the solution be

$$x^\alpha = A^\alpha_\beta X^\beta \tag{6}$$

Between the coefficients a^α_β and A^α_β excists a number of relations. We easily can find them, when we substitute the X^β of (5) in (6). We get

$$x^\alpha = A^\alpha_\beta a^\beta_\gamma x^\gamma \tag{7}$$

We had to replace the index α in (5) by γ, because otherwise both sums in (7) would not be defined uniqually. Indeed when we had not taken this precaution, the result would have been

$$x^\alpha = A^\alpha_\beta a^\beta_\alpha x^\alpha \ ?$$

and the expression is meaningless, because one of the sums is not properly defined.

The coefficient of x^α in (7) being 1 and all the other one 0 we have that

$$A^\alpha_\beta a^\beta_\gamma$$

is 1 as α and γ are equal and 0 as α and γ are different. We write it down as follows :

$$A^\alpha_\beta a^\beta_\gamma = \delta^\alpha_\gamma \tag{8}$$

In this expression δ^α_γ is called the " symbol of *Kronecker* " and it is defined by

$$\delta^\alpha_\beta = 0 \quad \text{if} \quad \alpha \neq \gamma$$

and

$$\delta^\alpha_\gamma = 1 \quad \text{if} \quad \alpha = \gamma$$

In substituting reversely x^α of (7) in (5) we get in the same way

$$a^\beta_\alpha A^\alpha_\gamma = \delta^\beta_\gamma \tag{9}$$

(8) and (9) represent each a set of $n \times n$ equations. If we separate from (8) those equations in which α is 5, we get

$$\begin{aligned}
A^5_\beta a^\beta_1 &\equiv A^5_1 a^1_1 + A^5_2 a^2_1 + \ldots + A^5_n a^n_1 = \delta^5_1 = o \\
A^5_\beta a^\beta_2 &\equiv A^5_1 a^1_2 + A^5_2 a^2_2 + \ldots + A^5_n a^n_2 = \delta^5_2 = o \\
&\vdots \\
A^5_\beta a^\beta_5 &\equiv A^5_1 a^1_5 + A^5_2 a^2_5 + \ldots + A^5_n a^n_5 = \delta^5_5 = 1 \\
&\vdots \\
A^5_\beta a^\beta_n &\equiv A^5_1 a^1_n + A^5_2 a^2_n + \ldots + A^5_n a^n_n = \delta^5_n = o
\end{aligned}$$

and with this set we may compute $A^5_1, A^5_2, \ldots A^5_n$ if the a^α_β are given. The above equations are one out of the n sets represented by (8). Each set deter-

mines n quantities A^{α}_{β}. In an analogous way the relations (9) offer the means to solve the a^{β}_{α}, if the A^{β}_{α} are known.

In his 1947 publication Tienstra explained the Ricci notation and his use of it (Tienstra 1947). This introduction, chapter 2 of his article, has been reproduced entirely as support for those people who wish to consult these early publications.

Appendix 8: The use of quaternions

In (Molenaar 1981), chapter IV, pp. 33-34 the following definition of quaternions is found.

A point i with coordinates

$$x_i\ ,\ y_i\ ,\ z_i$$

will be described by a quaternion:

$$q_i = 0 + x_i.i + y_i.j + z_i.k \qquad (4.1)$$

with product rules :

$$i.i = j.j = k.k = -1$$
$$i.j = -j.i = k$$
$$j.k = -k.j = i$$
$$k.i = -i.k = j$$

Point j is:

$$q_j = 0 + x_j.i + y_j.j + z_j.k$$

Substraction gives:

$$q_{ij} = q_j - q_i = 0 + (x_j - x_i).i + (y_j - y_i).j + (z_j - z_i).k$$

For q_{ij} the conjugate quaternion is :

$$q_{ij}^T = 0 - (x_j - x_i).i - (y_j - y_i).j - (z_j - z_i).k$$

and the norm :

$$N\{q_{ij}\} = q_{ij}.q_{ij}^T = q_{ij}^T.q_{ij} = l_{ij}^2$$
$$= (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2$$

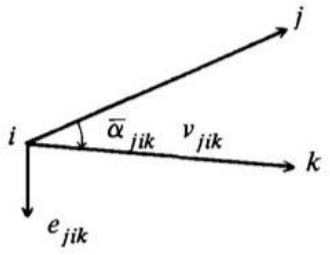
The inverse of q_{ij} with respect to multiplication is :

$$q_{ij}^{-1} = \frac{q_{ij}^T}{N\{q_{ij}\}}$$

where :

$$q_{ij}.q_{ij}^{-1} = q_{ij}^{-1}.q_{ij} = 1$$

Add point k with q_k and $q_{ik} = q_k - q_i$. The left division by q_{ij} is then:



$$Q_{jik} = q_{ik}.q_{ij}^{-1} \qquad (4.2.1)$$

which can also be written as :

$$Q_{jik} = v_{jik}.(cos\bar{\alpha}_{jik} + e_{jik}.sin\bar{\alpha}_{jik}) \qquad (4.2.2)$$

v_{jik} is the length ratio of side i,j and side i,k and $\bar{\alpha}_{jik}$ is the angle between these sides, whereas e_{jik} is the unit vector in the direction normal to plane j,i,k. This expression demonstrates in which way the measurable quantities can be entered in the coordinate computation when using quaternion algebra.

This description makes it clear that the spatial variate Q_{jik} has the same structure as the plane variate Π_{jik}. Because of the multiplication rules for quaternions, multiplications are not commutative, hence: $a \cdot b \neq b \cdot a$. That makes the application of quaternion algebra considerably more involved than complex number algebra in the field.

Quee describes in (Quee 1983), chapter 2, p. 32 how quaternions are linked to the measurement of distances and angles in the field.

By means of a theodolite established in P_i, the two "polar coordinates" of the spatial vector $P_i P_k$ can be measured : the zenith angle $\underline{\mathfrak{z}}_{ik}$ and the direction $\underline{r}_{ik}$:

$\underline{\mathfrak{z}}_{ik}$ is the angle between the vector $P_i P_k$ and the upward direction of the first axis of the theodolite set up in P_i ; (2.6)

$\underline{r}_{ik}$ is the angle between the zero direction of the horizontal circle and the projection of the vector $P_i P_k$ on the plane of that circle. (2.7)

Since the first axis of the theodolite is perpendicular to the horizontal circle, the local system (orthogonal trihedral i, j, k) can be defined by linking it up as follows with the theodolite : (see fig. 14)

- the i-vector lies in the plane of the horizontal circle, in the "zero direction"
- the k-vector lies in the part of the first axis "pointing downwards"
- the j-vector completes a right-handed system i, j, k.

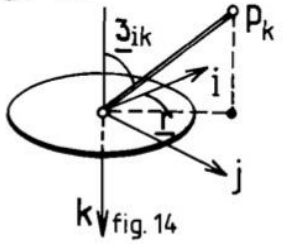


fig. 14

(2.8)

By (1.18), the spatial vector $P_i P_k$ can now be expressed as quaternion q_{ik} with a scalar part = 0, by conversion of the rectangular coordinate differences x, y, z to polar coordinates s, r, $\mathfrak{z}$:

$$\underline{q}^{(i)}_{ik} = 0 + i\, \underline{s}_{ik} \cos \underline{r}_{ik} \sin \underline{\mathfrak{z}}_{ik} + j\, \underline{s}_{ik} \sin \underline{r}_{ik} \sin \underline{\mathfrak{z}}_{ik} - k\, \underline{s}_{ik} \cos \underline{\mathfrak{z}}_{ik} \qquad (2.9)$$

$$\downarrow \sqrt{N\{\underline{q}^{(i)}_{ik}\}} = \underline{s}_{ik}$$

If measurements are also made from point P_i to point P_j the quaternion q_{ij} can be derived, which enables calculation of variate Q_{jik}. It must be noted that in situations mentioned above the angle α_{jik} and the length ratio v_{jik} are spatial observations between the points P_i, P_j and P_k. In other words, these measurements have not been projected on the horizontal plane, as was the case with Π_{jik} .

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Illustrations