NETHERLANDS GEODETIC COMMISSION

PUBLICATIONS ON GEODESY NEW SERIES

VOLUME 1 NUMBER 2

REPORT ON THE ADJUSTMENT OF THE UNITED EUROPEAN LEVELLING NET AND RELATED COMPUTATIONS

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1963

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PRINTED IN THE NETHERLANDS BY W. D. MEINEMA N.V., DELFT

PREFACE

The Symposium on the establishment of a United European Levelling Net, which Section II of the I.A.G. held in May 1955 in Florence, belongs already to the not-so-recent past. However, the report presented here goes back, in many respects, to the discussions held there, in particular with regard to the set-up to test the value of different methods of computation and of interpretation of the results of measurements and computations.

In Florence, Ir. A. WAALEWIJN in consultation with the undersigned proposed that the contribution of Delft to this project would be the practical trial of the adjustment by steps developed by the late Prof. J. M. TIENSTRA. Newer developments of adjustment theory and statistical lines of thought could also be tried and valuated in this application. For this field, the study of "linking up" and "unlinking" mathematical models in geodesy, offering possibilities for a sharper interpretation of results of measurements and computations, is, in particular, the field of work of the Computing Centre of the Delft Geodetic Institute.

I wish to express here my great appreciation of the enthusiasm with which all collaborators of the Computing Centre have worked on this project under my direction. Whereas, in particular, Ir. J. E. ALBERDA is to be thanked for many fundamental studies and computations and for the final editing of this report, we owe to Ir. B. G. K. KRIJGER the investigations and computations on the power of the different tests used and the systematic set-up which was needed to execute the chosen adjustment method on a modern computer. Ir. E. F. MEERDINK directed his attention, among other things, to the set-up of F-tests and the interpretation of their results. The collaborators mentioned had regular contacts with Ir. WAALEWIJN whose valuable advice contributed very much to a realistic interpretation of the material.

The most valuable aspects of the methods applied is that besides the results of the adjustment and of the tests of hypotheses, there could also be given a valuation of the results found and statements made. May this aspect receive the attention it deserves.

Prof. Ir. W. BAARDA

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1 INTRODUCTION

In 1955, at the Florence meeting of the Permanent International Commission on European Levellings, it was agreed among other things that the Computing Centre of the Delft Technological University would undertake to compute the adjustment of the United European Levelling Net by a method based on principles indicated by the late professor J. M. Tienstra. It was decided from the start that the Delft adjustment would be an adjustment by steps (or, in Tienstra's terminology in [13], an adjustment in phases).

Some small test-computations were made in 1956 to see what would be the most suitable set-up of the adjustment.

The data for the adjustment were received from the Commission's President, Dr. O. Simonsen, in May 1958.

The organization of the actual computations started in October 1958. The adjusted geo-potential numbers of bench marks were presented at the meeting of representatives of computing centres which was held at Delft in January 1959. Part of the weight coefficients of these geo-potential numbers were also ready by then; the rest of the weight coefficients followed soon.

The geo-potential numbers of mean sea level, as resulting from the net, were received at the end of August, 1959 from Dr. Simonsen. A statistical investigation of these values was carried out in September 1959 and a provisional report on the adjustment and connected investigations was presented to the Liverpool Symposium in October 1959; an outline of this report is published in [1]. Finally, some checks and re-computations were made in 1960. The re-computations were necessitated by the discovery of a gross error in the line 218→J–D–F–2. All results in this report are based on the corrected data.

2 THE ADJUSTMENT

2.1 **Method of adjustment**

It is well known that the two main forms in which an adjustment problem may be expressed are:

a. condition equations,

b. observation equations.

Any adjustment problem may be split up into different steps, and each step may be adjusted according to a. or b., so that even the most general adjustment problem can be solved with the rules of computation of a. and b., provided they are used in their general form, *i.e.* for correlated observations. A problem in the form of a. was called by TIENSTRA Standard Problem I, b. was called Standard Problem II, names which we shall also use here.

Adjustment by steps does not as a rule result in less computational work, but splitting up the total adjustment affords greater possibilities for the analysis of results, in particular when use is made of test methods of mathematical statistics. The different steps can usually be interpreted directly, so that a clear outline of the whole computation can be kept in mind. Smaller matrix inversions are involved in the computation of the precision of the final results. Alterations in certain parts of the adjustment or the addition of new parts can as a rule be effectuated easily.

The way adjustment by steps was used in the adjustment of U.E.L.N. can be briefly described by saying that the total net was split up in four partial nets, which were separately adjusted (first step) and later joined together (second step).

First step. Let Fig. 1 represent a partial net with the antennae that connect it to adjacent partial nets. The partial net is adjusted according to the method of observation equations. Let O_1 be a datum point in the partial net, and let the differences in geo-potential numbers (g.p.n.) between the nodal points and O_1 be denoted by $\underline{x}^\alpha$. The observed differences in g.p.n. between the nodal points are f^i, their least-squares corrections $\underline{v}^i$. (The underscore denotes a stochastic quantity.) The observation equations are then:

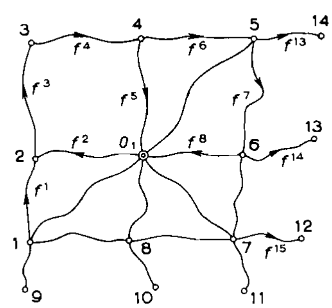

Figure 1.

$$-\underline{x}^1+\underline{x}^2 \quad = f^1+\underline{v}^1$$

$$+\underline{x}^2 \quad = f^2+\underline{v}^2$$

$$-\underline{x}^2+\underline{x}^3 = f^3+\underline{v}^3$$

etc.

or, in matrix notation:

$$\mathbf{A}\underline{\mathbf{x}} = \mathbf{f}+\underline{\mathbf{v}}$$

The weights of the observations are given; let their matrix be **G**. Since it is assumed that there is no correlation between observations, **G** is a diagonal matrix. The normal equations, which can be easily established without writing down the observation equations, are:

$$\mathbf{A}^T\mathbf{G}\mathbf{A}\underline{\mathbf{x}} = \mathbf{A}^T\mathbf{G}\underline{\mathbf{f}}$$

Let

$$(\mathbf{A}^T\mathbf{G}\mathbf{A})^{-1} = \mathbf{Q}$$

Then

$$\underline{\mathbf{x}} = \mathbf{Q}\mathbf{A}^T\mathbf{G}\mathbf{f}$$

and $\mathbf{Q}$ is the matrix of weight coefficients of the variates $\underline{x}^\alpha$.

For a point like 3 in Fig. 1, no unknown need be introduced in the adjustment, because the observations $\underline{f}^3$ and $\underline{f}^4$ can be replaced by a new observation. Let, in Fig. 2, q be such a point and r and s the adjacent nodal points.

We replace $\underline{f}^{rq}$ and $\underline{f}^{qs}$ by $\underline{f}^{rs}$:

$$\underline{f}^{rs} = \underline{f}^{rq} + \underline{f}^{qs}$$

If the weight coefficients are denoted by $\overline{\underline{f}^{rs}, \underline{f}^{rs}}$ etc., we have:

$$\overline{\underline{f}^{rs}, \underline{f}^{rs}} = \overline{\underline{f}^{rq}, \underline{f}^{rq}} + \overline{\underline{f}^{qs}, \underline{f}^{qs}} + 2\,\overline{\underline{f}^{rq}, \underline{f}^{qs}}$$

and, because $\underline{f}^{rq}$ and $\underline{f}^{qs}$ are not correlated:

$$\overline{\underline{f}^{rs}, \underline{f}^{rs}} = \overline{\underline{f}^{rq}, \underline{f}^{rq}} + \overline{\underline{f}^{qs}, \underline{f}^{qs}}$$

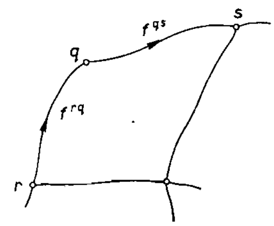

Figure 2.

The weight $g_{rs,\,rs}$ of $\underline{f}^{rs}$, which we can more simply denote by g_{rs} in this case is obtained by reciprocation:

$$g_{rs} = \frac{1}{\overline{\overline{\underline{f}^{rs}, \underline{f}^{rs}}}} = \frac{1}{\overline{\overline{\underline{f}^{rq}, \underline{f}^{rq}} + \overline{\underline{f}^{qs}, \underline{f}^{qs}}}} = \frac{g_{rq} \cdot g_{qs}}{g_{rq} + g_{qs}}$$

in which g_{rq} and g_{qs} are the weights of $\underline{f}^{rq}$ and $\underline{f}^{qs}$.

After the adjustment, corrections to $\underline{f}^{rq}$ and $\underline{f}^{qs}$ can be computed from the correction to $\underline{f}^{rs}$. According to a theorem indicated by J. M. Tienstra and later generally proved by W. Baarda we have:

$$\left.\begin{aligned} \underline{v}^{rq} + \underline{v}^{qs} &= \underline{v}^{rs} \\ \underline{v}^{rq} &= \frac{1}{g_{rq}}\, g_{rs} \cdot \underline{v}^{rs} \\ \underline{v}^{qs} &= \frac{1}{g_{qs}}\, g_{rs} \cdot \underline{v}^{rs} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2.1\text{–}1)$$

It is necessary to find the weight coefficients of the g.p.n. of points like q after the adjustment. We define:

$$\left.\begin{aligned} \frac{g_{rs}}{g_{rq}} &= a_s^q \\ \frac{g_{rs}}{g_{qs}} &= a_r^q \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2.1\text{–}2)$$

It is seen that

$$a_s^q + a_r^q = 1$$

The g.p.n. of q will be denoted by $\underline{y}^q$; it can be found in two ways

$$\underline{y}^q = \underline{x}^r + \underline{f}^{rq} + \underline{v}^{rq}$$

$$\underline{y}^q = \underline{x}^s - \underline{f}^{qs} - \underline{v}^{qs}$$

Multiplying the first equation by a_r^q and the second by a_s^q and adding the results, we obtain

$$\underline{y}^q = a_r^q \underline{x}^r + a_s^q \underline{x}^s + a_r^q \underline{f}^{rq} - a_s^q \underline{f}^{qs}$$

If $\underline{x}^t$ is the g.p.n. of an arbitrary nodal point in the net, $\underline{x}^t$ is a function of the observations, and in particular a function of $\underline{f}^{rs}$, so that we can write

$$\underline{x}^t = b^t \underline{f}^{rs} + \ldots$$

in which b^t is some coefficient.

For the non-diagonal weight coefficient $\overline{y^q, x^t}$ we get:

$$\overline{y^q, x^t} = a_r^q\, \overline{x^r, x^t} + a_s^q\, \overline{x^s, x^t} + \overline{x^t, (a_r^q f^{rq} - a_s^q f^{qs})}$$

We will show that the last term is zero.

Since $\underline{f}^{rq}$ and $\underline{f}^{qs}$ are not correlated to any observation but $\underline{f}^{rs}$, we obtain:

$$\overline{y^q, x^t} = a_r^q\, \overline{x^r, x^t} + a_s^q\, \overline{x^s, x^t} + b^t a_r^q\, \overline{f^{rq}, f^{rs}} - b^t a_s^q\, \overline{f^{qs}, f^{rs}}$$

According to (2):

$$\overline{f^{rq}, f^{rs}} = \overline{f^{rq}, f^{rq}} = \frac{1}{g_{rq}} = \frac{a_s^q}{g_{rs}}$$

Similarly:

$$\overline{f^{qs}, f^{rs}} = \frac{a_r^q}{g_{rs}}$$

Hence

$$\overline{y^q, x^t} = a_r^q\, \overline{x^r, x^t} + a_s^q\, \overline{x^s, x^t} + b^t (a_r^q \cdot a_s^q - a_s^q \cdot a_r^q) \frac{1}{g_{rs}}$$

$$\overline{y^q, x^t} = a_r^q\, \overline{x^r, x^t} + a_s^q\, \overline{x^s, x^t} \quad \ldots\ldots\ldots\ldots \quad (2.1\text{–}3)$$

If $\mathbf{D}$ is a matrix whose row-numbers correspond to the intermediate points, whose column-numbers correspond to the nodal points and whose elements are the numbers a_r^q, it is easily seen that, in matrix notation:

$$\overline{\mathbf{y}, \mathbf{x}} = \mathbf{DQ}$$

If q' is another intermediate point, we have to distinguish two cases:

a. q and q' are points on the same levelling line,

b. q and q' are situated on different lines.

In the first case we can compute $\overline{y^q, y^{q'}}$ as follows:

$$\overline{y^q, y^{q'}} = \overline{(a_r^q x^r + a_s^q x^s), (a_r^{q'} x^r + a_s^{q'} x^s)} + \overline{(a_r^q f^{rq} - a_s^q f^{qs}), (a_r^{q'} f^{rq} - a_s^{q'} f^{qs})}$$

$$\overline{y^q, y^{q'}} = a_r^q a_r^{q'}\, \overline{x^r, x^r} + (a_r^q a_s^{q'} + a_r^{q'} a_s^q)\, \overline{x^r, x^s} + a_s^q a_s^{q'}\, \overline{x^s, x^s} +$$
$$+ a_r^q a_r^{q'}\, \overline{f^{rq}, f^{rq'}} - a_r^q a_s^{q'}\, \overline{f^{rq}, f^{q's}} - a_s^q a_r^{q'}\, \overline{f^{qs}, f^{rq'}} + a_s^q a_s^{q'}\, \overline{f^{qs}, f^{q's}}$$

Now (see Fig. 3):

$$\overline{f^{rq},f^{rq'}} = \overline{f^{rq},f^{rq}} = a^q_s \overline{f^{rs},f^{rs}}$$

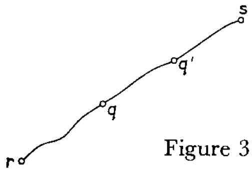

Figure 3.

$$\overline{f^{rq},f^{q's}} = 0$$

$$\overline{f^{qs},f^{rq'}} = \overline{f^{qq'},f^{qq'}} = \overline{f^{rq'},f^{rq'}} - \overline{f^{rq},f^{rq}} = (a^{q'}_s - a^q_s)\overline{f^{rs},f^{rs}}$$

$$\overline{f^{qs},f^{q's}} = \overline{f^{q's},f^{q's}} = a^{q'}_r \overline{f^{rs},f^{rs}}$$

Substituting these results, one obtains:

$$\overline{y^q,y^{q'}} = [a^q_r a^{q'}_r \overline{x^r,x^r} + (a^q_r a^{q'}_s + a^{q'}_r a^q_s)\overline{x^r,x^s} + a^q_s a^{q'}_s \overline{x^s,x^s}] + a^q_s a^{q'}_r \overline{f^{rs},f^{rs}} \quad . \; . \; . \quad (2.1\text{–}4)$$

When q and q' are situated on different lines $r \to s$ and $r' \to s'$ we obtain by a similar derivation

$$\overline{y^q,y^{q'}} = a^q_r a^{q'}_{r'} \overline{x^r,x^{r'}} + a^q_r a^{q'}_{s'} \overline{x^r,x^{s'}} + a^q_s a^{q'}_{r'} \overline{x^s,x^{r'}} + a^q_s a^{q'}_{s'} \overline{x^s,x^{s'}}$$

It is easily seen that, using the matrix **D** defined before, we have in general:

$$\overline{\mathbf{y},\mathbf{y}} = \mathbf{DQD}^{\mathrm{T}}$$

For points situated on the same line, an addition according to (2.1–4) has to be made.

We finally have to compute the weight coefficients of the g.p.n.'s of the junction-points with other partial nets, *e.g.* 14 in Figure 1. Denoting these g.p.n.'s by z, we see:

$$\underline{z}^{14} = \underline{x}^5 + \underline{f}^{13}$$

$$\overline{z^{14},z^{14}} = \overline{x^5,x^5} + \overline{f^{13},f^{13}}$$

$$\overline{z^{14},x^4} = \overline{(x^5+f^{13}),x^4} = \overline{x^5,x^4}$$

$$\overline{z^{14},z^{13}} = \overline{(x^5+f^{13}),(x^6+f^{14})} = \overline{x^5,x^6}$$

etc.

The matrix of weight coefficients of all the variates $\underline{x}$, $\underline{y}$ and $\underline{z}$ is finally

$$\mathbf{B} = \begin{pmatrix} \mathbf{Q} & \mathbf{S} & \mathbf{W} \\ \mathbf{S}^{\mathrm{T}} & \mathbf{R} & \mathbf{V} \\ \mathbf{W}^{\mathrm{T}} & \mathbf{V}^{\mathrm{T}} & \mathbf{T} \end{pmatrix}$$

in which

Q has been defined before

R or $\overline{\mathbf{y},\mathbf{y}}$ contains elements like $\overline{y^3,y^3}$

S or $\overline{\mathbf{x},\mathbf{y}}$ contains elements like $\overline{x^4,y^3}$

T or $\overline{\mathbf{z},\mathbf{z}}$ contains elements like $\overline{z^{14},z^{13}}$ and $\overline{z^{14},z^{14}}$

V or $\overline{\mathbf{y},\mathbf{z}}$ contains elements like $\overline{y^3,z^{14}}$

W or $\overline{\mathbf{x},\mathbf{z}}$ contains elements like $\overline{z^{14},x^4}$

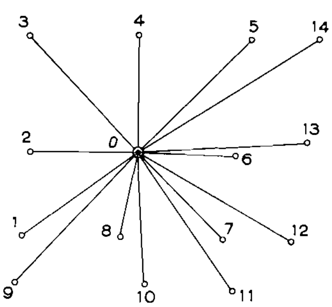

Figure 4.

All information concerning the adjusted partial net can now be expressed in the new variates $\underline{x}$, $\underline{y}$ and $\underline{z}$ and their matrix of weight coefficients **B**.

The situation is represented in Fig. 4.

Remark. If the partial net had been adjusted according to Standard Problem I, we would have arrived at exactly the same result by writing the variates $\underline{x}$, $\underline{y}$ and $\underline{z}$ as functions of adjusted observations, *e.g.*:

$$\underline{x}^5 = -(\underline{f}^5+\underline{v}^5) + (\underline{f}^6+\underline{v}^6)$$

$$\underline{x}^6 = -(\underline{f}^8+\underline{v}^8)$$

etc.

and then computing the weight coefficients of these functions by the appropriate formulae of least squares theory. The numerical establishment of the functions is an easy but rather tedious work, in which great care must be taken to avoid blunders. Standard Problem II was chosen for the first step mainly to avoid this work. The resulting matrix of normal equations had a higher order than the one that would have resulted from a treatment according to Standard Problem I, but since it could still be inverted, without partitioning, by the machine used, this offered no inconvenience.

Second step. The unification of the partial nets was done by applying Standard Problem I for correlated observations. In Fig. 5, three partial nets have been drawn. There is no more reason to distinguish the different types of observations $\underline{x}$, $\underline{y}$ and $\underline{z}$ from each other, and we shall denote by $\underline{p}$ the vector of all observations $\underline{x}$, $\underline{y}$ and $\underline{z}$ of all partial nets together.

The conditions are easily established, *e.g.*:

$$-(\underline{p}^{13}+\underline{\varepsilon}^{13}) + (\underline{p}^{14}+\underline{\varepsilon}^{14}) - (\underline{p}^{15}+\underline{\varepsilon}^{15}) + (\underline{p}^{16}+\underline{\varepsilon}^{16}) = 0$$

$$-(\underline{p}^{12}+\underline{\varepsilon}^{12}) + (\underline{p}^{13}+\underline{\varepsilon}^{13}) - (\underline{p}^{16}+\underline{\varepsilon}^{16}) + (\underline{p}^{17}+\underline{\varepsilon}^{17}) = 0$$

etc.

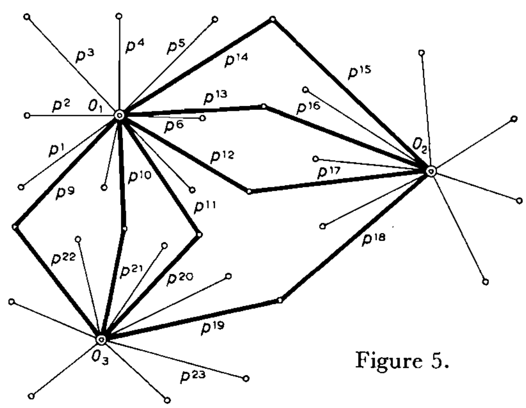

Figure 5.

or, in matrix notation:

$$\mathbf{U}(\underline{\mathbf{p}}+\underline{\boldsymbol{\varepsilon}}) = \mathbf{0}$$

Hence:

$$\mathbf{U}\underline{\boldsymbol{\varepsilon}} = \mathbf{0}-\mathbf{U}\underline{\mathbf{p}} = \underline{\mathbf{t}}$$

The matrix of weight coefficients $\mathbf{C}$ of the variates $\underline{p}$ is known from the first step:

$$\overline{\mathbf{p}, \mathbf{p}} = \mathbf{C} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_3 \end{pmatrix}$$

in which $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$, denote the matrices of weight coefficients of the "observations" $\underline{p}$ in the first, second and third net respectively.

There is no correlation between variates belonging to different partial nets, so the "non-diagonal" sub-matrices in $\mathbf{C}$ are zero.

The normal equations are

$$\mathbf{U}\mathbf{C}\mathbf{U}^{\mathrm{T}}\underline{\mathbf{k}} = \underline{\mathbf{t}}$$

From this we obtain the correlates

$$\underline{\mathbf{k}} = (\mathbf{UCU}^{\mathsf{T}})^{-1}\underline{\mathbf{t}}$$

The corrections are:

$$\underline{\boldsymbol{\epsilon}} = \mathbf{CU}^{\mathsf{T}}\underline{\mathbf{k}}$$

The matrix of weight coefficients of the correlates is

$$\overline{\mathbf{k}, \mathbf{k}} = (\mathbf{UCU}^{\mathsf{T}})^{-1}$$

and consequently that of the corrections

$$\overline{\boldsymbol{\epsilon}, \boldsymbol{\epsilon}} = \mathbf{CU}^{\mathsf{T}}\,\overline{\mathbf{k}, \mathbf{k}}\,\mathbf{UC}$$

The matrix of weight coefficients of the adjusted observations is (see *e.g.* [13], p. 112):

$$\overline{(\mathbf{p}+\boldsymbol{\epsilon}), (\mathbf{p}+\boldsymbol{\epsilon})} = \overline{\mathbf{p}, \mathbf{p}} - \overline{\boldsymbol{\epsilon}, \boldsymbol{\epsilon}} = \mathbf{C}-\mathbf{CU}^{\mathsf{T}}(\mathbf{UCU}^{\mathsf{T}})^{-1}\mathbf{UC}$$

The ultimate aim of the adjustment is to find the adjusted differences in g.p.n. c between each point and a certain datum point. Let this datum point, which serves as a reference point for the whole net, be O_1.

Then it is seen in Fig. 5 that, *e.g.*:

$$\underline{c}^{3} = \underline{p}^{3}+\underline{\varepsilon}^{3}$$

$$\underline{c}^{14} = \underline{p}^{14}+\underline{\varepsilon}^{14}$$

$$\underline{c}^{23} = (\underline{p}^{10}+\underline{\varepsilon}^{10}) - (\underline{p}^{21}+\underline{\varepsilon}^{21}) + (\underline{p}^{23}+\underline{\varepsilon}^{23})$$

etc.

In matrix notation:

$$\underline{\mathbf{c}} = \boldsymbol{\Lambda}(\underline{\mathbf{p}}+\underline{\boldsymbol{\epsilon}})$$

The matrix of weight coefficients $\overline{\mathbf{c}, \mathbf{c}}$ is then

$$\overline{\mathbf{c}, \mathbf{c}} = \boldsymbol{\Lambda}\,\overline{(\mathbf{p}+\boldsymbol{\epsilon}), (\mathbf{p}+\boldsymbol{\epsilon})}\,\boldsymbol{\Lambda}^{\mathsf{T}} = \boldsymbol{\Lambda}\mathbf{C}\boldsymbol{\Lambda}^{\mathsf{T}}-\boldsymbol{\Lambda}\mathbf{CU}^{\mathsf{T}}(\mathbf{UCU}^{\mathsf{T}})^{-1}\mathbf{UC}\boldsymbol{\Lambda}^{\mathsf{T}}$$

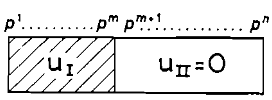

Figure 6.

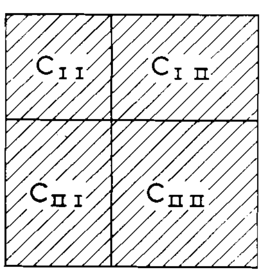

Figure 7.

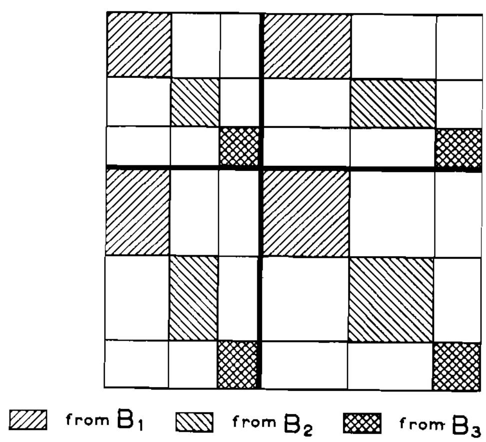

Figure 8.

Most of the variates $\underline{p}$ will have a coefficient zero in all the condition equations. Because they are correlated with variates whose coefficient is not zero in all the condition equations, they will receive corrections from the second step of the adjustment. Up to and including the computation of the correlates the second step can be treated as if it pertained only to the variates whose coefficient is not zero in all condition equations (in Fig. 3 those indicated by a heavy line). This is easily seen if we re-arrange the order of the variates, so that $p^1, \ldots, p^m$ are the ones that have non-zero coefficients, and $p^{m+1}, \ldots, p^n$ the ones with only zero coefficients. The matrix $\mathbf{U}$ is then built as in the diagram in Fig. 6. The matrix of weight coefficients is as in Fig. 7.

It is easily seen that $\mathbf{UCU}^T = \mathbf{U}_I\mathbf{C}_{II}\mathbf{U}_I^T$. In the computation one will of course as much as possible take advantage of the many zero elements in the different matrices. The matrix $\mathbf{C}$ deserves special attention in this respect. If the above-mentioned re-arrangement of variates is made, $\mathbf{C}$ is built up as in Fig. 8 (considering the example of three partial nets).

The formulae for the total adjustment are recapitulated on page 9 in matrix notation, for the case of four partial nets.

Remark. The second step could also have been performed by the method of Standard Problem II or rather by what TIENSTRA has called Standard Problem IV: condition equations containing unknowns.

If in Fig. 5 unknowns $\underline{X}_2$ and $\underline{X}_3$ are introduced for the differences in g.p.n. between O_2 and O_1 and between O_3 and O_1 respectively, we have:

$$\underline{X}_2 = (\underline{p}^{14}+\underline{\varepsilon}^{14}) - (\underline{p}^{15}+\underline{\varepsilon}^{15})$$

$$\underline{X}_2 = (\underline{p}^{13}+\underline{\varepsilon}^{13}) - (\underline{p}^{16}+\underline{\varepsilon}^{16})$$

. . . .

$$\underline{X}_3 = (\underline{p}^{9}+\underline{\varepsilon}^{9}) - (\underline{p}^{22}+\underline{\varepsilon}^{22})$$

$$\underline{X}_3 = (\underline{p}^{10}+\underline{\varepsilon}^{10}) - (\underline{p}^{21}+\underline{\varepsilon}^{21})$$

. . . .

$$\underline{X}_2-\underline{X}_3 = (\underline{p}^{19}+\underline{\varepsilon}^{19}) - (\underline{p}^{18}+\underline{\varepsilon}^{18})$$

etc.

Introducing new variates in the right hand members:

$$\underline{l}^1 = \underline{p}^{14}-\underline{p}^{15}$$

$$\underline{l}^2 = \underline{p}^{13}-\underline{p}^{16}$$

etc.

or, in general

$$\underline{\mathbf{l}} = \mathbf{V}\underline{\mathbf{p}}$$

we deduce for their matrix of weight coefficients

$$\overline{\mathbf{l},\mathbf{l}} = \mathbf{V}\,\overline{\mathbf{p},\mathbf{p}}\,\mathbf{V}^T$$

By inversion we obtain the matrix of weights, after which the whole problem can be treated as Standard Problem II for correlated observations. From the corrections to $\underline{\mathbf{l}}$, the corrections to $\underline{\mathbf{p}}$ can be derived, the final differences with respect to O_1 are, *e.g.*:

$$\underline{c}^5 = \underline{p}^5 + \underline{\varepsilon}^5$$

$$\underline{c}^{23} = \underline{X}_3 + \underline{p}^{23} + \underline{\varepsilon}^{23}$$

etc.

An application of the law of propagation of weight coefficients results in the weight coefficients of the variates $\underline{c}$. Although this application is simplified by the well-known properties that any $\underline{X}$ is not correlated to any $\underline{\varepsilon}$ and that

$$\overline{(p^i+\varepsilon^i),\,(p^k+\varepsilon^k)} = \overline{p^i,\,p^k} - \overline{\varepsilon^i,\,\varepsilon^k}$$

the number of matrices involved and their computation make this solution somewhat complicated from an organizational point of view. Therefore the method of condition equations was used in the second step. Test computations indicated that the total amount of purely computational work for both methods is about equal, at least in small nets.

RECAPITULATION OF FORMULAE

1st step	Observations	$\underline{\mathbf{f}}$
	Weights	$\mathbf{G}$ (diagonal)
	Observation equations	$\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{f}} + \underline{\mathbf{v}}$
	Normal equations	$\mathbf{A}^T\mathbf{G}\mathbf{A}\underline{\mathbf{x}} = \mathbf{A}^T\mathbf{G}\underline{\mathbf{f}}$
	Inversion	$(\mathbf{A}^T\mathbf{G}\mathbf{A})^{-1} = \mathbf{Q}$
	Solution	$\underline{\mathbf{x}} = \mathbf{Q}\mathbf{A}^T\mathbf{G}\underline{\mathbf{f}}$ (nodal points only)
	"Sum of squares"	$\underline{\mathbf{E}}_I = \underline{\mathbf{v}}^T\mathbf{G}\underline{\mathbf{v}} = \underline{\mathbf{f}}^T\mathbf{G}\underline{\mathbf{f}} - \mathbf{A}^T\mathbf{G}\underline{\mathbf{f}}\underline{\mathbf{x}}$
	Intermediate points	$\underline{\mathbf{y}}$
	Antennae-points	$\underline{\mathbf{z}}$
	Total partial net i:	$\underline{\mathbf{p}}_i = \begin{pmatrix}\underline{\mathbf{x}}\\ \underline{\mathbf{y}}\\ \underline{\mathbf{z}}\end{pmatrix}$
	Weight coefficients	$\mathbf{B} = \overline{\mathbf{p}_i,\,\mathbf{p}_i} = \begin{pmatrix}\overline{\mathbf{x},\mathbf{x}} = \mathbf{Q} & \overline{\mathbf{x},\mathbf{y}} = \mathbf{D}\mathbf{Q} & \overline{\mathbf{x},\mathbf{z}}\\ \overline{\mathbf{y},\mathbf{x}} & \overline{\mathbf{y},\mathbf{y}} = \mathbf{D}\mathbf{Q}\mathbf{D}^T & \overline{\mathbf{y},\mathbf{z}}\\ \overline{\mathbf{z},\mathbf{x}} & \overline{\mathbf{z},\mathbf{y}} & \overline{\mathbf{z},\mathbf{z}}\end{pmatrix}$
2nd step	Observations	$\underline{\mathbf{p}} = \begin{pmatrix}\underline{\mathbf{p}}_1\\ \underline{\mathbf{p}}_2\\ \underline{\mathbf{p}}_3\\ \underline{\mathbf{p}}_4\end{pmatrix}$

Weight coefficients
$$\mathbf{C} = \overline{\mathbf{p}, \mathbf{p}} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_4 \end{pmatrix}$$

Conditions
$$\mathbf{U}(\underline{\mathbf{p}}+\underline{\boldsymbol{\epsilon}}) = \mathbf{0}$$
$$\mathbf{U}\underline{\boldsymbol{\epsilon}} = -\mathbf{U}\underline{\mathbf{p}} = \underline{\mathbf{t}}$$

Normal equations
$$\mathbf{UCU}^{\mathrm{T}}\underline{\mathbf{k}} = \underline{\mathbf{t}}$$

Correlates
$$\underline{\mathbf{k}} = (\mathbf{UCU}^{\mathrm{T}})^{-1}\,\underline{\mathbf{t}}$$

Corrections
$$\underline{\boldsymbol{\epsilon}} = \mathbf{CU}^{\mathrm{T}}\,\underline{\mathbf{k}}$$

Weight coefficients
$$\overline{\mathbf{k}, \mathbf{k}} = (\mathbf{UCU}^{\mathrm{T}})^{-1}$$
$$\overline{\boldsymbol{\epsilon}, \boldsymbol{\epsilon}} = \mathbf{CU}^{\mathrm{T}}\,\overline{\mathbf{k}, \mathbf{k}}\,\mathbf{UC}$$

"Sum of squares"
$$\underline{\mathbf{E}}_{\mathrm{II}} = \underline{\boldsymbol{\epsilon}}^{\mathrm{T}}\mathbf{C}^{-1}\,\underline{\boldsymbol{\epsilon}} = \underline{\mathbf{k}}^{\mathrm{T}}\underline{\mathbf{t}}$$

Weight coefficients of adjusted observations
$$\overline{(\mathbf{p}+\boldsymbol{\epsilon}), (\mathbf{p}+\boldsymbol{\epsilon})} = \overline{\mathbf{p}, \mathbf{p}} - \overline{\boldsymbol{\epsilon}, \boldsymbol{\epsilon}}$$
$$\overline{(\mathbf{p}+\boldsymbol{\epsilon}), (\mathbf{p}+\boldsymbol{\epsilon})} = \mathbf{C}-\mathbf{CU}^{\mathrm{T}}(\mathbf{UCU}^{\mathrm{T}})^{-1}\mathbf{UC}$$

Functions
$$\underline{\mathbf{c}} = \boldsymbol{\Lambda}(\underline{\mathbf{p}}+\underline{\boldsymbol{\epsilon}})$$

Weight coefficients
$$\overline{\mathbf{c}, \mathbf{c}} = \boldsymbol{\Lambda}\,\overline{(\mathbf{p}+\boldsymbol{\epsilon}), (\mathbf{p}+\boldsymbol{\epsilon})}\,\boldsymbol{\Lambda}^{\mathrm{T}}$$
$$\overline{\mathbf{c}, \mathbf{c}} = \boldsymbol{\Lambda}\mathbf{C}\boldsymbol{\Lambda}^{\mathrm{T}}-\boldsymbol{\Lambda}\mathbf{CU}^{\mathrm{T}}(\mathbf{UCU}^{\mathrm{T}})^{-1}\mathbf{UC}\boldsymbol{\Lambda}^{\mathrm{T}}$$

2.2 **Execution and results**

The partition of the net was done as shown in Fig. 9 (see folding page at the end of this publication). This choice was made for several reasons:

a. These nets have a convenient size with a view to the capacity of the computing machine used.
b. It was expected that these parts were fairly homogeneous as far as their reliability is concerned.
c. This choice was also suggested by geographical features.

The datum points chosen were:

Partial net D: 301, the N.A.P. datum point at Amsterdam, which is also the datum point for the total net.
Partial net F: 339
Partial net C: 603
Partial net E: 429

The observational values were taken from the documents furnished by the President of the Permanent Commission on European Levellings. The weights of the observations were obtained from the same documents by inverting the weight coefficients $1/P$ given there. These data have been established by using the different values of the variance t^2 of 1 km levelling given by the various countries, expressed in mm²/km (see [12], page 25). But the adjustment is executed with observations expressed in geopotential units, so that the use of variances expressed in mm²/km is theoretically not correct. However, the difference has little practical significance.

The adjusted differences after the first step in g.p.n. between the points of each partial net and the respective datum points are given in Table I *), column 2. The corrections to these differences resulting from the second step are listed in Table I, column 3.

The final geo-potential numbers of the points with respect to the Amsterdam datum are listed in Table I, column 4.

The corrections from the second step indicate the influence which the partial nets exert on each other. A clear example is the partial net E, which is tilted by the connection to the other nets: the western part is depressed, the depression diminishes from 0.042 g.p.u. in 416 to 0.005 g.p.u. in JM-42. The eastern part is lifted by an amount diminishing from 0.048 g.p.u. in 420 to 0.002 g.p.u. in 405. A "neutral axis" may be imagined to lie between 417 and 418, between 421 and 422, through 429, between 427 and 428, ending between JM-44 and 406.

The corrections from the second step are biggest along the edges of the partial nets. It should be noted that the dividing lines often lie across polygons with large misclosures, in the Pyrenees and the Alps. The decrease of the corrections for points away from the edges is illustrative (corrections in 10^{-3} g.p.u.):

In partial net F:					
Point	313	312	333	332	
Correction from 2nd step	+36.8	+22.4	+6.2	+4.1	
In partial net C:					
Point	JM-47	518	JM-48	517	516
Correction from 2nd step	+105.3	+15.4	+8.8	+2.5	+1.3
In partial net E:					
Point	420	419	402	403	404
Correction from 2nd step	+48.3	+23.1	+23.7	+9.6	+4.9

*) The tables with roman numbering are printed on the pages 42 ff.

3 THE PRECISION AND THE ACCURACY

3.1 **Tests on model errors**

The four partial nets can be assumed to be fairly homogeneous as far as their accuracy*) is concerned. One may now ask the following questions:

1. Are there such differences between the results of the partial nets that a conclusion can be drawn on the occurrence of model- or systematic errors in levelling?
2. Have the different countries given a good estimation of t^2 (variance per km levelling line)?

An objective guide to the answers may be obtained by statistical tests.

A basic assumption is that our original observations are normally distributed and mutually independent in the probability sense. Let σ^i be the standard deviation of an observational quantity $\underline{p}^i$. Weights g_{ii} are determined by introducing the constant *variance factor* σ^2, according to:

$$(\sigma^i)^2 = \frac{\sigma^2}{g_{ii}}$$

The weight formula for the original observations is

$$g_{ii} = \frac{200}{L_i t^2}$$

t^2 is an estimate of the square of the standard deviation of 1 km levelling. The way it is evaluated is described in [12]. If t^2 has been computed from a large number of levelling observations, it is such a good estimate that it can be considered in practice as the "true" variance of 1 km levelling. For a discussion on the meaning of a statement of this kind, reference is made to [4], page 4. Lt^2 is then the variance of a section L km long. It follows when using this weight formula, that the variance factor is

$$\sigma^2 = 200$$

If the observations are adjusted by the method of least squares to fulfill b conditions, corrections $\underline{v}^i$ are found. The quantity

$$\underline{E} = [g_{ii}\underline{v}^i\underline{v}^i]$$

can then be computed; in the diagram on page 9 it has been called "sum of squares" and the well-known formulae for its computation have been added. Of course the notation $[g_{ii}\underline{v}^i\underline{v}^i]$ is not appropriate in the case of correlated observations.

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*) The terms *precision* and *accuracy* are used in accordance with the definitions given by CHURCHILL EISENHART in: The reliability of measured values, Photogrammetric Engineering, Vol. 18, page 545. See also: M. G. KENDALL and W. R. BUCKLAND, A dictionary of statistical terms, 2nd ed., Edinburgh, 1960.

As is well-known, see *e.g.*, reference [2], the quantity

$$\underline{\hat{\sigma}}^2 = \frac{\underline{E}}{b}$$

is an unbiased estimator of σ^2.

The stochastic quantity

$$\underline{F}_{b,\infty} = \frac{\underline{\hat{\sigma}}^2}{\sigma^2} \quad \text{(3.1–1)}$$

has a known probability distribution (a distribution of FISHER-SNEDECOR) which is dependent on b only; its mean (expectation) is 1.

If two stochastically independent estimates $\hat{\sigma}_p^2$ and $\hat{\sigma}_q^2$ of the same σ^2 have been made after adjusting on b_p and b_q conditions respectively, we can use the quantity

$$\underline{F}_{b_p, b_q} = \frac{\underline{\hat{\sigma}}_p^2}{\underline{\hat{\sigma}}_q^2} \quad \text{(3.1–2)}$$

which has also a FISHER-SNEDECOR distribution, that is dependent only on the number of degrees of freedom b_p and b_q.

Since the distributions corresponding to (3.1–1) and (3.1–2) are known and have been tabulated we can carry out F-tests using the found values of $\underline{\hat{\sigma}}^2$.

It can be shown that if model- or systematic errors occur, the expectation of $\underline{\hat{\sigma}}^2$ is greater than σ^2, so that in general the value found for $\underline{\hat{\sigma}}^2$ is greater than σ^2.

Therefore, using the relation (3.1–1) we will generally find in this case:

$$F_{b,\infty} > 1$$

Using a 5% rejection or critical region in the right-hand tail of the F-distribution, containing all F-values greater than a so-called critical value $F_{0.95;b,\infty}$, we are inclined to reject the hypothesis of non-occurrence of model errors if

$$F_{b,\infty} = \frac{\hat{\sigma}^2}{\sigma^2} > F_{0.95;b,\infty}$$

This type of test is called a one-sided test.

If on the other hand the estimation obtained for t^2 was too low, it can be shown that the expectation of $\underline{E}$ is too high, so that this may also give rise to a value of $\hat{\sigma}^2$ that is too high. Consequently the testing on model errors may be mixed up with the effect of a wrong estimation of t^2.

By comparing the values of $\hat{\sigma}^2$ in the different partial nets or in different steps of the adjustment, we obtain a possibility of investigating the effect of these two factors. For this investigation, (3.1–2) is used, which leads in principle to a two-sided test with a $2^1/_2$% critical region in the left and right tail of the F-distribution. By using the greatest of the two $\hat{\sigma}$'s in the numerator, we can use a one-sided test, with a $2^1/_2$% critical region in the right tail only.

The different values $\hat{\sigma}^2$ that were found, the resulting F-values and the corresponding critical F-values are listed in Tables A and B. The critical values were found in [7], Table VII. Table A contains one-sided tests, Table B two-sided tests; all tests are on the 5% level of significance.

TABLE A

Net	Number of conditions b	$[gvv]$	$\hat{\sigma}^2$	σ^2	$F_{b,\infty} = \frac{\hat{\sigma}^2}{\sigma^2}$	$F_{0.95;b,\infty}$
D	14	8038	574	200	2.87	1.68
E	14	5341	382	200	1.91	1.68
C	11	2983	271	200	1.36	1.80
F	15	4048	270	200	1.35	1.65
I	54	20410	378	200	1.89	1.33
II	14	4743	339	200	1.69	1.68
I+II	68	25153	370	200	1.85	1.30

I denotes the total first step of the adjustment.
II denotes the second step.
I+II denotes the total adjustment.

TABLE B

p–q	$\frac{\hat{\sigma}_p^2}{\hat{\sigma}_q^2} = F_{b_p, b_q}$	$F_{0.975; b_p, b_q}$
D–E	1.50	2.97
D–C	2.12	3.45
D–F	2.13	2.90
D–II	1.69	2.97
E–C	1.41	3.45
E–F	1.41	2.90
E–II	1.13	2.97
C–F	1.00	3.05
II–C	1.25	3.35
II–F	1.26	2.90
I–II	1.12	2.65

Starting with an analysis of Table A we see that the nets D and E, the total first step, the second step and the adjustment as a whole lead to a rejection of the hypothesis of non-occurrence of model-errors, or to the assumption that several of the estimates of t^2 given by the different nations, are too small. It is remarked that all F-values in Table A are greater than one, which fact also indicates that the conclusion drawn above is based on good evidence.

The test for the partial net F has been given for completeness only; it has no real meaning because according to [12], page 39, t^2 for this net has been computed from the polygons contributed to U.E.L.N. themselves. The resulting dependency invalidates the test.

If we now analyse Table B, it appears that the occurrence of model errors is not very likely and that the results of Table A are more probably due to the value given for t^2 being too small. The fact that all F-values are greater than one need not cause surprise; this is a consequence of the method of testing in which the greatest value of $\hat{\underline{\sigma}}^2$ is always put in the numerator. Nor does Table B give rise to the conclusion that the used formula for the weights, inversely proportional to the length of the levelling line, is wrong.

In Table B the different partial nets are compared among themselves and with the second step, and the whole first step with the second. The described test is not suited for comparing the partial nets with the first step as a whole or with the whole adjustment, since the required stochastic independence is not then present. The result of these tests is that there are no significant differences.

In conclusion, it is evident that the value $\sigma^2 = 200$ cannot be used for evaluating the precision of the results of the adjustment, since the estimate $\hat{\sigma}^2$ computed from the adjustment is significantly higher. This value $\hat{\sigma}^2 = 370$ will therefore be used in the sequel to compute standard deviations of adjusted geo-potential numbers.

Apart from the described tests, which follow from the method of adjustment, one can easily test each polygon separately. Let $\underline{t}^\varrho$ be the misclosure of polygon nr. ϱ. The weight coefficient $g^{\varrho\varrho}$ is the sum of the weight coefficients of the sections of the polygon; the weight of $\underline{t}^\varrho$ is $\frac{1}{g^{\varrho\varrho}} = g_{\varrho\varrho}$. From the adjustment of the polygon follows the estimator

$$\hat{\underline{\sigma}}^2 = \frac{g_{\varrho\varrho}\underline{t}^\varrho\underline{t}^\varrho}{1} = g_{\varrho\varrho}(\underline{t}^\varrho)^2$$

while

$$\frac{\hat{\underline{\sigma}}^2}{\sigma^2} = \underline{F}_{1,\infty}$$

The critical value for a test on the 5% level of significance is

$$F_{0.95;\,1,\infty} = 3.84$$

so that a model error in the polygon may be suspected if

$$g_{\varrho\varrho}(t^\varrho)^2 > 3.84 \times 200$$

$$|t^\varrho| > \sqrt{768 \cdot g^{\varrho\varrho}}$$

Nine out of 68 polygons turn out to have a significantly too large misclosure, namely

	t_ϱ	Critical value
Nr. 2001 (nodal points 201, 202, 226, 227, 228)	$+\ 39.31 \cdot 10^{-3}$ gpu	26.41
Nr. 2010 („ „ 207, 208, 209)	$-\ 27.85$	23.43
Nr. 3002 („ „ 305, 306, 307, 308, 309, 310, 329)	-204.34	154.60
Nr. 4003 („ „ 401, 402, 419, 420)	$+123.91$	122.34
Nr. 4012 („ „ 435, 436, 437, 438, JM-42)	$+\ 69.36$	36.95
Nr. 9003 („ „ 301, 302, 303, 219, 220, 221)	$-\ 65.11$	44.64
Nr. 9009 („ „ 313, 314, 519, 508, 507, 501)	-152.92	126.73
Nr. 9010 („ „ 314, 315, 316, 518, 519)	-179.76	129.81
Nr. 9013 („ „ 601, 602, 603, 511, 510, 505, 504)	$+\ 75.74$	69.37

3.2 Precision of results

The weight coefficients of the adjusted geo-potential numbers after the total adjustment were computed according to the method explained in Section 2.1. Not all the weight coefficients are printed in this report, but only the most important ones, those pertaining to the mareograph stations. These stations are denoted by RM, and their geo-potential numbers are obtained by adding the observed g.p.n. differences of the antennae JM–RM to the g.p.n. of the JM points as found from the adjustment. The weight coefficients of the g.p.n. of the RM points are found in the same way as described in Section 2.1 for antennae-points. They are given in Table II (see folding page at the end of this publication); the unit is 10^{-6} (g.p.u.)^2. The weight coefficients pertaining to the mareograph stations of the Northern Block have been computed by using the weights furnished by the Commission.

By multiplying the diagonal weight coefficients by the variance factor 370, one obtains the squares of the standard deviations of the geo-potential numbers of the mareograph stations with respect to the Amsterdam datum. The square of the standard deviation of the g.p.n.-difference between two mareographs, $\underset{\sim}{c}^{\alpha} - \underset{\sim}{c}^{\beta}$ is, according to the law of propagation of weight coefficients:

$$\sigma^2_{(c^\alpha - c^\beta)} = (g^{\alpha\alpha} + g^{\beta\beta} - 2g^{\alpha\beta}) \cdot 370$$

in which $g^{\alpha\alpha}$ and $g^{\beta\beta}$ denote the diagonal weight coefficients pertaining to $\underset{\sim}{c}^{\alpha}$ and $\underset{\sim}{c}^{\beta}$, and $g^{\alpha\beta}$ their non-diagonal weight coefficient.

When interpreting the thus obtained standard deviation it should be borne in mind that the value 370 is an estimate. We may obtain a 95% confidence interval for the mean $\tilde{\sigma}^2$ of $\hat{\sigma}^2$ by means of the following critical values obtained from [7], Table VII:

$$F_{0.975;\,68,\,\infty} = 1.36$$

$$F_{0.975;\,\infty,\,68} = 1.44$$

The resulting 95% confidence interval is

$$\frac{370}{1.36} < \tilde{\sigma}^2 < 1.44 \cdot 370$$

$$272 < \tilde{\sigma}^2 < 533$$

This illustrates the uncertainty of the computed standard deviations.

The use of 370 as variance factor in the Northern Block results in standard deviations of the g.p.n.-differences that are some 30% greater than the ones published by E. Kääriäinen in [8], Table 4. Dr. Kääriäinen used $(15.08)^2 \approx 227$ for the Finnish net and $(16.34)^2 \approx 267$ for the Norwegian net; these values include an allowance for the effect of land uplift and both are based on the estimate $\hat{\sigma}^2 = (14.55)^2 \approx 212$ resulting from the adjustment of the 23 loops of the Northern Block.

Since

$$\frac{370}{212} \approx 1.75 \quad \text{and} \quad F_{0.95;\,68,\,23} \approx 1.85$$

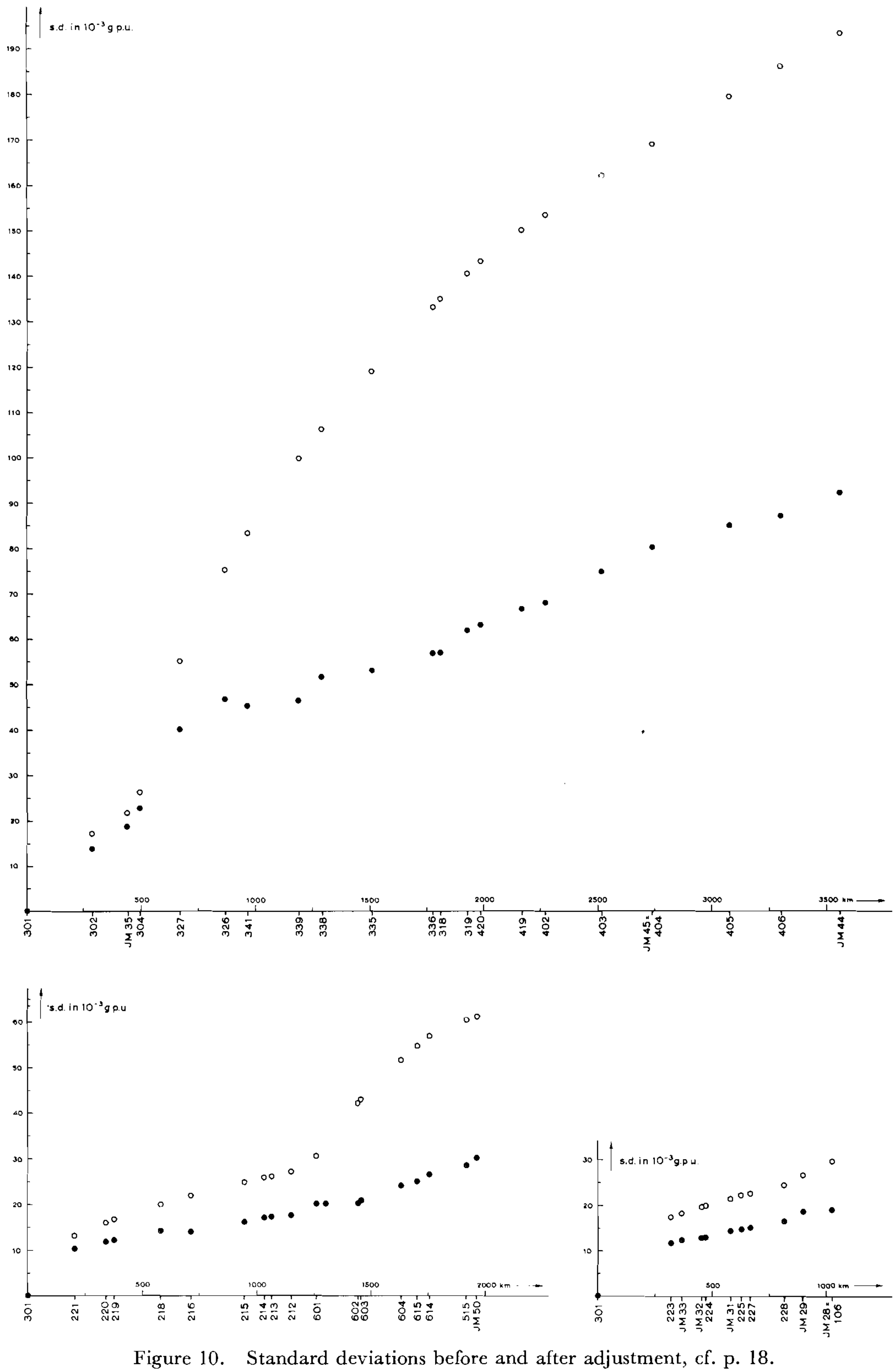

Figure 10. Standard deviations before and after adjustment, cf. p. 18.

the difference between the variance factors is not quite significant, so that the above-mentioned difference of 30% is rather acceptable in view of the uncertainties of the estimates. The resulting standard deviations are likely to be on the high side as far as the Northern Block is concerned, but in view of the isolated position of the constituent parts of this Block, this can be considered as a welcome safety-margin which prevents over-estimation of the accuracy.

In Fig. 10 the increase of the standard deviations in different directions, before and after the adjustment, is pictured. The values before the adjustment have been obtained by considering the observations along the most direct route between the points considered, multiplying the sum of their weight coefficients (as furnished by the Commission) by 370 and taking the square root.

Fig. 11 (see folding page at the end of this publication) pictures the standard deviations in the whole Central Block by means of contour lines of a surface whose height at a certain nodal point is equal to the standard deviation of the g.p.n. of that point. Its visualises roughly the precision of the net; outside the mentioned discrete points the surface has of course no significance.

3.3 **Power of the tests used**

In Section 3.1, *F*-tests were used to investigate whether model errors made themselves felt in the adjustment. The level of significance adopted was 5%, which means that there is a probability of 5% to make a "Type I error", *i.e.* to reject a true null hypothesis. We will now deal with the following question: if our null hypothesis is wrong, what probability do we have of nevertheless accepting it and make a "Type II error"?? In other words, what size can a model error attain before it leads (with a certain specified probability) to a significantly too high *F*-value? This question, which forms an essential part of the application of statistical tests, has been treated from a geodetic point of view by W. BAARDA in [4], page 21 ff. By "model error" we mean now a systematic error that manifests itself in the misclosures, and it is convenient to think of a gross error or blunder, because we will consider cases where one observation is "falsified". If we think of one polygon, a very large error in one of its sections will almost certainly lead to rejection of the null hypothesis that no model-errors are present, because it will result in a very large misclosure. A small error, that has the size of the standard deviation of the observation in question will very seldom lead to rejection: the probability β of detecting an error is dependent on its size. This probability β is called the *power* of the test with respect to the *alternative hypothesis* that an error of the given size occurs (this hypothesis is an alternative to the null hypothesis).

We may now fix β, *e.g.* $\beta = 0.8$ and derive the size of the corresponding error, which consequently is the size an error has to attain to be detected with 80% probability ("be detected" means here: lead to rejection in the test. The test cannot in general indicate the particular observation affected by the error.)

We will use the notation which is customary in tensor analysis, see [13]. Let us consider observations $\underline{p}^i (i = 1, \ldots, n)$, subject to b conditions. The weights are g_{ik}, the variance factor is σ^2.

The conditions are

$$u_i^\varrho(\underline{p}^i+\underline{\varepsilon}^i) = u_0^\varrho \qquad (i, k, l = 1, \ldots, m)$$

$$u_i^\varrho\underline{\varepsilon}^i = u_0^\varrho - u_i^\varrho\underline{p}^i = \underline{t}^\varrho \qquad (\varrho, \sigma = 1, \ldots, b; b < m)$$

The weights are g_{ik}, the weight coefficients g^{kl} if

$$g_{ik}g^{kl} = \delta_i^l = \begin{cases} 1 \text{ if } l = i \\ 0 \text{ if } l \neq i \end{cases}$$

Normal equations

$$g^{ik}u_i^\varrho u_k^\sigma \underline{k}_\sigma = \underline{t}^\varrho$$

Put

$$g^{ik}u_i^\varrho u_k^\sigma = g^{\varrho\sigma}$$

and

$$g^{\varrho\sigma}g_{\sigma\tau} = \delta_\tau^\varrho$$

Then

$$\underline{k}_\sigma = g_{\sigma\tau}\underline{t}^\tau$$

is the solution of the normal equations.

The estimator of the variance factor is:

$$\underline{\hat{\sigma}}^2 = \frac{g_{ik}\underline{\varepsilon}^i\underline{\varepsilon}^k}{b} = \frac{g_{\varrho\sigma}\underline{t}^\varrho\underline{t}^\sigma}{b}$$

One of the basic assumptions of least squares theory is that the means of the corrections $\underline{\varepsilon}^i$ are zero. If the observations are affected by systematic errors or blunders, ∇^i, the means of the $\underline{\varepsilon}^i$ will not be zero. The means of the misclosures will be $\nabla^\varrho = -u_i^\varrho\nabla^i$; the variate

$$\frac{\underline{\hat{\sigma}}^2}{\sigma^2} = \frac{g_{ik}\underline{\varepsilon}^i\underline{\varepsilon}^k}{b} = \frac{g_{\varrho\sigma}\underline{t}^\varrho\underline{t}^\sigma}{b}$$

has then a non-central F-distribution; its mathematical expectation can be shown to be

$$E\left\{\frac{\underline{\hat{\sigma}}^2}{\sigma^2}\right\} = 1 + \frac{1}{b}g_{\varrho\sigma}\nabla^\varrho\nabla^\sigma \geqslant 1$$

We define

$$\lambda = \frac{g_{\varrho\sigma}\nabla^\varrho\nabla^\sigma}{\sigma^2}$$

The distribution of $\frac{\underline{\hat{\sigma}}^2}{\sigma^2}$ is dependent on λ, hence we write:

$$\frac{\underline{\hat{\sigma}}^2}{\sigma^2} \equiv \underline{F}'_{b,\infty,\lambda}$$

Similar considerations hold for the type of F-test according to (3.1–2), where we have

$$\frac{\underline{\hat{\sigma}}^2{}_{b_{\mathrm{I}}}}{\underline{\hat{\sigma}}^2{}_{b_{\mathrm{II}}}} \equiv \underline{F}'_{b_{\mathrm{I}},b_{\mathrm{II}},\lambda}$$

in case the quadratic form in the numerator is non-central. For illustrative purposes we will only discuss the case that $b_{II} = \infty$.

The probability that $F'_{b,\infty,\lambda}$ is greater than the critical value $F_{1-\alpha;b,\infty}$ of the test is by definition the power of the test:

$$P\{\underline{F}'_{b,\infty,\lambda} > F_{1-\alpha;b,\infty}\} = \beta$$

β is a function of the significance level α adopted, of b and of λ; α and b being fixed, it is a monotonic function of λ. If we now suppose that there is only one ∇^i different from zero, *i.e.* that only one observation is affected by an error, we may find the size of that error that corresponds to $\beta = 0.8$ with the aid of tables or charts of the power function. Such charts are published *e.g.* in [9] and [10]. The latter are used here; they have been constructed for $\beta = 0.8$ and $\alpha = 0.05$.

The first test we will investigate is based on the estimate $\hat{\sigma}^2$ for each polygon separately. This adjustment can be considered as a first step of the adjustment of a net.

For the polygon numbered ϱ we get for the estimate

$$(\hat{\sigma}^2)_\varrho = g_{\varrho\varrho} t^\varrho t^\varrho$$

t^ϱ is then the misclosure of the polygon numbered ϱ, whereas $g_{\varrho\varrho} = \dfrac{1}{g_{\varrho\varrho}}$, and $g^{\varrho\varrho}$ equals the sum of the weight coefficients of the observations (sections or levelling lines) in the polygon.

Now suppose there is a model error ∇^ϱ in some section of the polygon. It is immaterial in which section it occurs, because all sections are in the same position with respect to the polygon. By establishing the condition we see that the error in the misclosure is:

$$\nabla t^\varrho = -\nabla^\varrho$$

Fixing $\alpha = 0.05$ and $\beta = 0.8$, we can compute ∇^ϱ from:

$$\lambda_{1,\infty} = \frac{1}{\sigma^2} g_{\varrho\varrho} \nabla t^\varrho \nabla t^\varrho = \frac{1}{\sigma^2} g_{\varrho\varrho} (\nabla^\varrho)^2$$

The indices 1 and ∞ of λ are explained by the fact that $\hat{\sigma}^2$, computed from one supernumerous observation, is tested against the variance factor σ^2 (= 200). With the aid of the nomogram [10] we find

$$(\nabla^\varrho)^2 = \frac{(\lambda_{1,\infty})^2}{g_{\varrho\varrho}} = \frac{7.85 \cdot 200}{g_{\varrho\varrho}} = 1570 g^{\varrho\varrho}$$

The result of this computation is found in Table III, column 2, giving the result per section. Most sections are part of two polygons; in that case the smallest of the two ∇'s is given.

We now consider an adjustment on b conditions, and we suppose that the adjustment is executed according to Standard Problem II (observation equations). For convenience we recall the formulas in the notation used in this section: $\underline{p}^i$ are the observations, $\underline{\varepsilon}^i$ the corrections, $\underline{h}^r$ the unknowns (in this case geo-potential numbers with respect to the datum).

Observation equations:

$$a^i_r \underline{h}^r = \underline{p}^i + \underline{\varepsilon}^i \qquad \begin{array}{l}(i = 1, \ldots, m)\\ (r = 1, \ldots, m-b)\end{array}$$

Weights:

$$g_{ik}$$

Normal equations:

$$g_{ik} a^i_r a^k_s \underline{h}^r = g_{ik} a^i_s \underline{p}^k$$

Put

$$g_{ik} a^i_r a^k_s = g_{rs}$$

$$g_{rs} g^{st} = \delta^t_r$$

and

$$g_{ik} a^i_s \underline{p}^k = \underline{F}_s$$

Then

$$\underline{h}^r = g^{rs} \underline{F}_s$$

Weight coefficients

$$\overline{h^r, h^s} = g^{rs}$$

Estimator of variance factor:

$$\underline{\hat{\sigma}}^2 = g_{ik} \underline{\varepsilon}^i \underline{\varepsilon}^k = g_{ik} \underline{p}^i \underline{p}^k - \underline{F}_s \underline{h}^s = g_{ik} \underline{p}^i \underline{p}^k - g^{rs} \underline{F}_r \underline{F}_s. \qquad (3.3\text{–}1)$$

If model errors ∇p^i occur in the p^i, it can be shown that (indicating the error in F_r by ∇F_r etc.):

$$\nabla F_r = g_{ik} a^k_r \nabla p^i$$

$$g_{ik} \nabla \varepsilon^i \nabla \varepsilon^k = g_{ik} \nabla p^i \nabla p^k - g^{rs} \nabla F_r \nabla F_s$$

By definition we have ($\alpha = 0.05$ and $\beta = 0.8$ being understood)

$$\lambda_{b,\infty} = \frac{1}{\sigma^2} g_{ik} \nabla \varepsilon^i \nabla \varepsilon^k$$

so that

$$g_{ik} \nabla \varepsilon^i \nabla \varepsilon^k = \sigma^2 \cdot \lambda_{b,\infty}$$

Now the model error ∇p^i is the error in the observation between $\underline{h}^r$ and $\underline{h}^s$ and therefore we indicate it by ∇^{rs}. We suppose there is a model error in *one* observation $\underline{p}^i$ so that

$$g_{ik} \nabla p^i \nabla p^k = g_{ii} (\nabla^{rs})^2 \qquad (3.3\text{–}2)$$

Furthermore we have

$$\nabla F_r = g_{ik} a^k_r \nabla p^i = g_{ii} a^i_r \nabla^{rs} \quad \text{(no summation on } r \text{ or } i\text{)}$$

The coefficients a^i_r are either $+1$ or -1, so that

$$\left.\begin{array}{l}\nabla F_r = -g_{ii} \nabla^{rs}\\ \nabla F_s = +g_{ii} \nabla^{rs}\end{array}\right\} \qquad (3.3\text{–}3)$$

From (3.3–1), (3.3–2) and (3.3–3) it follows that

$$g_{ik}\nabla\varepsilon^i\nabla\varepsilon^k = g_{ii}(\nabla^{rs})^2-(g_{ii})^2(\nabla^{rs})^2\{g^{rr}+g^{ss}-2g^{rs}\}$$

or

$$g_{ik}\nabla\varepsilon^i\nabla\varepsilon^k = (\nabla^{rs})^2 g_{ii}\{1-g_{ii}(g^{rr}+g^{ss}-2g^{rs})\}$$

and consequently

$$(\nabla^{rs})^2 = \frac{\sigma^2\cdot\lambda_{b,\infty}}{g_{ii}\{1-g_{ii}(g^{rr}+g^{ss}-2g^{rs})\}}$$

If we denote the relative weight coefficient of $\underline{h}^r$ and $\underline{h}^s$, *i.e.* the weight coefficient of their difference, by $g^{r-s,r-s}$ we see:

$$(\nabla^{rs})^2 = \frac{\sigma^2\cdot\lambda_{b,\infty}}{g_{ii}\{1-g_{ii}g^{r-s,r-s}\}} \quad \text{(3.3–4)}$$

g_{ii} is the weight of $\underline{p}^i$ before the adjustment, and $g^{r-s,r-s}$ can be computed easily if the matrix of weight coefficients of the unknowns is available.

Formula (3.3–4) shows clearly the relation existing between ∇^{rs}, g_{ii}, $\lambda_{b,\infty}$ and $g^{r-s,r-s}$. Before the adjustment $g^{r-s,r-s} = \frac{1}{g_{ii}}$ (assuming non-correlated original observations). Hence

$$(\nabla^{rs})^2 = \frac{\sigma^2\cdot\lambda_{b,\infty}}{g_{ii}(1-1)} = \infty$$

This confirms the self-evident truth that we cannot detect a model error if there is no adjustment. Further we see from (3.3–4) that the smaller the relative weight coefficient between the end points of a section, the smaller the model error that can be detected. We can also compute as follows a lower bound for ∇^{rs}:

$$g^{r-s,r-s} > 0$$

$$g_{ii}g^{r-s,r-s} > 0$$

$$1-g_{ii}g^{r-s,r-s} < 1$$

$$g_{ii}(1-g_{ii}g^{r-s,r-s}) < g_{ii}$$

$$(\nabla^{rs})^2 > \frac{\sigma^2\cdot\lambda_{b,\infty}}{g_{ii}}$$

Using the weight formula $g_{ii} = \frac{200}{Lt^2}$, we get

$$(\nabla^{rs})^2 > \frac{Lt^2}{2}\sigma^2\cdot\lambda_{b,\infty}$$

$$(\nabla^{rs})^2 > Lt^2\lambda_{b,\infty}$$

$$\nabla^{rs} > t\sqrt{L\lambda_{b,\infty}}$$

The best test is obviously the one in which the smallest model error leads to rejection. It is evident from (3.3–4) that to obtain this test we must not take the highest possible number of conditions (polygons), for by adding more and more conditions,

$g^{r-s,r-s}$ will finally hardly decrease, whereas $\lambda_{b,\infty}$ will increase with the number of conditions.

In order to verify this, the tests on

$$\frac{\hat{\sigma}^2}{\sigma^2} = \underline{F}_{b,\infty}$$

per partial net, as described in Section 3.1, have been investigated. The value of ∇^{rs} was computed for each levelling line (section). The result is found in column 4 of Table III. Of course, the lines between the partial nets had to be left out.

The result of this computation is remarkable. For each levelling line, the minimum model error leading to rejection has increased, in most cases considerably. Evidently the increase of $\lambda_{b,\infty}$ has much more influence than the decrease of $g^{r-s,r-s}$. Consequently it may be concluded that it is better to test per polygon than to test per partial net, if one wants to detect model errors of minimum size. But we may also use tests involving two, three or more polygons. Can we thus find a test that is still better than the test per single polygon? To answer this question we must try to find a test, in which the decrease of $g^{r-s,r-s}$ has a greater influence in (3.3–4) than the increase of $\lambda_{b,\infty}$. If we take a levelling line that occurs in two polygons, we see that the adjustment of these two polygons has a great influence on $g^{r-s,r-s}$. The obvious thing to do is therefore to examine the test in which $\hat{\sigma}^2$ is computed from two polygons having a line in common. Actually, an adjustment was necessary for each line; this adjustment has been done according to Standard Problem II. One of the end points of the line in question was taken as a reference point, the g.p.n. of the other end point being the only unknown in the adjustment. We simply get:

$$g^{r-s,r-s} = \frac{1}{\sum_i g_{ii}}$$

in which $\sum_i g_{ii}$ is the sum of the weights of the three levelling lines occurring in the problem.

The results of this computation are given in column 3 of Table III. In general we find indeed that the minimum model error that leads to rejection in 80% of the cases is smaller in the test on two polygons than it is in the test on a single polygon. In general, the improvement is not great; in several cases we even find that the minimum model error is larger in the two polygon case. This is mainly the case when the two polygons concerned are very different in precision. In this case the relative precision of the two points is only very slightly improved by the addition of the least precise polygon.

The computation has been done only for lines which are situated between two polygons, because it cannot be expected that the relative precision of the two end points of a line is much improved by adding a polygon which does not contain the line.

Since the relative weight coefficient $g^{r-s,r-s}$ of geo-potential numbers resulting from the adjustment of the total net are available, it is very easy to compute for every line ∇^{rs} for the test using $\hat{\sigma}^2$ of the total adjustment. The result can be found in Table III, column 5. The model error corresponding to a power of 0.8 is considerably larger except for the lines forming the edges of the partial nets. This is caused by the fact that the relative precision is increased most at those edges.

We might now want to investigate the power of the test when $\hat{\sigma}^2$ is computed from *e.g.* three polygons. It is not impossible that we find a ∇^{rs} that is still somewhat smaller than in the case for two polygons. But we meet some difficulties. In the first place, one can add a third polygon to two other ones in many different ways, and it is not self-evident which combination results in the highest power. In the second place, this investigation involves much extra computational work, therefore it has not been pursued.

One might investigate the power of the tests on $\hat{\sigma}^2_{b_1}/\hat{\sigma}^2_{b_2}$ concerning estimates of σ^2 from different partial nets. In that case formula (3.3–4) is still valid, when $\lambda_{b,\infty}$ is replaced by λ_{b_1,b_2}. In our case, $\lambda_{b_1,b_2} \approx 2\lambda_{b,\infty}$, so that the model error leading to rejection in 80% of the cases will become approximately $\sqrt{2}$ times as large. The power of this test is therefore much smaller than that of other tests. Nevertheless, this type of test has the advantage that the assumed value of σ^2 has no influence.

This investigation has not resulted in a statement saying which test has the greatest power: this question has not been answered fully. But the tests having greatest power are definitely not the tests involving two estimates, nor the tests on estimates obtained from a large number of supernumerous observations. It is well to test the estimate resulting from two contiguous polygons. In the line they have in common one can then detect a model error that is not likely to be much greater than the model error that can be detected by any other F-test. If the line to be investigated is only contained in one polygon, it is probably best to test the estimate resulting from that polygon. If the line is part of two polygons of very different quality, the best one can do is test the estimate from the most precise polygon. Tests on $\hat{\sigma}^2$ obtained from all observations will have minimum detectable errors that are nearly twice as large as the tests recommended above.

We give a small survey where for convenience the model errors are expressed in cm:

Number of conditions from which $\hat{\sigma}^2$ is computed	Model error ∇^{rs} leading to rejection with 80% probability ($\alpha = 0.05$; $\beta = 0.8$) varies from
1	2.5–28 cm
2	2.5–24 cm
11–15 (partial net)	3.5–40 cm
68 (total net)	5–54 cm

These values illustrate that even the hypotheses that were not rejected may be quite unreliable; the fact that a hypothesis is not rejected in a test does not imply that it is a true hypothesis.

It must be remarked that the above-mentioned investigations are based on the adopted 5 per cent level of significance, which is an arbitrary choice. Finally it is emphasized that, *e.g.*, n tests based on one degree of freedom are not equivalent to one test with n degrees of freedom because the critical regions used are not the same. In this respect, a closer investigation of the phenomena mentioned here must be made. The reader is referred to [3] for an outline of the underlying theory.

4 THE GEO-POTENTIAL OF MEAN SEA LEVEL

4.1 The data on mean sea level

In the preceding sections we have mentioned "mareograph stations". In reality this expression referred to bench marks in the immediate vicinity of mareographs; these bench marks are indicated in the Commission's notation by a number preceded by RM, *e.g.* RM 34. From the g.p.n. of an RM point, the g.p.n. of mean sea level (MSL) as indicated by the observations of the mareograph in question can be computed. This work was done by L. CAHIERRE and provisionally published in [5]. The results, with small corrections applied in 1960, are printed in Table IA.

We can summarize the connection between the g.p.n. c_{RM} of the RM point and the g.p.n. c_{MSL} of MSL as follows:

$$\underline{c}_{MSL} = \underline{c}_{RM} + \underline{a} + \underline{d}$$

The variate $\underline{a}$ corresponds to the levelled g.p.n.-difference between the RM bench mark and the reference plane for the sea level observations; $\underline{d}$ corresponds to the difference between MSL and this plane. The standard deviation of $\underline{a}$ can be neglected when it is compared to that of $\underline{c}_{RM}$; however, gross errors in a, caused by, *e.g.*, maintenance of the mareograph, seem to be possible. The value d actually used was a function of the annual means of sea level observation in a series of years: a straight trend line was fitted to these means and the ordinate for 1950 of this trend line was defined as MSL.

From the scatter of the annual values about the trend line one could formally estimate the standard deviation of $\underline{d}$ but such a computation has not been carried out, and it would indeed be difficult to interpret the resulting standard deviation. The problem belongs to the field of oceanography, as well as the problem of deciding whether the mareographs indicate the sea level without bias.

Consequently, we will in the present report leave the stochastic character of $\underline{a}$ and $\underline{d}$ out of consideration. This means that it is assumed that the matrix of weight coefficients of the variates $\underline{c}_{MSL}$ is the same as that of the variates $\underline{c}_{RM}$, *i.e.* the matrix in Table II. (When more is known about the distribution of the aforementioned variate $\underline{d}$, it may be possible to take it into account by simply increasing the diagonal elements of the matrix of weight coefficients by a corresponding amount.)

Under this assumption, we have at our disposal a number of observations of the g.p.n. of MSL and their matrix of weight coefficients. The mareographs concerned are those numbered from 1 to 50, excepted nrs. 40, 41 and 43, which are not attached to the net. From the report [5] it is concluded that some more mareographs must be left out, namely nrs. 32 and 33 for which no usable observations are available, and nr. 44, whose observations were stopped in the year 1923.

4.2 Tests of hypotheses on mean sea level

We will denote the observation of the g.p.n. of MSL at a mareograph numbered i, by $\underline{c}^i$ and the matrix of weight coefficients of the variates $\underline{c}^i$ by $||g^{ik}||$. For simplicity we will suppose that whenever a group of n mareographs is considered, they are numbered 1, 2, . . ., n so that the index i need not be the number actually given by the Commission. A hypothesis on the properties of MSL can be expressed in a number of condition equations, which have to be fulfilled by the means $\tilde{c}^i$ of the $\underline{c}^i$. For example, the hypothesis that MSL as defined above is an equipotential surface whose potential is equal to that of the N.A.P. (Amsterdam) datum, is expressed by:

$$\tilde{c}^i = 0 \qquad (i, j, k = 1, \ldots, n)$$

The actually observed values c^i do not fulfill these conditions; they have to be adjusted. The corrections $\underline{\varepsilon}^i$ resulting from the adjustment are immediately known, namely

$$\underline{\varepsilon}^i = -\underline{c}^i$$

The number n of mareographs considered may vary from one to the total number of mareographs attached to the net. It may be noted that the number of condition equations is equal to the number of observations, and in this case it is more evident than usual that the condition equations, forming the condition model, are the expression of certain hypotheses.

To test the hypothesis expressed by the conditions

$$\tilde{c}^i = 0$$

we use the estimate of the variance factor resulting from the adjustment. Denoting any estimate of the variance factor from mareograph observations by an index M, we can compute for any combination of mareographs

$$\underline{\hat{\sigma}}_M^2 = \frac{g_{ij}\underline{c}^i\underline{c}^j}{n}$$

The values c^i are given, and the matrix of weights $||g_{ij}||$ is obtained by inverting the n, n (partial) matrix of weight coefficients $||g^{ik}||$.

The estimate obtained can be tested with respect to the estimate found from the adjustment of the net:

$$\frac{\underline{\hat{\sigma}}_M^2}{\underline{\hat{\sigma}}^2} \equiv \underline{F}_{n,68}$$

because $\hat{\sigma}^2$ has been computed from an adjustment on 68 conditions.

4.2.1 *Tests concerning single mareographs*

The weight coefficient of the g.p.n. of each mareograph is directly available. By reciprocation one finds its weight. Each mareograph furnishes an estimate for the variance factor according to

$$\underline{\hat{\sigma}}_M^2 = g_{ii}\underline{c}^i\underline{c}^i$$

The test is based on the identity

$$\frac{\underline{\hat{\sigma}}_M^2}{\underline{\hat{\sigma}}^2} \equiv \underline{F}_{1,68}$$

The level of significance we use is 5%. For the practical computation we compare c^i with the critical value y^i which follows from

$$g_{ii}y^iy^i = \hat{\sigma}^2 \cdot F_{0.95;1,68} = 370 \cdot 3.98 = 1473$$

or

$$|y^i| = 38.4 \sqrt{g_{ii}}$$

The power of the tests can in principle be evaluated in the way described in Section 3.3, but here we meet a complication. In the actual adjustment the power of tests could be computed in the system based on $\sigma^2 = 200$. However, the adjustment resulted in an estimate $\hat{\sigma}^2$ that was significantly too high, which, as explained in Section 3.1, is probably not due to systematic errors but to the assumption of too high weights. We can now use the value $\hat{\sigma}^2 = 370$ instead of $\sigma^2 = 200$ in the formula

$$\lambda_{1,68} = \frac{1}{\sigma^2} g_{ii} \nabla c^i \nabla c^i$$

in which ∇c^i is the model error which on the average in 8 out of 10 tests will lead to rejection of the null hypothesis. The difficulty is, however, that to the authors' knowledge no tables are available for the power function of this more complicated non-central F-distribution. In the following we will therefore take $\sigma^2 = 370$ in the above formula and consider this as the true variance factor. Theoretically, the results will be more or less incorrect but it is hoped they give a sufficient practical indication.

We find

$$(\nabla c^i)^2 = g_{ii} \cdot \sigma^2 \cdot \lambda_{1,68} = g^{ii} \cdot 370 \cdot 8.2 = 3034\, g^{ii}$$

$$|\nabla c^i| = 55.0 \sqrt{g^{ii}}$$

The results are given in Table IV.

From the table it is evident that only 16 out of 44 g.p.n.'s of mareographs are significantly different from zero. It is very likely that several significant differences are caused by the fact that NAP is not identical with MSL. If we take the g.p.n. of MSL at the mareograph 34 (Den Helder) as our reference value, we find 12 significant differences from zero.

4.2.2 *Tests concerning pairs of mareographs*

From the geo-potential difference between MSL at any two mareographs numbered i and j one can estimate σ^2 by exactly the same formulae as used in the previous section. Instead of $\underline{c}^i$ one must introduce $\underline{c}^i - \underline{c}^j$ and instead of g^{ii} their relative weight coefficient, which may be denoted by $g^{(i-j)}$.

The results are given in Table V. Of course, not all combinations have been investigated, but it was tried to make a representative choice. Of 132 combinations considered, 59 show a significant difference.

A closer investigation shows that strong significance can be ascribed to some combination Gulf of Bothnia – North Sea (1–23, 12–23, 15–17, 15–24, 15–34), to several combinations North Sea - Mediterranean (28–48, 38–48, 38–50, 39–46, 39–48, 39–50, 42–46, 42–48, 42–50) and only to one combination Gulf of Bothnia - Mediterranean (15–48). It is to be noted that not all of these tests are mutually independent.

4.2.3 *Tests concerning groups of mareographs*

Up to now we have computed each estimate $\hat{\sigma}^2$ from one geo-potential difference only, and the tests concerned at most two mareographs. We will now include more mareographs in computing $\hat{\sigma}^2$. In view of the results of the previous section we compute three different estimates:

1. From mareographs situated on the North Sea or the Atlantic Ocean (nrs. 17 to 23, 29, 31, 34 to 39 and 42; group A).
2. From mareographs situated on the Mediterranean or the Adriatic Sea (nrs. 45 to 50; group MA).
3. From mareographs situated on the Mediterranean (nrs. 45 to 48; group MZ).

The mareographs situated on the Gulf of Bothnia will be discussed separately.

The three estimates will be denoted by $\hat{\sigma}_A{}^2$, $\hat{\sigma}_{MA}{}^2$ and $\hat{\sigma}_{MZ}{}^2$. The estimates $\hat{\sigma}_{MA}{}^2$ and $\hat{\sigma}_{MZ}{}^2$ pertain to only 6 and 4 mareographs respectively. Their computation involves inversions of matrices of the order 6 and 4, which were executed on a desk machine.
$\hat{\sigma}_A{}^2$ was computed by an electronic computer.

The tests are summarized in the following table:

TABLE C

M	$\hat{\sigma}_M{}^2$	$\dfrac{\hat{\sigma}_M{}^2}{\hat{\sigma}^2}$	n	$F_{0.95;n,68}$	Result
A	4886	13.2	16	1.79	reject
MA	10351	28.0	6	2.23	reject
MZ	11869	32.1	4	2.50	reject

The computations of $\hat{\sigma}_A{}^2$, $\hat{\sigma}_{MA}{}^2$ and $\hat{\sigma}_{MZ}{}^2$ are found in Table VI.

We see that all of these tests lead to rejection, and even with considerable significance. The null hypothesis was: the g.p.n. of all mareographs in a group is zero. The conclusion that the test indicates that *not* all mareographs in a group have a zero g.p.n. However, this does not all mean that the group as a whole lies higher or lower than NAP. We can illustrate this by considering the case that two mareographs a and b are tested with respect to NAP. Then we have

$$\hat{\sigma}_M{}^2 = \frac{g_{aa}c^a c^a + 2g_{ab}c^a c^b + g_{bb}c^b c^b}{2}$$

Differentiating with respect to c^a gives

$$\frac{\mathrm{d}\,\hat{\sigma}_M{}^2}{\mathrm{d}\,c^a} = g_{aa}c^a + g_{ab}c^b$$

Considering c^b as fixed and c^a as variable, $\hat{\sigma}_M{}^2$ reaches an extreme value for

$$c^a = -\frac{g_{ab}}{g_{aa}}c^b$$

This extreme value is a minimum, for

$$\frac{\mathrm{d}^2\,\hat{\sigma}_M{}^2}{\mathrm{d}\,(c^a)^2} = g_{aa} > 0$$

If a and b are not too far apart the computations show that

$$g_{ab} \approx -g_{aa}$$

Consequently, $\hat{\sigma}_M{}^2$ reaches a minimum if $c^a \approx c^b$. If, *e.g.*, $c^a = -c^b$, $\hat{\sigma}_M{}^2$ is much greater; the probability of rejecting the null hypothesis is also greater, meanwhile we can certainly not say that the combination of the two mareographs lies higher or lower than NAP.

We can also in this case compute ∇c^i, *i.e.* the model error in the mareograph i, which with 80% probability leads to rejection of the null hypothesis tested. The computation is done according to

$$\lambda_{n,68} = \frac{g_{ii}\nabla c^i \nabla c^i}{\sigma^2}$$

$$(\nabla c^i)^2 = \frac{\sigma^2 \lambda_{n,68}}{g_{ii}}$$

For σ^2 we use again the value $\hat{\sigma}^2 = 370$ obtained from the net adjustment, which, as indicated before, is theoretically wrong.

The computation is found in Table VI. On comparing the results with those in Table IV one notes that the power in the former case is in general smaller, but in some cases a little higher. An exception is formed by the mareographs 22 and 23, for which the "detectable" model error has become much smaller in Table VI. It is very likely that this can be explained by the fact that these mareographs are strongly dependent on each other.

An entirely different result is obtained if we do not assume a model error in a single mareograph, but in a group of mareographs, *e.g.*:

$$\nabla c^i = \nabla \qquad \text{for } i = a_1, \ldots, a_n \ (n \text{ mareographs})$$

We then get

$$\lambda_{n,68} = \frac{g_{ij}\nabla c^i \nabla c^i}{\sigma^2} = \frac{\nabla^2 \sum\limits_i \sum\limits_j g_{ij}}{\sigma^2} \qquad (4.2.3\text{–}1)$$

$$\nabla^2 = \frac{\lambda_{n,68}\sigma^2}{\sum\limits_i \sum\limits_j g_{ij}}$$

We find:

$$\left.\begin{aligned} \nabla_A^2 &= \frac{8695}{5.4562} = 1594 & \nabla_A &= 40\cdot 10^{-3}\,\text{gpu} \\ \nabla_{MA}^2 &= \frac{5550}{0.5863} = 9466 & \nabla_{MA} &= 97\cdot 10^{-3}\,\text{gpu} \\ \nabla_{MZ}^2 &= \frac{4736}{0.4413} = 10732 & \nabla_{MZ} &= 104\cdot 10^{-3}\,\text{gpu} \end{aligned}\right\} \quad . . \quad (4.2.3\text{–}2)$$

If a model error occurs in the assumed way, we can detect with 80% probability in the Atlantic group a model error of about 4 cm, in the Mediterranean one of about 1 dm.

4.2.4 *Computation and use of "adjusted" geo-potential numbers of mean sea level*

In all tests so far executed (except the testing of pairs of mareographs in Section 4.2.2) geo-potential numbers were used whose zero reference surface was defined by the NAP datum. However, it is a well-known fact that NAP differs considerably from MSL. This is of course a model error which contributed to the high significance with which many estimates $\hat{\sigma}^2$ exceeded their critical value in the tests described. We will have to use another model to compare the level of the different seas. We do not put the adjusted g.p.n.'s of MSL at the mareographs equal to zero but to a value c^M which is the same per group of mareographs. We then get three separate adjustments for the determination of c^A, c^{MA} and c^{MZ}. Each adjustment has the following form (we use Standard Problem II):

$$\underline{c}^i + \underline{\varepsilon}^i = \underline{c}^M \qquad (i = a_1, \ldots, a_n;\ n \text{ mareographs})$$

or

$$A^i \underline{c}^M = \underline{c}^i + \underline{\varepsilon}^i \qquad \text{with } A^i = 1 \text{ for } i = a_1, \ldots, a_n$$

$$g_{MM} = g_{ik} A^i A^k$$

$$\underline{F}^M = g_{ik} A^i \underline{c}^k$$

$$\underline{c}^M = \frac{\underline{F}^M}{g_{MM}}$$

From the simple form of A^i it follows that:

$$g_{MM} = \sum_{i=a_1}^{a_n} \sum_{k=a_1}^{a_n} g_{ik} \qquad \text{and} \qquad F^M = \sum_{i=a_1}^{a_n} g_{ik} c^k$$

The estimator for σ^2 is:

$$\underline{\hat{\sigma}}_{M'}^2 = \frac{g_{ik} \underline{\varepsilon}^i \underline{\varepsilon}^k}{n-1} = \frac{g_{ik} \underline{c}^i \underline{c}^k - \underline{F}^M \cdot \underline{c}^M}{n-1}$$

The estimator is denoted by $\underline{\hat{\sigma}}_{M'}^2$ to distinguish it from the previously computed $\underline{\hat{\sigma}}_M^2$. The computation of c^M and g_{MM} is found in Table VII. The computation of $\hat{\sigma}_{M'}^2$ is given in the following Table D.

TABLE D

M	$g_{ik}c^i c^k$	$F^M \cdot c_M$	$g_{ik}\varepsilon^i\varepsilon^k$	$n-1$	$\hat{\sigma}_{M'}{}^2$	$\frac{\hat{\sigma}_{M'}{}^2}{\hat{\sigma}^2}$	$F_{0.95;n-1,68}$	Result
A	78175	30818	47357	15	3157	8.53	1.81	reject
MA	62106	51797	10309	5	2062	5.57	2.35	reject
MZ	47475	38937	8538	3	2846	7.69	2.74	reject

It is seen that the significance is much lower than in Table C in Section 4.2.3, but still it is very high. This means that the result of the tests on $\hat{\sigma}_M{}^2$ cannot be explained completely by the difference in level between the seas compared, for the null hypothesis tested is now that all the mareographs of a group indicate the same MSL.

Another test can be designed by using the estimator

$$\underset{\sim}{\hat{\sigma}}_{M''}{}^2 = g_{MM}\underset{\sim}{c}^M\underset{\sim}{c}^M$$

using the identity:

$$\frac{g_{MM}\underset{\sim}{c}^M\underset{\sim}{c}^M}{\hat{\sigma}^2} \equiv \underset{\sim}{F}_{1,68}$$

The result is

TABLE E

M	$\hat{\sigma}_{M''}{}^2$	$\frac{\hat{\sigma}_{M''}{}^2}{\hat{\sigma}^2}$	$F_{0.95;1,68}$	Result
A	30694	83.0	3.98	reject
MA	51717	139.8	3.98	reject
MZ	38927	105.2	3.98	reject

The result is striking. We can draw the conclusion that the "mean" level of the Atlantic as well as that of the Mediterranean differs significantly from NAP. From the computations in Table VII it follows that these levels are $-75 \cdot 10^{-3}$ gpu and $-297 \cdot 10^{-3}$ gpu with estimated standard deviations of about $\sqrt{\frac{3157}{5.45649}} \approx 24 \cdot 10^{-3}$ gpu and $\sqrt{\frac{2062}{0.5863}} \approx 60 \cdot 10^{-3}$ gpu respectively (see also Table D).

We can examine the power of these tests too. The following relation is valid:

$$\lambda_{1,68} = \frac{g_{MM}\nabla c^M \nabla c^M}{\sigma^2}$$

$$(\nabla c^M)^2 = \frac{\sigma^2 \cdot \lambda_{1,68}}{g_{MM}} = \frac{370 \times 8.2}{g_{MM}} = \frac{3034}{g_{MM}} \quad \ldots\ldots\ldots\ldots \quad (4.2.4\text{–}1)$$

We find

$$\nabla c^A = 24 \cdot 10^{-3} \text{ gpu}$$

$$\nabla c^{MA} = 72 \cdot 10^{-3} \text{ gpu}$$

$$\nabla c^{MZ} = 83 \cdot 10^{-3} \text{ gpu}$$

On comparing these results with (4.2.3–2) we see that the power is increased considerably. This is also evident from a comparison of the formulae: if the ∇ from (4.2.3–1) is denoted by ∇_{I} and the ∇ from (4.2.4–1) by ∇_{II} it is evident that:

$$\frac{\nabla_{\text{II}}^2}{\nabla_{\text{I}}^2} = \frac{\dfrac{\sigma^2 \cdot \lambda_{1,68}}{g_{MM}}}{\dfrac{\sigma^2 \cdot \lambda_{n,68}}{g_{MM}}} = \frac{\lambda_{1,68}}{\lambda_{n,68}}, \text{ for } g_{MM} = \sum_i \sum_j g_{ij}$$

Consequently

$$\nabla_{\text{II}} = \nabla_{\text{I}} \sqrt{\frac{\lambda_{1,68}}{\lambda_{n,68}}}$$

4.2.5 *Mutual testing of "adjusted" geo-potential numbers of mean sea level*

We know the diagonal weight coefficients of the "adjusted" g.p.n.'s of MSL in different groups. If these g.p.n.'s are to be compared with each other, we must also know the correlation between these variates, *i.e.* we must have the disposal of their non-diagonal weight coefficients. These can be computed by the law of propagation.

Put

$$\frac{g_{ik}A^i}{g_{MM}} = B_k$$

Then

$$\underline{c}^M = \frac{\underline{F}^M}{g_{MM}} = \frac{g_{ik}A^i\underline{c}^k}{g_{MM}} = B_k\underline{c}^k$$

The non-diagonal weight coefficient $\overline{c^{M_1}, c^{M_2}}$ is found by

$$\overline{c^{M_1}, c^{M_2}} = B_k \cdot B_l \cdot \overline{c^k, c^l}$$

The matrix $||\overline{c^k, c^l}||$ is known; the coefficients B_k are easily computed. The computation is carried out in Table VIII. We find

$$\overline{c^A, c^{MA}} = +0.1335$$

$$\overline{c^A, c^{MZ}} = +0.1365$$

We can now compute the weight coefficient of $\underline{c}^{M_1} - \underline{c}^{M_2}$ from:

$$\overline{(c^{M_1} - c^{M_2}), (c^{M_1} - c^{M_2})} = g^{M_1M_1} + g^{M_2M_2} - 2g^{M_1M_2}$$

Table F gives the results.

Estimates for σ^2 can be computed from

$$(\hat{\sigma}_{M_1-M_2})^2 = g_{M_1-M_2,\, M_2-M_2} \cdot (\underline{c}^{M_1} - \underline{c}^{M_2})^2$$

and these estimates can again be compared to $\hat{\sigma}^2 = 370$, see Table G.

TABLE F

M_1	M_2	$g^{M_1M_1}$	$g^{M_2M_2}$	$g^{M_1M_2}$	$g^{M_1-M_2, M_1-M_2}$	$g_{M_1-M_2, M_1-M_2}$	$c^{M_1}-c^{M_2}$
A	MA	+0.1833	+1.7056	+0.1335	+1.6219	+0.6166	+222
A	MZ	+0.1833	+2.2660	+0.1365	+2.1763	+0.4595	+222

TABLE G

M_1–M_2	$(\hat{\sigma}_{M_1-M_2})^2$	$\frac{(\hat{\sigma}_{M_1-M_2})^2}{\hat{\sigma}^2}$	$F_{0.95;1,68}$	Result
A–MA	30389	82.1	3.98	reject
A–MZ	22646	61.2	3.98	reject

All tests result in rejection with very strong significance. These results indicate that there is a difference in level between the Atlantic and the Mediterranean.

4.3 **The Gulf of Bothnia**

In principle, the mareographs around the Gulf of Bothnia could have been included in the preceding investigations. Here, however, some peculiar difficulties present themselves, which are caused by the following facts (see Fig. 12).

1. All the RM-points nrs. 1 to 14 are connected by a single levelling line to the Stockholm point RM 15.
2. This single levelling line running from RM 15 to about RM 12 (more exactly, to nodal point 011) has a very low precision.

These facts imply that the very accurate Finnish net is connected to the remaining part of the U.E.L.N. by a very weak line. A glance at Table II reveals that in the matrix of weight coefficients, the rows (and columns) corresponding to mareographs 1 to 15 are practically proportional, so that the determinant of the partial matrix of weight coefficients of mareographs 1 to 15 practically vanishes. If, with the procedure of Section 4.2.3, we should want to test if the geo-potential of the Gulf of Bothnia is the same as that of the NAP datum (*i.e.* zero), we should have to invert this partial matrix which leads to entirely unreliable results, because it is extremely ill-conditioned. What we can do is: select a small number of mareographs, and base our test on these. To get an impression of how this works, we consider successively the following groups:

Nr. 15 (type of test considered in Section 4.2.1)
Nrs. 15 and 14
Nrs. 15, 14 and 3
Nrs. 15, 14, 10 and 3

The matrices of weights, being the reciprocals of the matrices of weight coefficients, are successively:

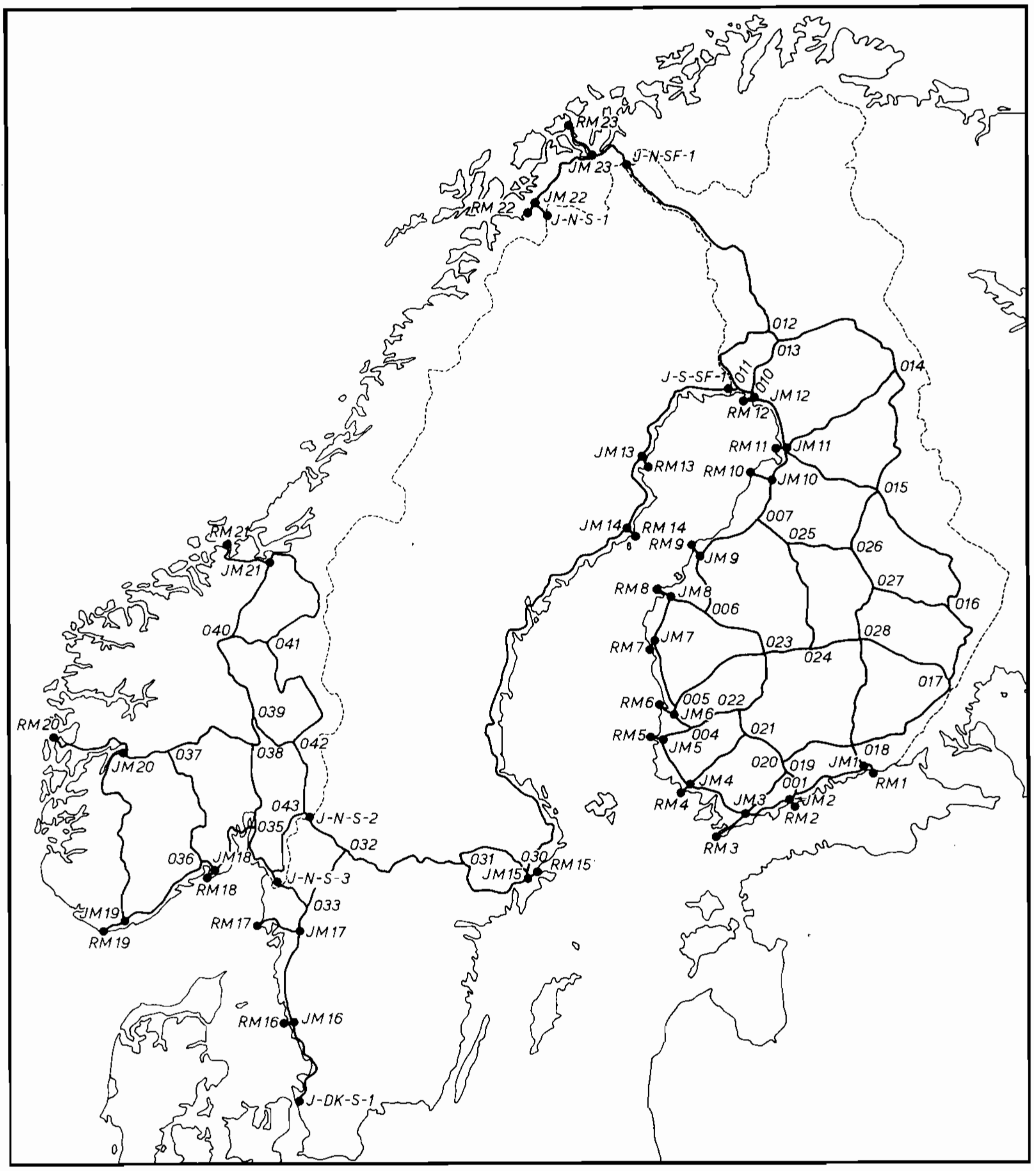

Figure 12.

$$
\begin{array}{r} 15 \end{array} \quad (4.268)^{-1} = 0.234302
$$

$$
\begin{array}{r} 15 \\ 14 \end{array} \begin{pmatrix} 4.268 & 4.253 \\ 4.253 & 90.991 \end{pmatrix}^{-1} = \begin{pmatrix} +0.245748 & -0.011486 \\ -0.011486 & +0.011527 \end{pmatrix}
$$

$$
\begin{array}{r} 15 \\ 14 \\ 3 \end{array} \begin{pmatrix} 4.268 & 4.253 & 4.253 \\ 4.253 & 90.991 & 90.990 \\ 4.253 & 90.990 & 132.322 \end{pmatrix}^{-1} = \begin{pmatrix} +0.245748 & -0.011486 & -0.000000 \\ -0.011486 & +0.035720 & -0.024193 \\ -0.000000 & -0.024193 & +0.024194 \end{pmatrix}
$$

$$
\begin{array}{r} 15 \\ 14 \\ 10 \\ 3 \end{array} \begin{pmatrix} 4.268 & 4.253 & 4.253 & 4.253 \\ 4.253 & 90.991 & 90.990 & 90.990 \\ 4.253 & 90.990 & 131.876 & 131.745 \\ 4.253 & 90.990 & 131.745 & 132.322 \end{pmatrix}^{-1} = \begin{pmatrix} +0.245728 & -0.011486 & -0.000000 & -0.000000 \\ -0.011468 & +0.035999 & -0.019944 & -0.004528 \\ -0.000000 & -0.019944 & +1.428685 & -1.408740 \\ -0.000000 & -0.004528 & -1.408740 & +1.413269 \end{pmatrix}
$$

The last row of the 4×4 matrix to be inverted is seen to be approximately equal to the third one, but the inversion of this matrix by the Gauss method (using ten digits) gives a satisfactory result, which was proved by re-inversion. It may be noted that in the inverse, the diagonal element of RM 15 remains the same in the last three cases.

From Table IA we take the observed geo-potential numbers of MSL at the mareographs involved:

Nr.	c_{MSL}
15	+0.137
14	+0.167
10	+0.242
3	+0.223

The resulting F-values in the four cases considered and the critical values of tests on the 5% level are

b Number of observations	$F_{b,68}$	$F_{0.95;b,68}$
1	$\frac{4397}{370} = 11.88$	3.98
2	$\frac{4408}{2 \cdot 370} = 5.96$	3.13
3	$\frac{4484}{3 \cdot 370} = 4.04$	2.74
4	$\frac{5044}{4 \cdot 370} = 3.41$	2.50

All these values lead to rejection in a test on the 5% level of significance; from the tables in [7] it follows that the approximate probabilities for the various F-values to exceed the values found, are respectively 0.001, 0.005, 0.01 and 0.015. A thorough investigation of this peculiar situation would be interesting from a general geodetic point of view but would lead beyond the scope of the present report.

By the procedures of Section 4.2.4 one might want to compute an "adjusted" MSL for the Gulf of Bothnia, but it can easily be shown that the "adjusted" MSL will be the value indicated by the Stockholm mareograph nr. 15: no amount of levelling around the Gulf of Bothnia can give information about its mean g.p.n. with respect to NAP if nr. 15 remains the only mareograph directly connected with the main net of U.E.L.N.

To prove this we consider a simplified case, consisting of the mareographs 15, 14 and 13. The ommission of the mareographs 1–12 and the fact that the Finnish net is not a single line are not essential.

Let the adjusted value sought for the geo-potential of the Gulf of Bothnia be $\underline{c}^M$. Calling the levelling observations $\underline{p}^1$ and $\underline{p}^2$ and their corrections $\underline{\varepsilon}^1$ and $\underline{\varepsilon}^2$, the observation equations take the form:

$$\underline{c}^M = \underline{c}^{15} + \underline{\varepsilon}^{15}$$

$$\underline{c}^M = \underline{c}^{14} + \underline{\varepsilon}^{14} = \underline{c}^{15} + \underline{\varepsilon}^{15} + \underline{p}^1 + \underline{\varepsilon}^1$$

$$\underline{c}^M = \underline{c}^{13} + \underline{\varepsilon}^{13} = \underline{c}^{15} + \underline{\varepsilon}^{15} + \underline{p}^1 + \underline{\varepsilon}^1 + \underline{p}^2 + \underline{\varepsilon}^2$$

Subtraction of the first equation from the other two results immediately in: $\underline{p}^1 + \underline{\varepsilon}^1 = 0$ and $\underline{p}^2 + \underline{\varepsilon}^2 = 0$, and consequently one finds $\underline{c}^M = \underline{c}^{15} + \underline{\varepsilon}^{15}$ for all three equations so that $\underline{\varepsilon}^{15} = 0$. A formal solution of the adjustment problem expressed by the three equations leads to the same result. Denoting the weight coefficients of $\underline{c}^{15}$, $\underline{p}^1$ and $\underline{p}^2$ by g^{15}, g^1 and g^2 respectively, the matrix of weight coefficients of the right-hand members is

	c^{15}	c^{14}	c^{13}
c^{15}	g^{15}	g^{15}	g^{15}
c^{14}	g^{15}	$g^{15}+g^1$	g^{15}
c^{13}	g^{15}	g^{15}	$g^{15}+g^1+g^2$

Inversion of this matrix and formation of the normal equation shows that the normal equation for the determination of $\underline{c}^M$ is

$$\frac{1}{g^{15}}\underline{c}^M = \frac{1}{g^{15}}\underline{c}^{15}$$

or

$$\underline{c}^M = \underline{c}^{15}$$

The computation of an "adjusted" mean sea level for some group of mareographs may be thought to smooth out the effect of individual deviations of each mareograph (*i.e.* the effect of the variate $\underline{d}$ mentioned in Section 4.1). The above computation shows that this is not the case here, unless the corresponding variate is included in the model, the result of which would be an increase in the diagonal elements of the above matrix of weight coefficients.

5 COMPUTATIONAL ASPECTS

Since it had been decided from the start that after the actual adjustment the final matrix of weight coefficients for all points would have to be computed, calculation by desk machines would have taken too much time and work. The numerical execution was done by the Computing Centre of the Central Organization for Applied Scientific Research (Afdeling Bewerking Waarnemingsuitkomsten van T.N.O.).

The electronic computer was a "ZEBRA" (a Dutch abbreviation for "Very simple binary computer"), then *) a new machine. It is a rather small but flexible computer; the memory consists of a magnetic drum with 8192 registers of 32 bits each. Two different codes are available, *viz.* the "simple code" and the "normal code". The simple code works in the floating decimal technique and takes up much room in the memory. Consequently, the maximum order of a matrix to be inverted without partitioning is about 30. In some computations, up to 75% of the memory is used to store instructions. Programming in the normal code is more economical but also more difficult because the numbers have to be normalized.

The basic matrices for the computation, as can be found in the recapitulation of formulae on pages 9 and 10 were furnished to the Computing Centre of T.N.O., where the programming and computation were carried out. Regular contacts were of course indispensable. Some difficulties were encountered owing to the newness of the machine and the consequent lack of experience of the staff, so that the complete results were not available at the time originally foreseen.

All computations were carried out with about nine significant figures. This may be somewhat more than is needed for keeping rounding-off errors under control, but the machine uses nine figures anyway so that this accuracy did not cost anything extra.

As pointed out before, the method of Standard Problem II (observation equations) was chosen for the first step of the adjustment, because the preliminary computations take somewhat less time. To have a check on the formulae and on the effect of rounding-off errors, and to be able to make a good comparison between the methods of Standard Problems I and II, the net F was also adjusted by the method of Standard Problem I (condition equations). The two results differed only ten units of the last decimal at most.

The partition of the net was such that no inversion or multiplication of the first step exceeded the capacity of the Zebra. In the second step, however, the computation of the final matrix of weight coefficients of adjusted g.p.n.'s necessitated partioning, and consequently much programming work of an organizational nature.

*) This was in 1958. In the meantime, a modified version has become available.

6 FINAL REMARKS

It seems useful to recapitulate some conclusions that can be drawn from the work described and to try to find suggestions for future similar undertakings and research.

There is no numerical specification of the desired accuracy of the final results of an international levelling net, the only realistic specification being that the net should be as accurate as possible. No doubt, the participating nations have taken all care to furnish the best possible data. However, in the future it will be important that the precision of the contribution is evaluated in a correct and uniform way from as much statistical data as possible. Otherwise no reliable and sharp statement about the precision of results can be given. In this respect, reference can be made to the important work of A. M. WASSEF, published in [14], [15] and [16] and to the report [6] of the discussions in Section II of the I.A.G. at the General Assembly of Helsinki, 1960. Statistical tests, which form an objective quide to reach conclusions about the presence of systematic errors, may be used, but an investigation of the power of these tests shows that in the present net large parts have such a low precision that large errors may be present without leading to rejection in tests.

From the matrix of weight coefficients (here published for the mareograph stations only) the variance-covariance matrix of the geo-potential numbers can be obtained by multiplying it by the estimate of the variance factor $\hat{\sigma}^2 = 370$. The result should give a reasonably good impression of the precision. As to the accuracy, the possibility of systematic errors cannot be excluded, so that one should not jump to conclusions even if the results show that there are highly significant differences in the geo-potential of MSL at different mareographs. For a discussion on this and other points, the reader is referred to [1].

In Section 4, the hypotheses on MSL were the rather obvious ones from a geodetic point of view. The formulating of more refined null- and alternative hypotheses on MSL belongs to the domain of oceanography; as an example can be seen the excellent paper [11] by J. R. ROSSITER, which deals with the reduction of sea level observations for secular variations and meteorological effects. Any "reduction" of the sea level observations on account of oceanographic considerations is essentially a hypothesis that the thus reduced MSL is an equipotential surface. In order to test statistically such a hypothesis by comparing the resulting surface with results of levelling, it is important that the precision of the reductions is evaluated (in fact the whole probability distribution of the reductions should be known). Especially in more refined hypothesis, the neglecting of the stochastic character of the reductions may be more dangerous to the results of tests than the neglecting of the stochastic character of $\underset{\sim}{d}$ in the tests of Section 4.

It will probably take a rather long time before a new European levelling net, consisting of a large number of polygons of high precision levelling, can be established. In the meantime, the fruitful collaboration between oceanographers and geodesists can be continued if attention is focused on areas where the results of the existing U.E.L.N. suggest further local investigations. International cooperation on a smaller scale than required for U.E.L.N. may result in the adjustment of networks in those restricted areas.

The rapid development of computing techniques should make it possible to establish in the future a much denser U.E.L.N., even to consider the joining together of complete national levellings. This work could be greatly simplified by letting each nation adjust its own contribution, so that only the organization and computation of the final step of the adjustment would require international cooperation.

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Table I. Partial Net D

1 Point Nr.	2 G.p.-difference with 301 after 1st step in g.p.u.	3 Correction from 2nd step in g.p.u.	4 Final g.p.-difference with 301 in g.p.u.	5 Final g.p.n. Datum N.A.P. in g.p.u.
201	+ 10.99467	−0.00030	+ 10.99437	+ 12.39547
202	+ 41.82635	−0.00031	+ 41.82604	+ 43.22714
203	+ 52.31319	−0.00033	+ 52.31286	+ 53.71396
204	+ 65.73514	−0.00035	+ 65.73479	+ 67.13589
205	+ 97.14861	−0.00048	+ 97.14813	+ 98.54923
206	+262.74520	−0.00083	+262.74437	+264.14547
207	+265.88709	−0.00135	+265.88574	+267.28684
208	+368.67591	−0.00201	+368.67390	+370.07500
209	+371.08420	−0.00139	+371.08281	+372.48391
210	+543.64668	−0.00117	+543.64551	+545.04661
211	+533.03064	−0.00101	+533.02963	+534.43073
212	+589.55354	−0.00066	+589.55288	+590.95398
213	+571.77482	+0.00046	+571.77528	+573.17638
214	+653.37735	+0.00005	+653.37740	+654.77850
215	+278.18345	−0.00043	+278.18302	+279.58412
216	+104.26188	−0.00064	+104.26124	+105.66233
217	+101.44437	−0.00066	+101.44371	+102.84481
218	+182.87344	−0.00065	+182.87279	+184.27389
219	+160.18637	−0.00007	+160.18630	+161.58740
220	+123.53091	−0.00014	+123.53077	+124.93187
221	+ 28.17834	−0.00015	+ 28.17819	+ 29.57929
222	+ 20.71191	−0.00024	+ 20.71167	+ 22.11277
223	+ 6.14606	−0.00023	+ 6.14583	+ 7.54693
224	+ 9.30875	−0.00028	+ 9.30847	+ 10.70957
225	− 0.59881	−0.00030	− 0.59911	+ 0.80199
226	+ 12.51741	−0.00030	+ 12.51711	+ 13.91821
227	+ 0.34074	−0.00030	+ 0.34044	+ 1.74154
228	+ 7.35248	−0.00030	+ 7.35218	+ 8.75328
229	+ 2.90662	−0.00029	+ 2.90633	+ 4.30743
230	+ 91.27613	−0.00029	+ 91.27584	+ 92.67694
231	+105.12709	−0.00035	+105.12674	+106.52784
301	0	0	0	+ 1.40110
302	+ 94.59528	+0.00046	+ 94.59574	+ 95.99684
303	+493.67715	+0.00108	+493.67823	+495.07933
306	+248.50704	+0.00254	+248.50958	+249.91068
307	+207.99102	−0.00268	+207.98834	+209.38944
308	+136.25873	−0.00194	+136.25679	+137.65789
503	+434.93761	+0.00449	+434.94210	+436.34320
601	+419.03538	−0.00385	+419.03153	+420.43263
602	+617.77381	−0.00319	+617.77062	+619.17172
606	+343.69743	−0.00026	+343.69717	+345.09828
607	+321.39186	−0.00469	+321.38717	+322.78828
JM28	+ 20.76958	−0.00030	+ 20.76928	+ 22.17038
JM29	+ 6.56254	−0.00030	+ 6.56224	+ 7.96334
JM30	+ 14.93318	−0.00031	+ 14.93287	+ 16.33397
JM31	+ 3.20138	−0.00029	+ 3.20109	+ 4.60219
JM32	+ 0.99243	−0.00027	+ 0.99216	+ 2.39326
JM33	+ 3.96919	−0.00024	+ 3.96895	+ 5.37005
JM34	+ 6.51452	−0.00009	+ 6.51443	+ 7.91553
JM35	+ 2.86213	+0.00028	+ 2.86241	+ 4.26351

Table I. Partial Net F

1 Point Nr.	2 G.p.-difference with 339 after 1st step in g.p.u.	3 Correction from 2nd step in g.p.u.	4 Final g.p.-difference with 339 in g.p.u.	5 Final g.p.n. Datum N.A.P. in g.p.u.
304	−118.52200	+0.00663	−118.51537	+ 2.76546
305	+ 93.26132	+0.00422	+ 93.26554	+214.54637
306	+128.62598	+0.00387	+128.62985	+249.91068
307	+ 88.09546	+0.01315	+ 88.10861	+209.38944
308	+ 16.35363	+0.02343	+ 16.37706	+137.65789
309	+113.15639	+0.01034	+113.16673	+234.44756
310	+165.14956	+0.01045	+165.16001	+286.44084
311	+156.99621	+0.01196	+157.00817	+278.28900
312	+348.91450	+0.02242	+348.93692	+470.21775
313	+279.81470	+0.03677	+279.85147	+401.13230
314	+635.33033	−0.01866	+635.31167	+756.59250
315	+ 85.04256	−0.04452	+ 84.99804	+206.27887
316	+ 52.27807	−0.05492	+ 52.22315	+173.50398
317	−109.92133	−0.03820	−109.95953	+ 11.32130
318	− 12.56585	−0.03491	− 12.60076	+108.68007
319	+583.04256	−0.04503	+582.99753	+704.27836
320	+ 53.61706	−0.00165	+ 53.61541	+174.89624
321	− 80.92655	+0.00875	− 80.91780	+ 40.36303
322	− 99.04297	−0.00662	− 99.04959	+ 22.23124
323	−112.11620	−0.00461	−112.12081	+ 9.16002
324	− 48.72443	−0.00226	− 48.72669	+ 72.55414
325	−115.15108	+0.00109	−115.14999	+ 6.13084
326	− 64.50668	+0.00356	− 64.50312	+ 56.77771
327	− 88.96913	+0.00540	− 88.96373	+ 32.31710
328	− 1.15924	+0.00606	− 1.15318	+120.12765
329	+ 15.57322	+0.00614	+ 15.57936	+136.86019
330	− 15.51998	+0.00612	− 15.51386	+105.76697
331	+146.99739	+0.00841	+147.00580	+268.28663
332	+ 76.26221	+0.00413	+ 76.26634	+197.54717
333	+ 43.41451	+0.00623	+ 43.42074	+164.70157
334	− 26.77916	−0.01386	− 26.79302	+ 94.48781
335	+420.28533	−0.01398	+420.27135	+541.55218
336	+ 40.66934	−0.02975	+ 40.63959	+161.92042
337	− 89.95447	−0.01027	− 89.96474	+ 31.31609
338	+ 80.79153	−0.00447	+ 80.78706	+202.06789
339	0	0	0	+121.28083
340	− 35.61041	−0.00082	− 35.61123	+ 85.66960
341	− 85.23454	+0.00354	− 85.23100	+ 36.04983
417	+690.72018	−0.00467	+690.71551	+811.99635
420	+997.33884	−0.05937	+997.27947	+1118.56029
JM35	−117.02417	+0.00685	−117.01732	+ 4.26351
JM36	− 84.24583	+0.00087	− 84.24496	+ 37.03587
JM37	−100.40031	−0.00063	−100.40094	+ 20.87989
JM38	−116.49282	−0.00355	−116.49637	+ 4.78446
JM39	−115.77405	+0.02799	−115.74606	+ 5.53477
JM46	−100.03459	−0.03810	−100.07269	+ 21.20814
JM47	+ 80.08400	−0.05955	+ 80.02445	+201.30528

TABLE I. Partial Net E

1 Point Nr.	2 G.p.-difference with 429 after 1st step in g.p.u.	3 Correction from 2nd step in g.p.u.	4 Final g.p.-difference with 429 in g.p.u.	5 Final g.p.n. Datum N.A.P. in g.p.u.
401	−404.36715	+0.03888	−404.32827	+ 52.33405
402	−450.14897	+0.02366	−450.12531	+ 6.53701
403	−415.65360	+0.00955	−415.64405	+ 41.01827
404	−453.40726	+0.00491	−453.40235	+ 3.25997
405	−447.77791	+0.00180	−447.77611	+ 8.88621
406	−452.47755	+0.00009	−452.47746	+ 4.18486
408	+ 78.89674	−0.00315	+ 78.89359	+535.55591
409	−245.94522	−0.00254	−245.94776	+210.71456
410	− 54.72784	−0.00298	− 54.73082	+401.93150
411	+333.04979	−0.00574	+333.04405	+789.70637
412	+169.79100	−0.00743	+169.78357	+626.44589
413	+222.71282	−0.01161	+222.70121	+679.36353
416	−451.22079	−0.04202	−451.26281	+ 5.39951
417	+355.34658	−0.01255	+355.33403	+811.99635
418	− 81.76924	+0.00406	− 81.76518	+374.89714
419	−307.54976	+0.02307	−307.52669	+149.13563
420	+661.84966	+0.04831	+661.89797	+1118.56029
421	+240.91486	−0.00030	+240.91456	+697.57688
422	+ 63.50649	+0.00137	+ 63.50786	+520.17018
423	+402.24556	+0.00452	+402.25008	+858.91240
424	+459.43851	+0.00261	+459.44112	+916.10344
425	+208.27216	+0.00285	+208.27501	+664.93733
426	+229.08106	+0.00117	+229.08223	+685.74455
427	+239.28079	+0.00026	+239.28105	+695.94337
428	+ 74.55349	−0.00095	+ 74.55254	+531.21486
429	0	0	0	+456.66232
430	+ 25.05668	+0.00034	+ 25.05702	+481.71934
435	−331.48274	−0.00460	−331.48734	+125.17498
436	+539.92824	−0.00458	+539.92366	+996.58598
437	−128.29418	−0.00443	−128.29861	+328.36371
JM39	−451.08139	−0.04616	−451.12755	+ 5.53477
JM42	−443.46149	−0.00451	−443.46600	+ 13.19632
JM44	−453.80406	−0.00109	−453.80515	+ 2.85717
JM45	−453.40726	+0.00491	−453.40235	+ 3.25997
JM46	−435.49299	+0.03882	−435.45417	+ 21.20814

TABLE I. Partial Net C

1 Point Nr.	2 G.p.-difference with 603 after 1st step in g.p.u.	3 Correction from 2nd step in g.p.u.	4 Final g.p.-difference with 603 in g.p.u.	5 Final g.p.n. Datum N.A.P. in g.p.u.
501	− 78.57484	−0.02972	− 78.60456	+484.49141
502	−169.78729	−0.02181	−169.80910	+393.28687
503	−126.72783	−0.02494	−126.75277	+436.34320
504	−151.96862	−0.01293	−151.98155	+411.11442
505	− 46.17025	−0.01127	− 46.18152	+516.91445
506	+849.56317	−0.01450	+849.54867	+1412.64464
507	+901.55841	−0.01443	+901.54398	+1464.63995
508	−363.37211	−0.00223	−363.37434	+199.72163
509	−357.39641	−0.00149	−357.39790	+205.69807
510	+305.79797	−0.00319	+305.79478	+868.89075

TABLE I. Partial Net C (*continued*)

1 Point Nr.	2 G.p.-difference with 603 after 1st step in g.p.u.	3 Correction from 2nd step in g.p.u.	4 Final g.p.-difference with 603 in g.p.u.	5 Final g.p.n. Datum N.A.P. in g.p.u.
511	+361.29800	−0.00075	+361.29725	+924.39322
512	+158.83391	−0.00042	+158.83349	+721.92946
515	−554.06216	+0.00152	−554.06064	+ 9.03533
516	−505.72171	+0.00131	−505.72040	+ 57.37557
517	−506.72325	+0.00252	−506.72073	+ 56.37524
518	−560.04400	+0.01542	−560.02858	+ 3.06739
519	−346.08196	+0.00440	−346.07756	+217.01841
520	—	—	—	+334.26603
601	−142.65322	−0.01012	−142.66334	+420.43263
602	+ 56.07569	+0.00006	+ 56.07575	+619.17172
603	0	0	0	+563.09597
604	+ 66.65143	+0.00195	+ 66.65338	+629.74935
605	− 28.15377	+0.00248	− 28.15129	+534.94468
606	−218.00171	+0.00402	−217.99769	+345.09828
607	−240.31447	+0.00678	−240.30769	+322.78828
608	−302.84046	+0.00524	−302.83522	+260.26075
612	− 90.88530	+0.00370	− 90.88160	+472.21437
614	− 68.09463	+0.00167	− 68.09296	+495.00301
615	− 14.74865	+0.00138	− 14.74727	+548.34870
616	− 35.77169	+0.00374	− 35.76795	+527.32802
309	−328.63138	−0.01703	−328.64841	+234.44756
313	−161.92521	−0.03846	−161.96367	+401.13230
314	+193.49218	+0.00435	+193.49653	+756.59250
JM47	−361.90273	+0.11204	−361.79069	+201.30528
JM48	−562.50257	+0.00880	−562.49377	+ 0.60220
JM49	−562.00468	+0.00141	−562.00327	+ 1.09270
JM50	—	—	—	+ 0.40505

TABLE IA.

Mareograph Nr.	c_{MSL}	Mareograph Nr.	c_{MSL}
1	+0.250	23	−0.111
2	+0.212	24	−0.056
3	+0.223	25	−0.015
4	+0.225	26	−0.036
5	+0.242	27	−0.062
6	+0.240	28	−0.088
7	+0.241	29	−0.038
8	+0.243	30	−0.079
9	+0.233	31	+0.027
10	+0.242	34	−0.089
11	+0.265	35	−0.142
12	+0.272	36	−0.168
13	+0.243	37	−0.012
14	+0.167	38	+0.008
15	+0.137	39	+0.142
16	−0.041	42	+0.142
17	−0.077	45	−0.030
18	−0.062	46	−0.217
19	−0.114	47	−0.169
20	−0.060	48	−0.329
21	−0.078	49	−0.291
22	−0.011	50	−0.330

TABLE III. Model error in line i, which with 80% probability would lead to rejection in a test after adjustment on b condition equations ($\alpha = 0.05$). See page 20. Unit 10^{-3} g.p.u.

1 i	2 $b=1$	3 $b=2$	4 b per partial net	5 $b=68$
201–202	38		56	76
201–228	38	33	44	60
201–228	43		66	91
202–203	42		58	79
202–226	38	31	43	60
203–204	34		46	62
203–229	34	33	44	61
204–205	49		66	89
204–230	34	33	42	58
205–206	44		60	82
205–231	44	42	54	75
206–207	46		66	88
206–217	44	37	48	66
207–208	34		50	68
207–209	34	33	45	61
208–209	34	34	50	63
208–607	53			104
209–210	46	41	66	70
210–211	41	39	61	72
210–215	41	38	51	69
211–212	41	41	61	73
211–602	74	67		106
211–606	53	52		93
212–213	41	39	61	69
212–601	58	56		99
213–214	41	43	61	75
213–503	58	58		98
214–215	41	43	61	76
214–308	102	90		163
215–216	46	49	66	87
216–217	44	36	47	65
216–218	44	47	61	81
217–231	44	40	55	75
218–219	44	46	61	80
218–307	89	86		160
219–220	44	41	54	73
219–303	64	61	94	108
220–221	49	46	59	81
220–231	44	39	52	71
221–222	49	43	57	78
221–301	52	50	68	93
222–223	35	34	46	63
222–230	35	33	44	60
223–224	35		50	68
223–301	52		75	103
224–225	42		58	84
224–229	35	30	42	57
225–226	25	26	35	48
225–227	25		35	48
226–227	25	25	34	47
227–228	38		56	76
229–230	34	30	41	56
301–302	64		94	125
302–303	64	68	94	121
302–304	158			252
303–306	89	88		148
304–305	158	176	340	328
304–327	238		340	339
305–306	158	144	317	227
305–329	221	193	259	284
306–307	89	98	317	179
307–308	121	135	317	248
308–309	102	113	317	204
309–310	172	161	317	266
309–502	102	104		183
310–311	172	157	273	279
310–329	203	193	259	326
311–312	172	169	285	301
311–331	203	173	227	300
312–313	172	160	324	266
312–333	210	182	241	302
313–314	181	185	324	328
313–501	172	141		233
314–315	186	173	324	312
314–519	181	162		290
315–316	186	172	341	296
315–334	224	207	280	344
316–317	260		341	435
316–518	186			331
317–318	164	161	341	278
317–401	164			314
318–319	164	161	332	294
318–336	228	195	262	331
319–320	213	201	332	357
319–420	164	153		279
320–321	154	149	332	266
320–417	154	152		279
321–322	228		332	390
321–416	154			292
322–323	238		300	399
322–337	228	188	246	319
323–324	242		318	424
323–338	238	220	292	394
324–325	280		401	540
324–340	242	220	287	386
325–326	259		347	461
325–340	259	241	326	441
326–327	219		299	370
326–341	219	197	254	336
327–328	219	207	280	354

Table III. (*continued*)

1 *i*	2 *b*=1	3 *b*=2	4 *b* per partial net	5 *b*=68
328–329	203	177	237	289
328–330	203	179	237	303
330–331	203	182	239	312
330–341	219	191	251	336
331–332	210	190	255	336
332–333	210	199	264	342
332–339	220	197	250	336
333–334	224	199	265	336
334–335	250	228	301	403
335–336	206	203	266	357
335–337	206	197	258	347
335–338	238	212	273	363
336–337	206	193	255	336
338–339	242	203	264	347
339–340	242	217	286	386
339–341	220	207	268	350
401–402	175		267	331
401–420	164	149	267	276
402–403	203		289	382
402–419	175	158	211	285
403–404	195		260	357
403–423	195	171	224	303
404–405	215		283	386
404–425	195	177	233	319
405–406	226		292	399
405–426	215	201	266	363
406–408	237		332	454
406–428	226	210	277	382
408–409	169	159	204	280
408–437	169		226	310
409–410	169	156	208	285
409–428	187	175	223	306
410–411	169	165	219	300
410–429	187	179	235	321
411–412	171	158	207	284
411–436	169	149	198	271
412–413	227		281	378
412–435	171		252	344
413–416	233		336	424
413–421	227	201	257	347
416–417	154	159	336	292
417–418	213	186	336	323
418–419	203	176	289	308
418–422	203	183	246	333
419–420	175	167	267	303
421–422	183	164	212	287
421–430	183	180	233	321
422–423	183	165	212	290
423–424	183	162	213	296
424–425	183	165	219	300

1 *i*	2 *b*=1	3 *b*=2	4 *b* per partial net	5 *b*=68
424–430	183	159	210	287
425–426	183	171	224	308
426–427	183	166	214	294
427–428	187	172	219	300
427–429	183	161	207	284
429–430	183	161	209	287
435–436	53	57	79	108
435–437	53		81	111
436–437	53	58	79	108
501–502	91	96	128	166
501–507	91	98	128	173
502–503	90	81	120	172
502–506	90	81	104	141
503–504	58	61	120	111
504–505	90	79	101	133
504–601	58	58	133	104
505–506	90	79	100	139
505–510	92	85	110	147
506–507	91	71	91	134
507–508	92	96	119	155
508–509	92	77	96	188
508–519	101	104	146	188
509–510	87	79	100	141
509–517	87	78	98	138
510–511	87	77	99	130
511–512	87	69	91	121
511–603	93	80	99	123
512–516	87	80	101	143
512–615	93	86	111	150
516–614	98		131	182
516–517	87		115	159
517–518	101		146	198
518–519	101	107	146	198
601–602	74	80	133	147
602–603	89	74	133	119
603–604	89	86	127	156
604–605	89	79	151	140
604–615	93	85	107	146
605–606	89	87	148	136
605–616	111	100	128	182
606–607	53	57	148	103
607–608	111		148	190
608–612	100		141	201
608–616	100	94	122	168
612–614	111		151	207
612–616	100	84	110	158
614–615	98	86	108	155

Table IV. Tests of mareographs relative to N.A.P. See page 28.
c^i, $|y^i|$ and $|\Delta c^i|$ in 10^{-3} g.p.u.

i	g^{ii}	$\sqrt{g^{ii}}$	Observed c_{MSL} c^i	Critical value $\|y^i\|$	Model error $\|\nabla c^i\|$
1	132.2	11.50	+250	442	633
2	132.2	11.50	+212	442	633
3	132.3	11.50	+223	442	634
4	132.2	11.50	+225	442	633
5	132.2	11.50	+242	442	633
6	132.2	11.50	+240	442	633
7	132.2	11.50	+241	442	633
8	132.2	11.50	+243	441	633
9	132.1	11.49	+233	441	633
10	131.9	11.48	+242	441	633
11	131.7	11.48	+265	441	632
12	131.5	11.47	+272	440	632
13	106.0	10.30	+243	395	567
14	91.0	9.54	+167	366	525
15	4.3	2.07	+137	79	114
16	2.1	1.44	− 41	55	79
17	2.9	1.72	− 77	65	95
18	8.1	2.85	− 62	109	157
19	11.7	3.42	−114	131	189
20	13.4	3.67	− 60	141	202
21	12.1	3.48	− 78	134	192
22	138.0	11.75	− 11	451	647
23	135.8	11.65	−111	447	642
24*	1.5	1.22	− 56	47	67
25	1.5	1.21	− 15	46	66
26	1.2	1.08	− 36	41	59
27*	1.2	1.08	− 62	41	59
28*	1.0	1.00	− 88	38	55
29	1.0	1.00	− 38	38	55
30*	0.7	0.84	− 79	32	46
31	0.6	0.75	+ 27	29	41
34*	0.2	0.50	− 89	19	27
35*	0.9	0.97	−142	37	53
36*	12.3	3.50	−168	134	193
37	16.3	4.04	− 12	155	222
38	10.2	3.19	+ 8	122	176
39*	12.6	3.55	+142	136	195
42*	18.7	4.32	+142	166	238
45	17.4	4.17	− 30	160	230
46*	10.8	3.29	−217	126	181
47*	6.1	2.46	−169	95	136
48*	2.5	1.57	−329	60	86
49*	2.2	1.49	−291	57	82
50*	2.5	1.58	−330	61	87

Observations marked with * exceed the critical value.

Table V. *(continued)*

i–j	$g^{(i-j)}$	$\sqrt{g^{(i-j)}}$	Observed difference c^i–c^j	Critical value $\|y^{(i-j)}\|$	Model error $\|\nabla^{(i-j)}\|$
37–38	12.954	3.599	− 20	138	198
37–39	20.527	4.531	−154	174	250
37–42	26.957	5.192	−154	199	286
37–45	25.728	5.072	+ 18	195	279
37–46*	19.787	4.448	+205	171	245
37–48*	16.928	4.114	+317	158	227
38–39*	11.697	3.420	−134	131	188
38–42	18.297	4.277	−134	164	236
38–45	17.112	4.137	+ 38	159	228
38–46*	11.481	3.388	+225	130	187
38–48*	10.558	3.249	+337	125	179
38–50*	11.128	3.336	+338	128	184
39–42	10.284	3.207	0	123	177
39–45*	9.837	3.136	+172	120	173
39–46*	7.854	2.802	+359	108	154
39–48*	12.607	3.551	+471	136	196
39–50*	13.361	3.655	+472	140	201
42–45*	8.211	2.865	+172	110	158
42–46*	12.764	3.573	+359	137	197
42–48*	18.671	4.321	+471	166	238
42–50*	19.445	4.410	+472	169	243
45–46*	11.115	3.334	+187	128	184
45–48*	17.344	4.165	+299	160	229
45–50*	18.126	4.257	+300	163	235
46–47	10.070	3.173	− 48	122	175
46–48	10.685	3.269	+112	125	180
46–50	11.511	3.393	+113	130	187
47–48*	5.307	2.304	+160	88	127
47–50*	6.483	2.546	+161	98	140
48–49	1.952	1.397	− 38	54	77
48–50	2.422	1.556	+ 1	60	86
49–50*	0.962	0.981	+ 39	38	54

Observations marked with * exceed the critical value.

TABLE V. Tests of differences between mareographs. See page 28.
$c^i - c^j$, $|y^{(i-j)}|$ and $|\nabla^{(i-j)}|$ in 10^{-3} g.p.u.

i–j	$g^{(i-j)}$	$\sqrt{g^{(i-j)}}$	Observed difference $c^i - c^j$	Critical value $\|y^{(i-j)}\|$	Model error $\|\nabla^{(i-j)}\|$
1–2*	0.243	0.493	+ 38	19	27
1–4*	0.367	0.606	+ 25	23	33
1–12	0.730	0.854	− 22	33	47
1–15	127.994	11.313	+113	434	623
1–18	134.439	11.595	+312	445	639
1–21	138.312	11.761	+328	451	648
1–23*	4.977	2.231	+361	86	123
1–34	132.298	11.502	+339	441	634
1–36	144.036	12.001	+418	460	661
1–42	150.414	12.264	+108	471	676
1–48*	134.135	11.582	+579	444	638
2–3	0.233	0.483	− 11	19	27
3–4	0.307	0.554	− 2	21	31
4–5	0.193	0.439	− 17	17	24
4–12*	0.727	0.853	− 47	33	47
4–15	127.991	11.313	+ 88	434	623
5–6	0.216	0.465	+ 2	18	26
6–7	0.283	0.532	− 1	20	29
7–8	0.193	0.439	− 2	17	24
8–9	0.346	0.588	+ 10	23	32
9–10	0.400	0.632	− 9	24	35
9–14	41.063	6.408	+ 66	246	353
10–11*	0.207	0.455	− 23	17	25
11–12	0.197	0.444	− 7	17	24
12–13	26.498	5.148	+ 29	198	284
12–14	40.535	6.367	+105	244	351
12–15	127.286	11.282	+135	433	621
12–23*	4.327	2.080	+383	80	115
13–14	15.005	3.874	+ 76	149	213
13–18	108.201	10.402	+305	399	573
13–23*	30.759	5.546	+354	213	305
13–34	106.060	10.299	+332	395	567
13–36	117.798	10.853	+411	416	598
13–42	124.176	11.143	+101	428	614
13–48*	107.897	10.387	+572	399	572
14–15	86.753	9.314	+ 30	357	513
14–18	93.198	9.654	+229	370	532
14–21	97.071	9.852	+245	378	543
14–23*	44.796	6.693	+278	257	369
14–34	91.057	9.542	+256	366	526
14–36	102.795	10.139	+335	389	558
14–42	109.173	10.449	+ 25	401	576
14–48*	92.894	9.638	+496	370	531
15–17*	1.915	1.384	+214	53	76
15–18*	6.475	2.545	+199	98	140
15–21*	10.348	3.217	+215	123	177
15–23	131.547	11.469	+248	440	632
15–24*	2.837	1.684	+193	65	93
15–34*	4.334	2.082	+226	80	115
15–36*	16.072	4.009	+305	154	221

i–j	$g^{(i-j)}$	$\sqrt{g^{(i-j)}}$	Observed difference $c^i - c^j$	Critical value $\|y^{(i-j)}\|$	Model error $\|\nabla^{(i-j)}\|$
15–42	22.450	4.738	− 5	182	261
15–48*	6.171	2.484	+466	95	137
16–17	0.888	0.942	+ 36	36	52
16–19	9.669	3.110	+ 73	119	171
16–20	11.375	3.373	+ 19	129	186
16–23	133.728	11.564	+ 70	444	637
16–24	0.652	0.807	+ 15	31	44
17–18	5.752	2.398	− 15	92	132
18–19	5.439	2.332	+ 52	89	128
18–21	11.677	3.417	+ 16	131	188
19–20	9.618	3.101	− 54	119	171
19–23	141.601	11.900	− 3	457	655
20–21	16.520	4.064	+ 18	156	224
20–23	143.291	11.970	+ 51	459	659
20–34	13.498	3.674	+ 29	141	202
20–36	25.236	5.024	+108	193	277
20–42	31.614	5.623	−202	216	310
20–48*	15.335	3.916	+269	150	216
21–22	144.073	12.003	− 67	461	661
22–23*	5.200	2.280	+100	87	126
22–24	136.562	11.686	+ 45	448	644
22–34	138.059	11.750	+ 78	451	647
22–36	149.797	12.239	+157	470	674
22–42	156.175	12.497	−153	480	688
22–48	139.896	11.828	+318	454	651
24–25*	0.181	0.425	− 41	16	23
25–26	0.299	0.547	+ 21	21	30
26–27*	0.013	0.114	+ 26	4	6
27–28*	0.208	0.456	+ 26	17	25
28–29*	0.357	0.597	− 50	23	33
26–30*	0.653	0.808	+ 43	31	45
28–34	1.060	1.030	+ 1	40	57
28–36	12.798	3.577	+ 80	137	197
28–42*	19.176	4.379	−230	168	241
28–48*	2.897	1.702	+241	65	94
29–31*	0.531	0.729	− 65	28	40
30–31*	0.264	0.514	−106	20	28
31–34*	0.620	0.787	+116	30	43
34–35*	1.130	1.063	+ 53	41	59
35–36	12.358	3.515	+ 26	135	194
35–39*	12.912	3.593	−284	138	198
35–42*	19.024	4.362	−284	167	240
35–48*	2.991	1.729	+187	66	95
36–37*	10.763	3.281	−156	126	181
36–38*	11.771	3.431	−176	132	189
36–39*	17.824	4.222	−310	162	233
36–42*	24.178	4.917	−310	189	271
36–45	22.929	4.788	−138	184	264
36–47	14.890	3.859	+ 1	148	213
36–48*	13.027	3.609	+161	138	199

Observations marked with * exceed the critical value.

TABLE VI. Testing groups of mareographs. See page 29. c^i and $|\nabla c^i|$ in 10^{-3} g.p.u.

Group A (Atlantic–North Sea)

i	c^i	$g_{ik}c^i$	g_{ii}	$\frac{\sigma^2.\lambda_{16,68}}{g_{ii}}$	$\|\nabla c^i\|$
17	− 77	− 26	0.624	13934	118
18	− 62	+ 8	0.339	25649	160
19	−114	− 11	0.206	42209	205
20	− 60	+ 3	0.136	63934	253
21	− 78	− 1	0.111	78333	280
22	− 11	+ 19	0.193	45052	212
23	−111	− 19	0.196	44362	211
29	− 38	− 95	2.099	4142	64
31	+ 27	+254	3.608	2410	49
34	− 89	−395	4.331	2008	45
35	−142	−156	1.109	7840	89
36	−168	− 21	0.143	60804	247
37	− 12	+ 9	0.114	76272	276
38	+ 8	+ 1	0.170	51147	226
39	+142	+ 14	0.169	51450	227
42	+142	+ 3	0.101	86089	293

$g_{ik}c^ic^k = 78175$
$\hat{\sigma}_A{}^2 = 4886$
$\sigma^2 = 370$
$\lambda_{16,68} = 23.5$
$\sigma^2.\lambda_{16,68} = 8695$

Group MA (Mediterranean–Adriatic)

	g_{ij}										
i \ j	45	46	47	48	49	50	c^i	$g_{ij}c^i$	g_{ii}	$\frac{\sigma^2.\lambda_{6,68}}{g_{ii}}$	$\|\nabla c^i\|$
45	+0.0946	−0.0713	−0.0087	−0.0035	−0.0018	−0.0008	− 30	+16.04	0.0946	58668	242
46	−0.0713	+0.1672	−0.0509	−0.0140	−0.0049	−0.0030	−217	−18.52	0.1672	33194	182
47	−0.0087	−0.0509	+0.2307	−0.1061	−0.0170	−0.0078	−169	+14.72	0.2307	24057	155
48	−0.0035	−0.0140	−0.1061	+0.6881	−0.3032	−0.0720	−329	−93.32	0.6881	8066	90
49	−0.0018	−0.0049	−0.0169	−0.3032	+1.4015	−0.8877	−291	−11.16	1.4015	3960	63
50	−0.0008	−0.0030	−0.0078	−0.0720	−0.8877	+1.1095	−330	−82.14	1.1095	5002	71

$g_{ij}c^ic^j = 62106$
$\hat{\sigma}_{MA}{}^2 = 10351$
$\sigma^2 = 370$
$\lambda_{6,68} = 15$
$\sigma^2.\lambda_{6,68} = 5550$

Group MZ (Mediterranean)

	g_{ij}								
i \ j	45	46	47	48	c^i	$g_{ij}c^i$	g_{ii}	$\frac{\sigma^2.\lambda_{4,68}}{g_{ii}}$	$\|\nabla c^i\|$
45	+0.0946	−0.0713	−0.0087	−0.0048	− 30	+ 15.68	0.0946	50063	224
46	−0.0713	+0.1671	−0.0512	−0.0180	−217	− 19.55	0.1671	28342	168
47	−0.0087	−0.0512	+0.2299	−0.1187	−169	+ 11.57	0.2299	20600	144
48	−0.0048	−0.0180	−0.1187	+0.4951	−329	−138.78	0.4951	9566	98

$g_{ij}c^ic^j = 47475$
$\hat{\sigma}_{MZ}{}^2 = 11869$
$\sigma^2 = 370$
$\lambda_{4,68} = 12.8$
$\sigma^2.\lambda_{4,68} = 4736$

TABLE VII. Computation of "adjusted" *MSL* per group of mareographs. See page 31.

A (Atlantic – North Sea)

i	$g_{ik}A^k$
17	0.00702
18	0.00027
19	0.00002
20	0.00006
21	0.00017
22	0.00001
23	0.00001
29	0.08405
31	0.93924
34	3.58024
35	0.80643
36	0.00916
37	0.00195
38	0.01305
39	0.01113
42	0.00368

$$g_{AA} = g_{ik}A^iA^k = 5.45649$$
$$F^A = g_{ik}A^ic^k = -410.9$$
$$c^A = \frac{F^A}{g_{AA}} = -75 \cdot 10^{-3} \text{ g.p.u.}$$

MA (Mediterranean – Adriatic)

i	$g_{ik}A^k$
45	0.0085
46	0.0231
47	0.0403
48	0.1893
49	0.1869
50	0.1382

$$g_{MA,MA} = g_{ik}A^iA^k = 0.5863$$
$$F^{MA} = g_{ik}A^ic^k = -174.4$$
$$c^{MA} = \frac{F^{MA}}{g_{MA,MA}} = -297 \cdot 10^{-3} \text{ g.p.u.}$$

MZ (Mediterranean)

i	$g_{ik}A^k$
45	0.0098
46	0.0266
47	0.0513
48	0.3536

$$g_{MZ,MZ} = g_{ik}A^iA^k = 0.4413$$
$$F^{MZ} = g_{ik}A^ic^k = -131.1$$
$$c^{MZ} = \frac{F^{MZ}}{g_{MZ,MZ}} = -297 \cdot 10^{-3} \text{ g.p.u.}$$

TABLE VIII. Computation of coefficients B_k. See page 33.

i	$(B_k)_A$	i	$(B_k)_{MA}$	i	$(B_k)_{MZ}$	i	$(B_k)_A \cdot g^{kl}$
17	0.00129	45	0.0145	45	0.0222	45	0.17881
18	0.00005	46	0.0394	46	0.0603	46	0.16860
19	0	47	0.0687	47	0.1162	47	0.14406
20	0.00001	48	0.3229	48	0.0013	48	0.13181
21	0.00003	49	0.3188			49	0.12971
22	0	50	0.2357			50	0.12933
23	0						
29	0.01540						
31	0.17213						
34	0.65614						
35	0.14779						
36	0.00168						
37	0.00036						
38	0.00239						
39	0.00204						
42	0.00067						

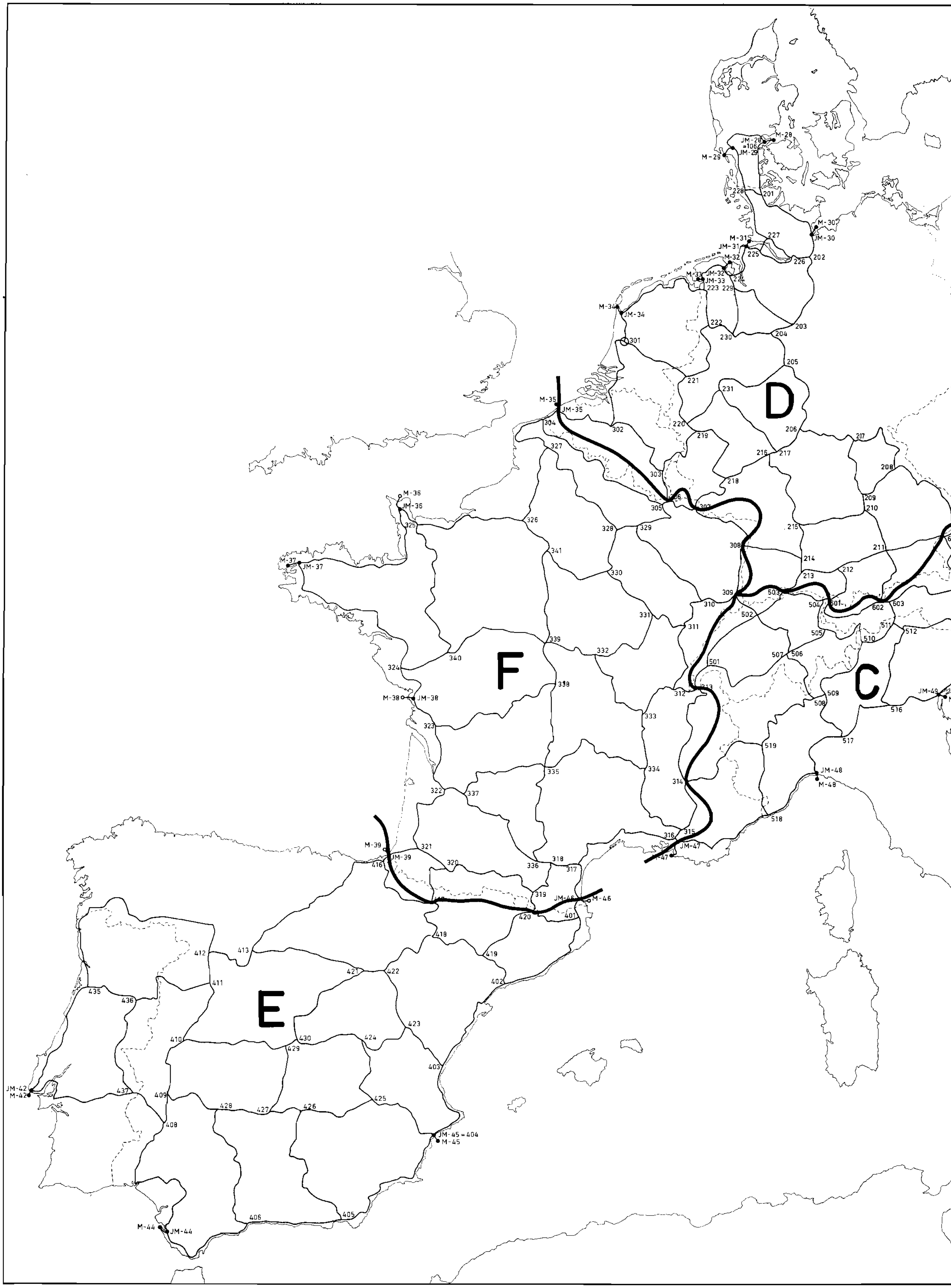
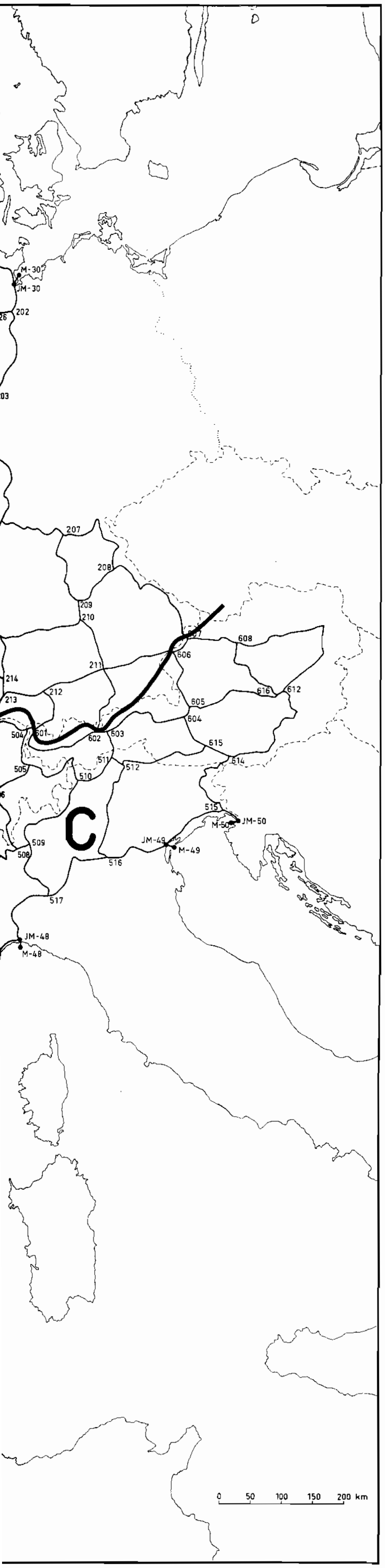


Figure 9. Partitioning of the net.



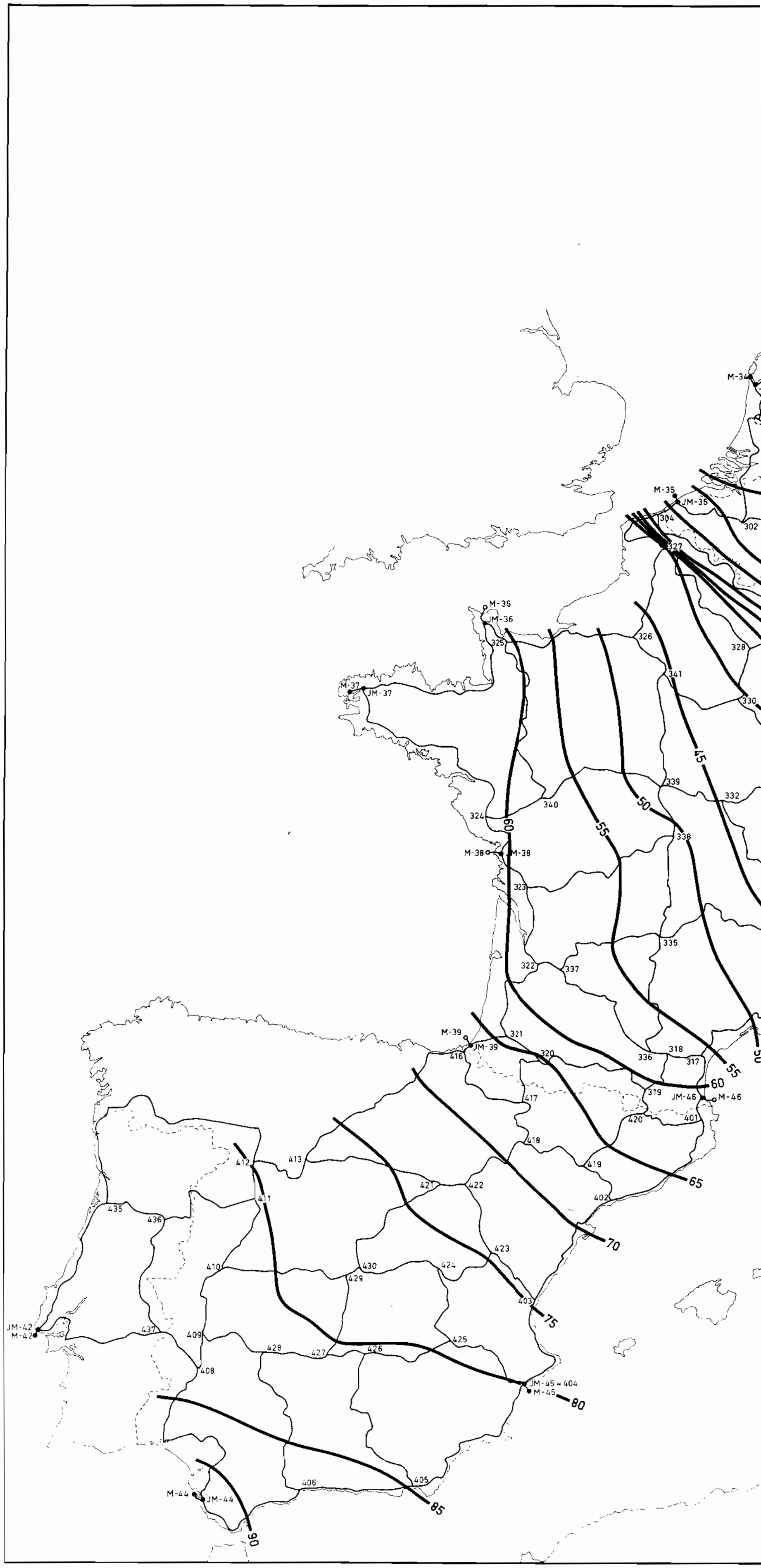


Figure 11. Lines of "equal standard deviations"

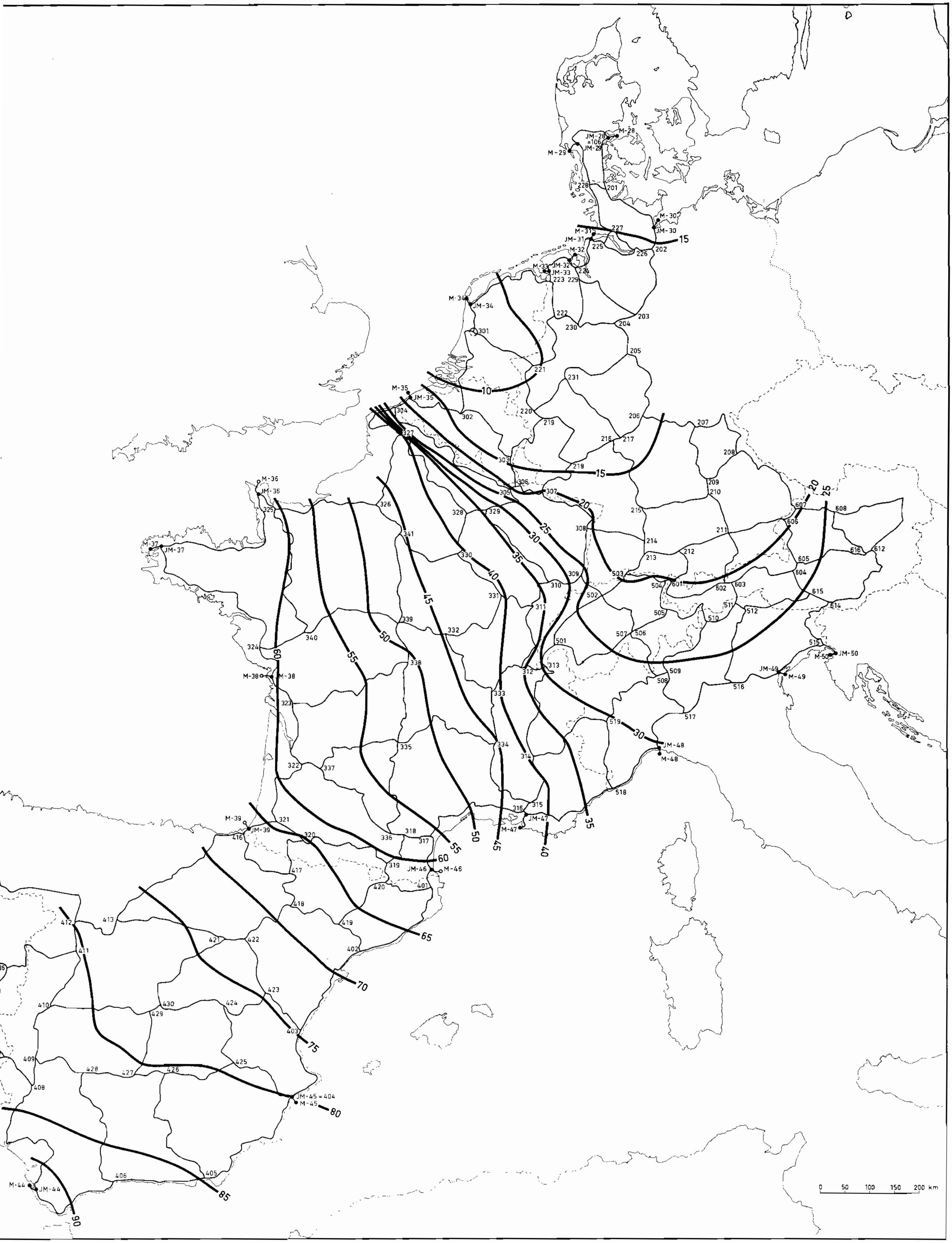


Figure 11. Lines of "equal standard deviations", cf. p. 18. Unit 10^{-3} g.p.u.

TABLE II. Matrix of weight coefficients of geo-potential numbers of MSL. Unit 10^{-6} (gpu)^2

r \ s	RM 1	RM 2	RM 3	RM 4	RM 5	RM 6	RM 7	RM 8	RM 9	RM 10	RM 11	RM
RM 1	132.232											
RM 2	132.102	132.215										
RM 3	132.082	132.152	132.322									
RM 4	132.047	132.095	132.122	132.229								
RM 5	132.021	132.060	132.077	132.125	132.214							
RM 6	131.994	132.023	132.035	132.064	132.099	132.200						
RM 7	131.961	131.984	131.992	132.013	132.036	132.068	132.219					
RM 8	131.945	131.964	131.971	131.988	132.005	132.029	132.106	132.186				
RM 9	131.873	131.883	131.886	131.895	131.902	131.911	131.934	131.946	132.052			
RM 10	131.743	131.744	131.745	131.747	131.748	131.749	131.752	131.754	131.764	131.876		
RM 11	131.676	131.676	131.677	131.677	131.677	131.677	131.678	131.678	131.679	131.685	131.701	
RM 12	131.513	131.513	131.513	131.513	131.513	131.513	131.513	131.513	131.513	131.514	131.514	131
RM 13	105.510	105.510	105.510	105.510	105.510	105.510	105.510	105.510	105.510	105.510	105.510	105
RM 14	90.990	90.990	90.990	90.990	90.990	90.990	90.990	90.990	90.990	90.990	90.990	90
RM 15	4.253	4.253	4.253	4.253	4.253	4.253	4.253	4.253	4.253	4.253	4.253	4
RM 16	2.070	2.070	2.070	2.070	2.070	2.070	2.070	2.070	2.070	2.070	2.070	2
RM 17	2.649	2.649	2.649	2.649	2.649	2.649	2.649	2.649	2.649	2.649	2.649	2
RM 18	2.949	2.949	2.949	2.949	2.949	2.949	2.949	2.949	2.949	2.949	2.949	2
RM 19	2.955	2.955	2.955	2.955	2.955	2.955	2.955	2.955	2.955	2.955	2.955	2
RM 20	2.963	2.963	2.963	2.963	2.963	2.963	2.963	2.963	2.963	2.963	2.963	2
RM 21	3.030	3.030	3.030	3.030	3.030	3.030	3.030	3.030	3.030	3.030	3.030	3
RM 22	131.520	131.520	131.520	131.520	131.520	131.520	131.519	131.519	131.518	131.516	131.514	131
RM 23	131.520	131.520	131.520	131.520	131.520	131.520	131.519	131.519	131.518	131.516	131.514	131
RM 24	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1
RM 25	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1
RM 26	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1
RM 27	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	
RM 28	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	
RM 29	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	
RM 30	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	
RM 31	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	
RM 32	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	
RM 33	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	
RM 34	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	
RM 35	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	
RM 36	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	
RM 37	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	
RM 38	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	
RM 39	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	
RM 42	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	
RM 44	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	
RM 45	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	
RM 46	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	
RM 47	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	
RM 48	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	
RM 49	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	
RM 50	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	

11	RM 12	RM13	RM14	RM15	RM16	RM17	RM18	RM19	RM20	RM21	RM22	RM23	RM24	RM25	RM2
.701															
.514	131.524														
.510	105.510	105.994													
.990	90.990	90.990	90.991												
.253	4.253	4.253	4.253	4.268											
.070	2.070	2.070	2.070	2.070	2.083										
.649	2.649	2.649	2.649	2.649	2.070	2.945									
.949	2.949	2.949	2.949	2.949	2.070	2.649	8.105								
.955	2.955	2.955	2.955	2.955	2.070	2.649	7.196	11.726							
.963	2.963	2.963	2.963	2.963	2.070	2.649	6.518	7.770	13.432						
.030	3.030	3.030	3.030	3.030	2.070	2.649	4.284	4.395	4.526	12.140					
.514	131.491	105.510	90.990	4.253	2.070	2.649	2.949	2.955	2.963	3.030	137.993				
.514	131.491	105.510	90.990	4.253	2.070	2.649	2.949	2.955	2.963	3.030	134.289	135.785			
.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.456	1.481		
.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.378	1.456	
.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1.161	1,161	1.16
.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.157	1.15
.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.974	0.97
.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.818	0.81
.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.607	0.60
.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.511	0.51
.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.403	0.40
.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.35
.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.09
.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.121	0.12
.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.22
.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.231	0.23
.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.241	0.24
.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.252	0.25
.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.25
.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.25
.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.253	0.25
.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.255	0.25
.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.269	0.26
.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.280	0.28
.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.28
.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.283	0.28

25	RM26	RM27	RM28	RM29	RM30	RM31	RM32	RM33	RM34	RM35	RM36	RM37	RM38	RM39	RM4
56															
61	1.165														
57	1.157	1.162													
74	0.974	0.974	0.994												
18	0.818	0.818	0.818	0.999											
07	0.607	0.607	0.607	0.602	0.702										
11	0.511	0.511	0.511	0.512	0.497	0.556									
03	0.403	0.403	0.403	0.404	0.399	0.416	0.470								
50	0.350	0.350	0.350	0.350	0.347	0.358	0.376	0.417							
90	0.090	0.090	0.090	0.090	0.089	0.091	0.094	0.100	0.246						
21	0.121	0.121	0.121	0.121	0.122	0.119	0.115	0.107	0.029	0.942					
24	0.224	0.224	0.224	0.224	0.226	0.219	0.210	0.195	0.052	0.418	12.252				
31	0.231	0.231	0.231	0.230	0.232	0.225	0.216	0.200	0.053	0.391	8.889	16.289			
41	0.241	0.241	0.241	0.241	0.243	0.235	0.225	0.208	0.056	0.349	5.320	6.747	10.159		
52	0.252	0.252	0.252	0.252	0.254	0.246	0.236	0.218	0.058	0.307	3.506	4.173	5.523	12.584	
3	0.253	0.253	0.253	0.253	0.255	0.247	0.236	0.218	0.058	0.303	3.381	4.010	5.275	10.494	18.688
3	0.253	0.253	0.253	0.253	0.255	0.247	0.237	0.219	0.058	0.303	3.370	3.995	5.253	10.337	16.286
3	0.253	0.253	0.253	0.253	0.255	0.247	0.237	0.219	0.058	0.302	3.348	3.967	5.210	10.060	13.925
5	0.255	0.255	0.255	0.255	0.257	0.249	0.239	0.220	0.059	0.294	3.103	3.651	4.739	7.765	8.362
9	0.269	0.269	0.269	0.269	0.271	0.262	0.251	0.232	0.062	0.245	1.710	1.927	2.331	3.086	3.178
0	0.280	0.280	0.280	0.280	0.282	0.273	0.261	0.241	0.064	0.207	0.844	0.912	1.032	1.220	1.240
3	0.283	0.283	0.283	0.282	0.285	0.276	0.264	0.243	0.065	0.197	0.671	0.715	0.789	0.894	0.905
3	0.283	0.283	0.283	0.283	0.285	0.276	0.264	0.243	0.065	0.196	0.647	0.688	0.757	0.853	0.863

The matrix is symmetric

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33	RM34	RM35	RM36	RM37	RM38	RM39	RM42	RM44	RM45	RM46	RM47	RM48	RM49	RM50
417														
00	0.246													
07	0.029	0.942												
95	0.052	0.418	12.252											
200	0.053	0.391	8.889	16.289										
208	0.056	0.349	5.320	6.747	10.159									
218	0.058	0.307	3.506	4.173	5.523	12.584								
18	0.058	0.303	3.381	4.010	5.275	10.494	18.688							
19	0.058	0.303	3.370	3.995	5.253	10.337	16.286	23.275						
19	0.058	0.302	3.348	3.967	5.210	10.060	13.925	14.789	17.373					
20	0.059	0.294	3.103	3.651	4.739	7.765	8.362	8.419	8.529	10.800				
32	0.062	0.245	1.710	1.927	2.331	3.086	3.178	3.186	3.202	3.394	6.058			
41	0.064	0.207	0.844	0.912	1.032	1.220	1.240	1.242	1.246	1.289	1.607	2.463		
43	0.065	0.197	0.671	0.715	0.789	0.894	0.905	0.906	0.908	0.930	1.089	1.358	2.205	
43	0.065	0.196	0.647	0.688	0.757	0.853	0.863	0.864	0.865	0.886	1.029	1.262	1.863	2.483