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# THE GLOBAL MAPPING OF GRAVITY WITH TWO SATELLITES

by

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ABSTRACT.

This is a report on a method for the detailed and global mapping of the planetary gravitational field using two artificial satellites that carry devices to track each other along the same near-circular, near-polar orbit, separated by a few hundred kilometers and as low as the atmosphere would allow. They are kept aloft, in spite of air drag, by the action of small rocket engines that maintain a proof-mass inside each spacecraft in constant free fall. The signal is the relative line-of-sight velocity of the proof masses, averaged over several seconds.

A specially tailored analytical perturbation theory, where the reference orbit is periodical and obtained by numerical integration in a low-degree zonal reference field, is used here to derive a linearized model for the signal. The "lumped coefficients" of the perturbations can be calculated very efficiently with a technique that relies heavily on the Fast Fourier Transform algorithm, and whose principle is similar to that of Gauss' method for integrating Lagrange's planetary equations. Computer simulations of the relative motion of the satellites in a field whose potential is the sum of zonal spherical harmonics up to degree 300, suggest that the model is accurate to better than 1% at most frequencies present in the spectrum of the signal. The programs used for the simulations are explained and listed in an appendix.

Detailed consideration is given to a method for estimating from the data all potential coefficients up to a high degree and order (such as 300). This method is based on the choice of a common orbit that closes upon itself after enough days have elapsed to resolve all the unknown coefficients. This orbit gives a rotationally symmetrical distribution of data. After taking care of the non-periodical component of the signal (due to orbit estimation errors and secular resonant effects) by introducing extra unknowns, the normal equations of the adjustment become very sparse. With a suitable ordering of unknowns, it shows an "arrow" structure, the "shaft" consisting of diagonal blocks. It is feasible to solve such a system (in spite of its great size) with ordinary modern computers, and also to find the formal accuracies of the results by a partial inversion of the normal matrix.

KEYWORDS: satellite goedesy; satellite-to-satellite tracking; Earth model; analytical perturbations; sparse matrices; Hill's equations; celestial mechanics; GRAVSAT; GRM; adjustment; least squares collocation.

A NOTE OF THANKS.

A research visitor fellowship from the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.), the Dutch research foundation, has allowed me to spend a whole year in Holland, working exclusively on the study of the charting of gravity by space techniques. During this time, I have accumulated reasons for being thankful to a number of people from in and outside this green and level land. First of all, I want to express gratitude to those who made my life easier for twelve months: Reiner and Renate Rummel and their neighbours, Ate and Adjie Bos, who fed me, housed me, provided me with transport, cheered me up, showed me around, and took with equanimity my grumblings about the weather peculiar to the place. Outside Holland, my appreciation includes the three Chrises in Munich who made my escapades there so pleasant, as well as scientifically productive: Chris Reigber, and Chris and Christine Rizos. Concerning the work itself, several people in Europe and in the USA contributed information, advise, and encouragement. In Europe, Reiner Rummel and Chris Reigber gave my ideas the benefit of the doubt and listened to them patiently, while Diderick van Daalen showed me how to factor "arrow" matrices, a key step in the implementation of such ideas. My research involved a lot of computing, and here I was helped greatly by Chris Reigber, who provided me with a copy of the subroutine COWELL for integrating orbits, and by Chris Rizos, who showed me the way to use it. Boudewijn van Gelder computed some orbits for me with a version of the program GEODYN, which I repeated with my own software, thus checking it before embarking on extensive and expensive calculations.

Fritz Brouwer gave me a good deal of practical help in getting familiar with the interactive system at the Technical University of Delft, and so did several other members of the staff of the Afdeling der Geodesie. Outside Europe, Peter Bender in Colorado, Dick Rapp in Ohio, Steve Klosko, Clyde Goad and, last but not least, Carl Wagner in Maryland, kept me in touch with developments, made suggestions, asked questions, and made criticisms. These were all constructive, and kept my morale up.

We are fellow sufferers of the same obsession, which is to put "paid" to certain claims by the members of the Flat Earth Society, and it is very nice to know that, somewhere below the curve of this globular world, some people actually care about what one is doing.

When I wrote the draft of this report, it did not look at all like the eye-pleasing object now in front of you. The transformation has been the product of hard and careful work by Wil Coops-Luijten, who typed the clean copy. Also some of the credit must go to Brett Saunders, who did the illustrations. Nothing we ever do is totally of our own making, and this is most true of any serious effort in science, however small the outcome. I have many thanks to give to many people; I have mentioned some here, others escape me right now. To all, my gratitude is real.

TO THE READER.

After a long year working on the problem of how to map the gravitational field by means of satellite-to-satellite tracking, all I claim is that my results, though preliminary, look to me encouraging. The real situation is so much more complicated and untidy than what can be simulated in any computer, or can be expressed by any set of formulas. The ideas put forward here should be tested further, and harder, than I have had time for. When writing this report, I have done my best to weed out inconsistencies and errors; I hope that those that remain are only minor ones that the reader will be able to find and put aright without much inconvenience. The mathematical principle at the heart of the matter, rotational symmetry, is simply beautiful, although in this work, because of the limitations of the writer, this quality may not be easy to see. Beauty alone, unfortunately, is no guarantee of goodness, but I do believe that any solution to this great puzzle must have harmony and grace. Whatever their ultimate fate, if the ideas proposed here can help to clarify the problem and to move the discussion forward, they will have served their purpose well.

The quest to know the shape of the Earth is as old as thought. Fulfilling it has long been a task in the overall enterprise of understanding the world. It is like a thread in a very long rope, spun by the hands of countless men through history, who have joined in and done their job for all sorts of reasons, but always out of curiosity as well. This thread runs from the sunny days of Anaxagoras and Ptolomei to our own interesting times. In a year of mixed seasons and storms, as the days grow darker, my wish is that we may continue to spin it for a while yet.

Delft, Autumn of 1983.

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## INTRODUCTION.

"There is nothing too little for so little a creature as man. It is by studying little things that we attain the great knowledge of having as little misery and as much happiness as possible".

Samuel Johnson
(quoted by Boswell in his *London Journal* of 1762-63).

Over thousands of years, the understanding of the shape of the Earth based on scientific observations gradually evolved from the recognition of its essentially spherical character to the measurement of its mean radius and, eventually, of its flattening. Slowly, additional information became available through the surveying of the land masses and the very sparse coverage, mostly with ship measurements, of the oceans. In the late nineteen-fifties, the first artificial satellites were put in orbit and, from then on, the situation began to change drastically and at a lively pace. Analysis of radar and optical tracking data soon showed that the Earth is slightly pear-shaped, with the southern half a bit larger than the northern, and revealed some considerable departures from the general ellipsoidal shape with longitude as well as latitude: the first "Earth's models" had appeared. Those early gravity maps only displayed very broad features, like the great depression in the geoid south of India. Over the years, the "models" (global maps in the form of truncated spherical harmonic expansions of the potential) were improved continuously, as more data from satellites became available. In the early Seventies their global accuracy stood at better than 10 m, and their resolution at some 1000 km. Then, between 1975 and 1978, two satellites (GEOS 3 and SEASAT) were launched carrying very accurate radar altimeters. Their measurements were used, among other things, for mapping the mean sea surface, which departs from a true level surface (the ocean geoid) by no more than two meters. As a consequence, the oceans became the best known parts of the globe to geodesists and those scientists interested in the irregularities of the gravity field, reversing drastically the previous situation. At the same time, the patient and careful accumulation of gravimetry from all over the world, made possible by improvements in technique, continuing exploration and better scientific exchanges, produced large sets of more conventional terrestrial data. The combination of the satellite models, the altimetry, and the terrestrial data, resulted in global maps with a resolution close to 100 km, such as those made by Lerch et al. (1981), Rapp (1982(a)), etc. These recent maps are particularly good over the oceans (thanks to the use of satellite altimetry), where they are believed to be accurate to about one meter. The picture they give of the fine detail of oceanic anomalies is truly impressive, showing clearly the wrinkles in the geoid caused by the mountain ranges, trenches and faults marking the boundaries of the plates that make up the Earth's crust (as an example, see Rapp, 1982(b)). The very broad details, on the other hand, have become better known thanks to the use of laser ranging to very high (and, thus, very visible) satellites like LAGEOS and STARLETTE. Most of the problems today lie, probably, in the medium range, from 3000 km to 300 km. There, sea surface departures from the horizontal due to currents, errors in the oceanic maps caused by uncertainties in the orbits of the altimeter satellites, and poor, inconsistent or unavailable data on land, contribute to the unreliability of existing gravity field models. Those obtained purely from satellite tracking are affected by the uneven distribution of tracking stations, which can observe most satellites, the very highest excepted, only when they are in their proximity.

The limitations of present day gravity field maps, in spite of the impressive progress they represent over the situation twenty, and even ten years ago, has maintained alive the interest of geodesists and geophysicists in the development of new methods for charting the gravity field on a global basis. The main problem with tracking satellites from the ground is that, in order to observe them for long periods of time to get good coverage, they have to be in very high orbits. But the gravitational anomalies, and the orbital perturbations that they cause and that are the source of the information in the tracking data, decrease rapidly with altitude, particularly the finer details. To obtain a sharp picture, the satellite has to be low, and to get this picture over a large area, it has to be tracked from afar. The solution is to track it with another satellite. Tracking a low spacecraft from a high one can achieve this, although coverage may be limited to less than the whole world. This idea has been tried already twice, using first the ATS/6 geosynchronous satellite to track the combined Apollo-Soyusz craft in a low orbit, and then ATS/6 and GEOS-3. Both experiments were conducted in the mid-Seventies. Perhaps the most impressive demonstration of what can be achieved by tracking satellites over very large areas came a decade earlier, and was not part of the study of the Earth, but of the Moon.

Muller and Sjogren (1968) used data from the Apollo Lunar Orbiters to show evidence for the existence of very large and extense mass concentrations, or "mascons", under the lunar plains (the maria) of the visible side. In order to cover the whole Moon (or the whole Earth), instead of having two spacecraft far apart, one low and one high (in the case of the lunar orbiters, the Earth was the "high spacecraft"), we could have two that would follow each other along much the same orbit, and at as low an altitude as atmospheric friction would allow. Among the first to propose this principle, known as "low-low satellite-to-satellite tracking", was Wolff (1969). This is one way of trying to obtain a homogeneous set of measurements from a single source with global coverage. The alternative, and it has remained the only one over the years, is to put a gravity gradiometer in orbit (for a review of this idea, see Forward, 1973).

The deficiencies of existing gravity maps become clearer when one compares the spherical harmonic coefficients of the same degree and order obtained by different groups of scientists. The latest such comparison that I know of has been made by Reigber (1983), and the discrepancies among the various sets of results appear to be considerable. More reliable models are needed to improve the calculation of spacecraft orbits; to provide a better geoid as a reference surface for heights (mean sea level); to remove trends from the data before making detailed regional maps of the field at the Earth's surface; to determine the tilt of the sea due to the currents, and in this way to plot such currents (averaged over some length of time); etc. Also the study of both gross and fine structures within the solid Earth, its crust and upper mantle in particular, can be done better if a reliable and comprehensive gravity survey is available. Gravitational anomalies are the product of the irregular distribution of matter inside the planet, so a knowledge of them can help to unravel this distribution when combined with other information about the interior, such as seismic and geological data. For this reason, the National Academy of Science of the USA (NAS) has sponsored the discussion of means and objectives regarding future gravity surveys using satellites. A report (1979) was produced after a workshop under NAS auspices and the main conclusions, summarized, were: (a) for geophysical and geological studies, the gravity field at the surface has to be known with a resolution of 100 km and an accuracy of between 2.5 and 10 mgals, depending on the applications, (b) oceanographers need a geoid accurate to 10 cm for features between 100 km and 3000 km in size.

To understand the technical difficulties in fulfilling these requirements, one should consider that a 10 mgal anomaly on the ground, covering a square area of some 100 km on each side, would change the relative velocity of two low satellites, a few hundred kilometers from each other and at a height of some 200 km, by less that 10 microns per second ($10^{-6}$m $s^{-1}$). The same anomaly at the same height would cause a change in the gradient of gravity of a few hundredths of Eötvös units, or less than $10^{-11}$ of the normal acceleration of gravity per meter. This means that extremely sensitive instruments are required, beyond what has been in use until now. For satellite-to-satellite tracking, a study commissioned by the European Space Agency in the 1970's proposed a laser tracking system placed on the Space Shuttle or on an independent orbiter, ranging to two reflecting targets in its vicinity (SLALOM report, 1978). More recent information on this idea can be found in the CSTG Bulletin, No. 2, of 1980. The purpose of this experiment, if conducted from the Shuttle, would be to cover some regions over Eurasia and the Mediterranean. The National Aeronautics and Space Administration of the US, for its part, has nurtured for years plans for putting two satellites in orbit, capable of tracking each other by a radar interferometric system being designed and tested at the Applied Physics Laboratory, which is said to be able, eventually, to reach accuracies of better than 1 micron per second (Pisacane and Yonoulis, 1980). As one way to increase sensitivity is to keep the satellites as low as possible, the US idea would have the two spacecraft provided with small rocket engines, or thrusters, able to fire in all directions to compensate with their impulses for the loss of velocity caused by air drag, so the pair may stay in orbit for some six months, to ensure that enough data is gathered. At the same time, by using a closed-loop control system that senses the position of a proof-mass inside each satellite, such mass would be kept in permanent free fall, affected only by gravitational forces, simplifying the analysis of the measurements, which record the relative velocity of one proof-mass respect to the other.

The alternative to satellite-to-satellite tracking is satellite gradiometry. In Europe, the French have considered this possibility, as shown by a recent study from a group of scientists from various organizations (Balmino et al., 1981). Their idea is to perfect a three-axial accelerometer already used in France's space programme, and to put an ensemble of several of these in orbit. In the US, present efforts are concentrated on developing a completely new instrument, where the sensors are superconducting accelerometers inmersed in liquid helium, and are derived from the design ideas for absolute gravimeters and gravity-wave detectors perfected over the last decade. The aim is to build a device with an accuracy of one thousandth of an Eötvös or better (Paik, 1981).

There seem to be three main difficulties in the way to the detailed mapping of gravity from space: building adequate instruments, financing the whole operation, and analyzing the data to create the actual maps. All I can say about the first two is that they are quite considerable. The last one, data-processing, is also on the formidable side; of the three, it is the only one with which this work is concerned. To map the field globally to a resolution of one hundred kilometers or so, the mathematical representation of the map must have, regardless of the actual type of base functions chosen (spherical harmonics, area means, etc.), in the order of $10^5$ parameters. The values of these paramaters are unknowns to be estimated from several millions of observations (assuming a six-month's mission and a sampling rate of a few seconds). One possibility is to take data covering a certain region and map that part of the world separate from the rest. These local maps, or solutions, have been produced already using some of the existing satellite-to-satellite data from last decade's experiments (see, for example, Marsh and Marsh, 1977, and Kahn and Wells, 1979). No counterpart to these exist for gradiometry, as no gradiometer has been used in a mapping mission yet. The methods for producing such local maps already exist, in principle, though a good deal of "tunning up" is needed before they can be used with confidence, a major problem being numerical instabilities when solving the adjustment equations. The logical alternative would be producing a world-wide, or global, map. Here one must deal with the huge numbers of unknowns and observations mentioned already. In addition to the parameters describing the gravitational anomalies, one must include others to account for the imperfectly known position of the satellites in their orbits. In the case of the gradiometer, assuming that several second derivatives of the geopotential can be measured simultaneously, it appears possible to split the problem into two independent parts: mapping the field and estimating the orbits, and that both parts can be carried out very efficiently, so global estimation may be feasible even with present day computers (Rummel and Colombo, 1983). The likely accuracy of the estimated gravitational parameters (in the form of spherical harmonic potential coefficients) and of the orbital parameters may be very high with an instrument like the superconducting gradiometer (Colombo and Kleusberg, 1982). The problem is basically more difficult with satellite-to-satellite tracking.

The main difference between satellite gradiometry and satellite-to-satellite tracking stems from the latter involving two, instead of one, spacecraft. There is also the nature of orbital perturbations, which have a more complicated mathematical representation than the second derivatives of the potential. The key to a feasible estimation of the gravitational parameters is an ordered spatial arrangement of the data, and a relatively simple linearized model that approximates the effect of the field anomalies on the signal. The latter is readily available for gradiometry, from the differentiation, twice over, of the spherical harmonic expansion of the potential. In the case of the satellite pair, both members must retain their relative positions within rigid limits for long periods of time, *as they together form the actual instrument*, and changing their distance, relative heights, etc., is equivalent to changing the instrument, something quite foreign to the gradiometer. A realistic mathematical model must be found which is also simple enough to be used in calculations of reasonable length, and to allow the exploitation of symmetries in the measurements to cut down the computing effort. The search for models for satellite-to-satellite tracking that are both handy enough and accurate enough has been going on for quite some time. Activity over the last few years has been centered around the discussions of the "GRAVSAT working group" (GRAVSAT was the original name for NASA's future gravity mission) in the US. A number of simplified models for the signal have been considered, including Wolff's old idea of equating relative velocity changes to the changes in gravitational potential between the satellites (divided by the mean velocity), or that of taking the time-derivative of the range-rate as equal to the difference between the gravitational accelerations of the two spacecraft projected along the direction of their line-of-sight. These models were proposed to side-step the supposedly intractable description of the signal in terms of orbital mechanics (such description seemed useful only in local solutions; C. Schwartz (1970) was one of the first to study it; see also Douglas et al (1980)).

In 1982, computer simulations of relative line-of-sight velocities were conducted by Lerch and others at Goddard Space Flight Center, USA, and the computed signal was used to assess the various simplified models then under study. This showed that none of them could be described as better than "fair", and that probably none was "good enough". On the other hand, these simplistic models have been very useful for error analysis, i.e. to guess how good the results of a mission could be, given certain characteristics (separation, height, etc.) and a certain quality of data.

For the global charting of the field, of many studies conducted over the years, those by Breakwell (1979), and Rummel (1980), are representative. The reassuring thing about them is that they all tend to arrive at similar conclusions, even when the approximations are quite different. A search for a better model, incorporating principles of orbital mechanics, become clearly necessary after the simulations at Goddard. Some attempts were being made even before, as exemplified by the work of Gaposhkin and Kaula as members of the "GRAVSAT group" (Kaula (1983) has presented recently his own ideas on this matter [1]). Finally, a breakthrough occurred when, in December of 1982, Wagner and Goad delivered to the Fall Meeting of the American Geophysical Union a joint paper suggesting the possibility of estimating vast numbers of potential coefficient using a linear model derived from classical analytical orbital perturbations' theory.

[1] See also his paper in J.G.R. (Red), pp. 8345-9349, Oct. 1983, Vol. 88, No. B10, "Inference of Variations in the Gravity Field from Satellite-to-Satellite Range Rate".

Their solution to such an enormous estimation problem depends on a principle that is both simple and beautiful. Imagine that two satellites follow each other a few hundred kilometers apart and virtually along the same orbit. To obtain the highest resolution, this orbit is low, near-circular and close to polar. In order to have some structure in the signal, the "one instrument" principle is enforced by using the thrusters to keep the spacecraft "flying" in a tight formation. Imagine, further, that their common height is chosen so that the orbital period is congruent with a whole number of revolutions of the Earth, so, after some months, the satellites return to the same places where they started from, as seen by an observer fixed to the Earth. From the point of view of this observer, the common orbit is a helix that wraps itself around a nearly spherical surface of revolution until it comes back to its start. If the mission goes on afterwards, the same places will be reached again and the same signal will be measured once more, except for some secular phenomena kept on check by the corrective manoeuvres[(1)]. The whole mission becomes periodical, "biting its own tail". To ensure this, the mean orbit is also "frozen", so it does not precess in its own plane (though the plane itself may move). The temporal periodicity of such an orbit corresponds to a beautifully symmetrical pattern in space. The "tail biting" orbit, and the points where measurements are taken at regular intervals along it, are such that an observer looking from any of these points can tell his latitude, if the position of the pole is visible as well, but not his longitude: each successive turn of the helix is identical to any of the others. This means that both the helix and the pattern of measurement have *rotational symmetry* around the Earth's axis. This is the principle at the heart of the global solution.

The idea of using a rotationally symmetrical orbit to speed up computations was tried for the first time, in connection to satellite-to-satellite tracking, in an error analysis that I did some years ago using one of the more naïve models (Colombo, 1981a). The great merit of Wagner and Goad has been to remove a collective mental block, showing that this symmetry also works its wonders when a much more realistic model is used, and that there is no unsurmountable problem in trying to base such a model on analytical perturbations' theory. This approach is, of course, only one of many alternatives, but, in my opinion, it is the best proposed

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(1) The spelling adopted is mostly that of the Concise Oxford Dictionary.

up to now for global mapping, and so I have followed it in the work reported here. Models based on orbital mechanics, until the time when I started my own research, have relied on the "literal" formulation of analytical perturbations. This is ideal for *understanding* what happens when the main characteristics of an orbit (semi-major axis, eccentricity, inclination, etc.) are changed, but has the drawback of being chock-full with the beautifully long and complex expansions that celestial mechanicists are so fond of. This makes *computing* more than a bit awkward. There is a very clever alternative, if the latter is the goal, attributed to Gauss and known as the "numerical" formulation, where certain quantities that have very complicated "literal" forms, requiring many arithmetic operations, can be calculated by much simpler, but equivalent, methods. As the purpose of computing either "literal" or "numerical" formulas is to obtain the Fourier coefficients of the perturbations and, eventually, of the signal, it seems a good idea to try to couple the "numerical" approach with powerful mathematical tools such as the Fast Fourier Transform for harmonic analysis. The advantage of this is that one does not need to exclude all terms above the first power in the eccentricity, as it has been the case until now because of the difficulties posed by the "literal" formulation (and one has to include higher powers if a very high resolution of the field is the objective).

In addition to having joined together a "numerical" formulation to the Fast Fourier Transform for computing the analytical perturbations' Fourier coefficients, I have adopted a reference orbit that is not merely "frozen" in the sense that its mean ellipse does not change its shape and orientation, except for a slow precession around the terrestrial axis, but that actually closes upon itself in its plane, so it appears to an observer on this plane to be perfectly periodical.
This orbit, integrated numerically in a reference field comprising the central force term and the low degree zonals of the spherical harmonic expansion (according to one of the existing satellite models), is not the usual Keplerian ellipse along which the perturbations are linearized in the classical approach (Kaula, 1966) known as "variation of constants" in astronomy. Being a true orbit in the zonal reference field,

it contains both long and short period perturbations, and the effect of higher degree zonals in addition to those of the oblatness. As a result, it comes much closer to the actual path of each satellite than the classical ellipse (and the satellites can follow it for longer periods without being forced to do so by the action of their engines). This means a greater accuracy in the model than it is possible with the conventional ellipse and a formulation limited to first power in the eccentricity. That a greater accuracy is really needed to deal with high degree spherical harmonics (about 100 km wavelength) is apparent from the results of some of the calculations reported in this work. Simpler models, truncated at the first power, have performed reasonably well, but only when tested up to degree and order eight or thereabouts, which does not show the actual difficulties at much finer resolutions. Abandoning the classical ellipse for the sake of accuracy requires giving up also on the use of the classical theory. For this reason I have developed an approach specially tailored to the problem, based on the solution of the approximate variational equations sometimes known as Hill's equations, which for low, near-circular orbits are correct to the order of the Earth's flattening. This means an ultimate accuracy of a few parts per thousand in the model. These differential equations are related to the ones derived by Hill (1878) for the study of perturbations in the lunar orbit caused by the Sun. As in Hill's original work, the formulas correspond to a system of Cartesian coordinates rotating uniformly in the orbital plane. They have some application in studying the rendezvous of two spacecraft (see Kaplan, 1976).

The general plan of this work is as follows: section 1 explains the main concepts relevant to the linearized model, and presents the derivations leading to it. Hill's equations, whose solutions are part of the model, are obtained by linearizing the equations of motion, and the main assumptions underlying the model are discussed in detail. Section 2 deals first with the periodical reference orbit and how to compute it; then it gives a complete "literal" treatment of the forcing terms of Hill's equations, and finally it arrives at the Fourier-series form of these terms, a form that simplifies greatly the analytical solutions of the variationals. The mathematical details of how these solutions are obtained are shown in Appendix I (Appendix II covers some aspects not relevant to the central problem, but that complete the orbit theory). There is, also in section 2, an explanation of how the "numerical" approach can be used to implement the theory in a computer (without actually employing the lengthy formulas that took me so much trouble to derive and may require so much patience of the reader who tries to follow them).

Section 3 gets down to the central business, which is the estimation of a huge number of potential coefficients from an enormous number of measurements. Here the methods of estimation compatible with the approach (least squares adjustment, least squares collocation) are presented in their more relevant aspects; the linearized observation equations are derived from the model of section 1 and the perturbation theory of section 2; and the choice of orbit that brings about the rotational symmetry of the data points is considered in some detail. This is followed by an explanation of the structure that this choice and the resulting symmetry induce in the normal matrix. As I consider the adjustment of some additional parameters, known here as "arc parameters", that soak up all the main aperiodic fluctuations in the signal, the normal matrix is not block diagonal, as in my old error analysis of 1981 or in Wagner and Goad's paper, but an "arrow" matrix, with a "shaft" of diagonal blocks and two side "wings" (at the lower and at the right edges of the matrix) corresponding to the extra parameters.

This matrix, which is huge but very sparse, can be set up directly, without having to create first a matrix of observation equations (which is nearly full and much larger) and the arrow-shaped normals can be solved quite easily by Cholesky decomposition. Also, it is possible to calculate the formal variances of the estimated potential coefficients by a partial inversion of this matrix. The overall procedure can be implemented in existing sequential processing machines to estimate spherical harmonic coefficients up to degrees as high as 300 in a few hours and without the risk of undue accumulation of rounding errors. After discussing several problems associated with obtaining and using such large global models, the section closes with the consideration of the *complementary role* of local and global mapping. Section 4 presents the results of computer simulations that test the quality of the model of the satellite-to-satellite tracking signal, using a very high degree field of zonals (up to $n = 300$), as far as it has been practicable within the time at my disposal. The listings of the computer programs that I have used for these simulations, together with some explanation of how they work, can be found in Appendix III. Finally, Appendix IV shows in full detail, frequency by frequency, the spectra of purely gravitational perturbations of the signal for satellite separations of 100 km and 300 km, respectively.

# 1. THE MATHEMATICAL MODEL.

## Preliminary comments and overview.

This section introduces the basic concepts needed to develop a mathematical model of the signal for a satellite-to-satellite tracking mission. As treated here, this signal is the result of substracting, from the relative velocity between two spacecraft subject only to the Earth's gravitational field, the value of this velocity computed using a field model with incorrect parameters and erroneous estimates of the initial state (position and velocity) of each satellite. This difference is a function of the unknown corrections needed to set both the field model and the initial states right. The model proposed here is the result of linearizing this function about known parameter values, so that it can be used in a conventional least squares adjustment, plus some approximations that simplify the mathematical formulas.

The two spacecraft are supposed to follow each other along nearly circular and polar orbits that are almost identical. The orbits are as low as possible, to allow the greatest sensitivity to the irregularities in the field that the mission must map. Each satellite carries a proof-mass inside, which is maintained in permanent free-fall by the action of small rocket engines that fire intermitently to avoid the mass touching the walls that enclose it, thus eliminating the effect of aerodynamic drag and other non-gravitational forces on the mass. The relative motion measured is that of the two proof-masses, using radar to find the changes in separation between both spacecraft while also sensing the varying positions of the proof-masses inside them. Depending on the way the measurements are taken, the signal may be either the instantaneous relative line-of-sight velocity, or its average over a number of seconds, which is directly proportional to the change in distance over the same period, also known as change in "biased range". As one model is a simple time integral of the other, all formulas derived for the instantaneous velocity can be modified very simply to deal with the other cases. For this reason only the model for the instantaneous velocity is treated in this section. The extension to obtain the model for the averaged velocity is done in section 3, which deals with the actual observation equations and the details of the adjustment.

The main objective in deriving the model has been to reflect the physical reality of the problem as accurately as possible, while respecting the constraint that the formulas should be susceptible of practical implementation. The result is of considerable complexity, as the reader will appreciate when looking at sections 2 and 3. This complexity is not incompatible with practical use, but makes the writing of computer programs more difficult than the simpler models proposed so far. This extra programming increases the overall effort and cost of a mission by a negligible amount, so it is a mere nuisance. The important thing is that the model must describe the signal accurately from its broadest aspects to its finest details. Simpler models have failed to do this in computer simulations, and may need upgrading before they can be of practical use. This could prove to be much more difficult than deriving a more realistic model from the start.

The model is based on Newtonian orbit mechanics. Because of the absence of non-gravitational forces, the use of analytical perturbation theory, which avoids expensive numerical integrations when setting up the normal matrix for the adjustment, is a natural choice. While following the example of previous authors in making this choice, I am not starting from the Lagrangian planetary equations, which describe the motion of a satellite in inertial space in terms of its Keplerian elements, linearized about an approximate orbit consisting of an ellipse precessing according to the secular perturbations caused by the Earth's oblatness. This is today the standard form of perturbation theory used in satellite geodesy, but it demands a good deal of familiarity and experience before one can start making the simplifications needed to obtain practical formulas, and initially it is difficult to grasp intuitively.
Lacking myself such familiarity, I prefer to build my theory on something I understand better. For this reason I start this section by discussing a general, kinematic model for the relative velocity, and then introduce the dynamical aspects, not in inertial space, but in

a geocentric coordinates' frame that rotates with the satellites. This leads from Newton's equations of motion to the variational equations known as Hill's equations, which describe the changes in a nearly circular orbit that take place when there are small changes in the initial state and the gravitational field departs slightly from that of a perfect sphere. These are linear differential equations with the perturbations for unknowns. Because of the simplifications introduced to obtain tractable analytical, or closed, expressions for their solution, the variationals are accurate to about the order of the flattening, or better than one percent, at most frequencies in the signal. The linearization of the equations of motion into variationals is done, essentially, along the trajectory that a spacecraft would follow if the gravitational field consisted only of the main zonals, including the second (oblatness) and third (pear-shape) and so on up to a chosen degree N.

As explained at the beginning of section 2, this orbit is closed, or periodical, from the point of view of an observer fixed to the instantaneous orbital plane. The reason for this is made clear in section 3, where the resulting periodicities in the structure of the observation equations and the normal matrix make the adjustment of an enormous number of unknowns feasible. The details of the analytical solution of the variational equations are given in section 2. The use of a reference orbit which includes both the secular and the short-term effects of the main zonals, the second and third in particular, eliminates those perturbations from the residual signal. Compared to the precessing ellipse of the standard theory, which does not contain the short-term effects, the more realistic reference orbit must result in smaller perturbations and, therefore, in a better linear approximation.

This first section ends with the detailed formulation of the linear model, which was sketched at the beginning, as obtained with the help of the physical and mathematical concepts introduced in between.

Note on mathematical symbols.

Listed below is the basic notation used in this work. When needed, further symbols shall be introduced in the text.

| | |
|---|---|
| $\underline{r}$ | is the geocentric, 3-dimensional position vector; |
| $\dot{a}$, $\underline{\ddot{a}}$ | indicate the first and second time-derivative of a scalar or vector; |
| $D_x\underline{a}$ | is the partial derivative of $\underline{a}$ with respect to x; |
| $\underline{h}^o$ | is a unit 3-D vector pointing in the positive "h" direction (when necessary, the sense is made plain by means of a drawing); |
| $\underline{a}.\underline{b}$ | is the internal, or scalar, product of two vectors; |
| $\underline{a}\times\underline{b}$ | is the external, or vector, product of two vectors; |
| $\lvert a\rvert$ | is the modulus of $\underline{a}$; |
| $s_{(obs.)}$ | is the observed, or measured, value of s; |
| $A\underline{b}$ | is the product of matrix A by vector $\underline{b}$. |

## 1.1 The relative line of sight velocity.

The measured quantity is the rate of change in the distance between two drag-free satellites, or line-of-sight velocity s. The distance $\rho$ is

$$\rho = |\underline{r}_1 - \underline{r}_2|$$

$$= ((\underline{r}_1 - \underline{r}_2).(\underline{r}_1 - \underline{r}_2))^{\frac{1}{2}} \qquad (1.1.1)$$

and its time derivative is

$$\dot{\rho} = s = (\underline{\dot{r}}_1 - \underline{\dot{r}}_2).(\underline{r}_1 - \underline{r}_2)\ ((\underline{r}_1 - \underline{r}_2).(\underline{r}_1 - \underline{r}_2))^{-\frac{1}{2}}$$

$$= (\underline{\dot{r}}_1 - \underline{\dot{r}}_2).(\underline{r}_1 - \underline{r}_2)\rho^{-1}$$

$$= (\underline{\dot{r}}_1 - \underline{\dot{r}}_2).\underline{e}^0 \qquad (1.1.2)$$

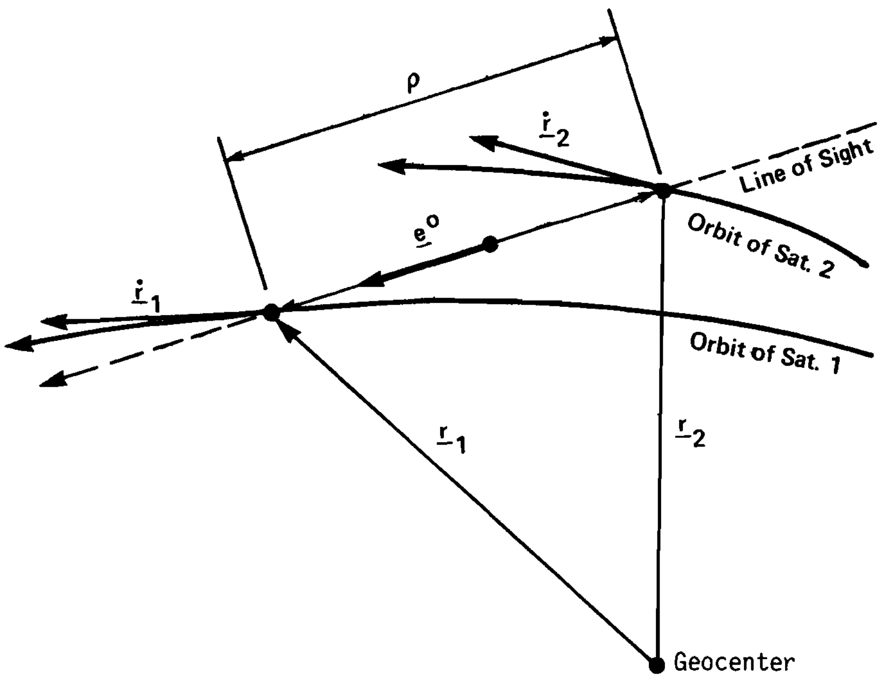


Fig. 1.1.1 Geometry of s.s.t.

where

$$\underline{e}^0 = (\underline{r}_1 - \underline{r}_2)\rho^{-1} \tag{1.1.3}$$

The pair of vectors $\underline{r}$, $\dot{\underline{r}}$ associated with each satellite constitute its *state*: the value of the six components (three for each vector) at a given time $t_0$ and the forces acting on the spacecraft from then on fully determine its future trajectory for all $t > t_0$.
In general, the position and velocity of a satellite are functions of time and of a *parameter vector* $\underline{p}$. This vector consists of the initial state, or initial conditions, and of a number of coefficients that determine the forcing function, such as the spherical harmonic potential coefficients of the gravity field. Therefore, the observation equation is of the form

$$s_{(\text{observed})} = s(\underline{r}_1(\underline{p}),\underline{r}_2(\underline{p}),\dot{\underline{r}}_1(\underline{p}),\dot{\underline{r}}_2(\underline{p}))+n \tag{1.1.4}$$

where $\underline{p}$ includes both known and unknown parameters, and n is the *measurement noise*, which here is supposed to be random, and of known variance. From (1.1.2) follows that (1.1.4) indicates a nonlinear relationship between s and $\underline{p}$. To solve for the unknown components of $\underline{p}$, such as the potential coefficients, (1.1.4) must be linearized about some approximation $\underline{p}_0$ of $\underline{p}$ in order to start an iterative parameter estimation consisting of successive linear least squares adjustments, such as the Gauss-Newton procedure. The linearized observation equation has the form

$$s_{(obs.)} - s(\underline{p}_0) = \sum_{i=1}^{2} D_{\underline{r}_i} s.(D_{\underline{p}}\underline{r}_i \Delta\underline{p}) + $$

$$+ D_{\dot{\underline{r}}_i} s.(D_{\underline{p}}\dot{\underline{r}}_i \Delta\underline{p}) + n \qquad (1.1.5)$$

where

$$\Delta\underline{p} = \underline{p} - \underline{p}_0 \qquad (1.1.6)$$

and (1.1.5) is a first order approximation to (1.1.4). The orbits of the two satellites computed with the parameter values $\underline{p}_0$ are known as the *reference orbits*. Derivatives such as $D_{\underline{r}_i} s$ and $D_{\dot{\underline{r}}_i} s$ can be obtained directly from (1.1.2) once the reference orbits are known; they are calculated from purely geometrical considerations. On the other hand $D_{\underline{p}}\underline{r}_i$ and $D_{\underline{p}}\dot{\underline{r}}_i$ can be obtained only after solving a system of differential equations intimately related to the equations of motion of free-falling bodies (such as the two satellites) known as the *variational equations*. To understand what these equations are and how they can be solved analytically, it is necessary to study first the equations of motion.

*Note*: In general, the number of parameters needed to describe the gravitational field of a planet is infinite. However, only a finite number of those parameters affect sensibly the motion of the spacecraft and the signal s, so their effects can be distinguished from the measurement noise. This is why s is assumed throughout to depend only on a *finite* number of parameters.

## 1.2 The equations of motion.

The relative velocity s is independent of the system of coordinates in which the individual positions and velocities of the two satellites are defined. Therefore $\underline{r}_i$ and $\dot{\underline{r}}_i$ in (1.1.2) can be given in any convenient system. In this paragraph the equations of motion will be derived in a rotating system, because this results in simple variational equations that can be integrated analytically if the reference orbits are almost circular. The choice of coordinates made here resembles somewhat that in G.W. Hill's theory of the lunar orbit (see, for example, Brower and Clemence (1961), Ch. 12).

Consider a system of coordinates revolving about the planetary center of mass (geocenter) with a time-varying angular velocity vector $\underline{N}$. Assume that the geocenter coincides with the origin of coordinates 0 at all times, and that it is free from acceleration in inertial space. The newtonian equations of motion of a particle in such a system, summed up in vector form, are

$$\boxed{\ddot{\underline{r}} = \underline{a} - \underline{N} \times (\underline{N} \times \underline{r}) - 2\underline{N} \times \dot{\underline{r}} - \dot{\underline{N}} \times \underline{r}} \tag{1.2.1}$$

(see, for example, Spiegel (1967), Ch. 6).

Here $\underline{a}$ is the acceleration in inertial space,

$- \underline{N} \times (\underline{N} \times \underline{r})$ is the centrifugal acceleration,

$- 2\underline{N} \times \dot{\underline{r}}$ is the Coriolis acceleration,

$- \dot{\underline{N}} \times \underline{r}$ is the so-called linear acceleration,

and $\underline{N}$ is independent of the unknown parameters $\underline{p}$ mentioned in the previous paragraph and of the motion of the particle, in contrast to $\underline{r}$.

If $\underline{a}$ is due only to the gravitational field, represented by its potential V, then

$$\underline{a} = \nabla V \tag{1.2.2}$$

where $\nabla \equiv D_{\underline{r}}$ is the gradient operator. In spherical coordinates $(r, \varphi, \lambda)$ where r is the geocentric distance, $\varphi$ the latitude and $\lambda$ the longitude, the potential can be expanded in external spherical harmonics (at least outside any sphere containing all planetary masses)

$$V = \frac{GM}{a} \sum_{\alpha=0}^{1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} \bar{C}_{nm}^{\alpha} \, \bar{Y}_{nm}^{\alpha}(\varphi,\lambda) \tag{1.2.3}$$

where

$$\bar{Y}^{\alpha}_{nm}(\varphi,\lambda) = \bar{P}_{nm}(\sin\varphi)\left\{{\cos \atop \sin}\right\}^{\alpha=0}_{\alpha=1} m\lambda \qquad (1.2.4)$$

and $\bar{P}_{nm}$ is the associated Legendre function of the first kind, degree n and order m. Here $\bar{P}_{nm}$, $\bar{Y}^{\alpha}_{nm}$ and $\bar{C}^{\alpha}_{nm}$ are *fully normalized* in the sense that

$$(\bar{C}^{\alpha}_{nm})^2 = \frac{1}{4\pi}\int_0^{2\pi} d\lambda \int_0^{\pi} (\bar{C}^{\alpha}_{nm}\ \bar{Y}^{\alpha}_{nm}(\varphi,\lambda))^2 \cos\varphi\ d\varphi \qquad (1.2.5).$$

Also G is the universal constant of gravitation, M is the mass of the planet, compared to which that of the satellite is negligible, and "a" is the mean equatorial radius. In what follows the spherical harmonic expansion will be truncated at a degree $n = N_{max}$ so high that the influence on the signal of all terms of greater degree can be neglected. This makes the number of parameters $\bar{C}_{nm}$ needed to describe the field a finite one, as explained in the Note in the preceeding paragraph. The normalized Legendre functions can be calculated with the help of the following recursive formulas:

(a) for m = 0

$$\bar{P}_{00}(\sin\varphi) = 1,\ \bar{P}_{10}(\sin\varphi) = \sqrt{3}\sin\varphi \text{ (starting values)}$$

and

$$\bar{P}_{n\,0}(\sin\varphi) = \frac{1}{n}\{[(2n+1)(2n-1)]^{\frac{1}{2}}\sin\varphi\ \bar{P}_{n-1\,0}(\sin\varphi) -$$

$$- (n-1)\left[\frac{2n+1}{2n-3}\right]^{\frac{1}{2}} \bar{P}_{n-2\,0}(\sin\varphi) \qquad (1.2.6)$$

(b) m = n

$$\bar{P}_{nn}(\sin\varphi) = \left[\frac{2n+1}{2n-1}\right]^{\frac{1}{2}} \cos\varphi\ \bar{P}_{n-1\,n-1}(\sin\varphi) \qquad (1.2.7)$$

(c) $m \neq n$

$$\bar{P}_{nm}(\sin\varphi) = \left[\frac{(2n+1)(2n-1)}{(n-m)(n+m)}\right]^{\frac{1}{2}} \sin\varphi\, \bar{P}_{n-1\ m}(\sin\varphi) -$$

$$- \left[\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}\right]^{\frac{1}{2}} \bar{P}_{n-2\ m}(\sin\varphi) \quad (1.2.8)$$

(notice that these are recursions in n, with m fixed).
Since the forces acting between the two satellites are negligible, knowing how a single spacecraft moves is enough to understand the behaviour of the pair. Because the two satellites are supposed to be made "drag free" by the use of small rocket engines, so their proof-masses are in constant free-fall, all non-gravitational forces such as aerodynamic friction, electromagnetic drag, and solar radiation pressure are excluded from this treatment. Gravitational forces that do not originate from the Earth or vary with time, such as the attractions of the Sun, Moon and major planets and of the solid, oceanic and atmospheric tides raised by those attractions, are also kept out. Their effects on the motion of the spacecraft can be calculated to a large extent by using existing models, so they are likely to be eliminated from the data when the computed values of the signal, $s(\underline{p}_0)$, are subtracted from the measurements to set up observation equations in accordance to the linearized expression (1.1.5). If this elimination is not complete, due to imperfections in the models, these may be corrected separately, from the residuals, in an iterative process.
Because of the attractions of other components of the solar system, the geocenter is accelerated in inertial space and, along with it, the geocentric coordinates used in this work. This accelerated system can be treated as an inertial system by making use of the concept of *tidal potential*. For each attracting body, this potential is the sum of the gravitational potential of this body and of that of a fictitious uniform field whose gravitational acceleration is everywhere the same in magnitude and direction, but opposite in sign, to the acceleration of the geocenter caused by that body. The total tidal potential acting on the Earth is simply the sum of those of the individual bodies that have appreciable influence on the geocenter's movement. Polar motion, precession, nutation and changes in length of day are ignored here because they result in variations in the

horizontal position of the satellite with respect to the Earth of a few meters, which is much less than the shortest wavelenght component of the signal that can be distinguished from the noise (about 100 km). Any cumulative effects can be eliminated in the same way as those of the attractions of the Sun, Moon etc., discussed above. The Newtonian character of the equations of motion (1.2.1) precludes the consideration of relativistic effects. These may be introduced as "relativistic corrections" to be added to the computed signal $s(\underline{p}_0)$.

## 1.3 The variational equations.

The solutions of the variationals are the components of the matrices $D_{\underline{p}}\underline{r}_i$ and $D_{\underline{p}}\dot{\underline{r}}_i$ that appear in the linearized signal equation (1.1.5). They relate individual changes $\Delta p_k$ in the elements $p_k$ of the vector of unknown parameters $\underline{p}$ to the perturbations in $\underline{r}$ and $\dot{\underline{r}}$ caused by those changes. They are the derivatives of the components of the state vector with respect to the $p_k$. The following reasoning shows how, through the introduction of some approximations, it is possible to arrive to a form of the variationals that can be solved analytically.

A change $\Delta p_k$ in $p_k$ results in a perturbation of the orbit. If u is a continuous function of $p_k$, either directly or through the state vector (which depends on $p_k$) and if $u_0(t)$ and $u_\Delta(t)$ are the values of u at time t corresponding to $p_k = p_{k0}$ and to $p_k = p_{k0} + \Delta p_k$, respectively, then

$$D_{p_k} u(t) = \lim_{\Delta p_k \to 0} \left\{ \frac{u_\Delta(t) - u_0(t)}{\Delta p_k} \right\}$$

As both the unperturbed variables ($\underline{r}_0$, $\dot{\underline{r}}_0$, $\ddot{\underline{r}}_0$, $\underline{a}_0$) and the perturbed variables ($\underline{r}_\Delta$, $\dot{\underline{r}}_\Delta$, $\ddot{\underline{r}}_\Delta$, $\underline{a}_\Delta$) satisfy the equations of motion, then, as $\underline{N}$ is independent of $p_k$,

$$\ddot{\underline{r}}_\Delta - \ddot{\underline{r}}_0 = \underline{a}_\Delta - \underline{a}_0 - \underline{N} \times (\underline{N} \times (\underline{r}_\Delta - \underline{r}_0)) - 2\underline{N} \times (\dot{\underline{r}}_\Delta - \dot{\underline{r}}_0) - \\ - \dot{\underline{N}} \times (\underline{r}_\Delta - \underline{r}_0) \qquad (1.3.1)$$

This is the vector form of the *perturbation equations*. These are *nonlinear*, because $\underline{a}_\Delta$ and $\underline{a}_0$ are, in general, nonlinear functions of $\underline{r}$.

Dividing both members by $\Delta p_k$ and taking limits:

$$\lim_{\Delta p_k \to 0} \left\{ \frac{\ddot{\underline{r}}_\Delta - \ddot{\underline{r}}_0}{\Delta p_k} \right\} = \lim_{\Delta p_k \to 0} \{ [\underline{a}_\Delta - \underline{a}_0 - \underline{N} \times (N \times (\underline{r}_\Delta - \underline{r}_0)) -$$

$$- 2\underline{N} \times (\dot{\underline{r}}_\Delta - \dot{\underline{r}}_0) - \dot{\underline{N}} \times (\underline{r}_\Delta - \underline{r}_0)] \frac{1}{\Delta p_k} \}$$

which, using the notation of the previous page, can be written as

$$D_{p_k} \ddot{\underline{r}} = D_{p_k} \underline{a} + D_{\underline{r}} \underline{a} D_{p_k} \underline{r} - \underline{N} \times (N \times D_{p_k} \underline{r}) - 2\underline{N} \times D_{p_k} \dot{\underline{r}} -$$

$$- \dot{\underline{N}} \times D_{p_k} \underline{r}$$

Finally, since $\Delta p_k$ is not a function of time,

$$\boxed{\begin{aligned} \frac{d^2}{dt^2} D_{p_k} \underline{r} = D_{p_k} \underline{a} + D_{\underline{r}} \underline{a} D_{p_k} \underline{r} - \underline{N} \times (\underline{N} \times D_{p_k} \underline{r}) - \\ - 2\underline{N} \times \frac{d}{dt} D_{p_k} \underline{r} - \dot{\underline{N}} \times D_{p_k} \underline{r} \end{aligned}} \qquad (1.3.2)$$

This is the vector form of the *variational equations* in the rotating frame introduced in the previous paragraph, and $D_{p_k} \underline{r}$ is the unknown. The symbol $D_{\underline{r}} \underline{a}$ represents a 3×3 matrix of second derivatives of the gravitational potential V (Marussi's tensor). Two terms involving derivatives of $\underline{a}$ appear because, in general, $\underline{a}$ is both a direct function of $p_k$ and also a function of $\underline{r}$, which depends on $p_k$. The variationals are *linear* equations.

If $\underline{N}$ is chosen perpendicular to the instantaneous plane of the unperturbed (reference) orbit and oriented so that the system turns in the same direction as the satellite, and if the magnitude of $\underline{N}$ is a constant $n_0$ (as yet unspecified), then $\dot{\underline{N}}$ is zero (or close to zero) for a polar (near-polar) reference orbit of the type described in paragraph (2.1), which is shaped by a reference gravity field made up of the zero harmonic of the potential and its main zonals.
In this case the *linear acceleration* term in the equations of motion (1.2.1) and the related last term in the variationals (1.3.2) can be

dropped because they are either zero or much smaller than the others. To continue this reasoning it is better to replace the vector form of the variationals with a set of three scalar equations in cartesian coordinates. Choosing an orthogonal triad of unit vectors $\underline{z}^0$, $\underline{r}^0$ and $\underline{u}^0$, with $\underline{z}^0$ having the same direction and sense as $\underline{N}$, and both $\underline{r}^0$ and $\underline{u}^0$ lying in the plane of the reference orbit and pointing so ($\underline{z}^0$, $\underline{r}^0$, $\underline{u}^0$) is left-handed, as shown in figure 1.2.1, the position vector can be written

$$\underline{r} = \underline{z}^0 z + \underline{r}^0 r + \underline{u}^0 u$$

where z, r and u are the scalar components of $\underline{r}$ in the system defined by the triad and whose origin is the center of mass 0. Vectors $\underline{\dot{r}}$, $\underline{\ddot{r}}$, and $\underline{a}$ can be written similarly in component form. Since $\underline{a} = \nabla V \equiv D_{\underline{r}}V$ according to (1.2.2),

$$\underline{a} = \underline{z}^0 a_z + \underline{r}^0 a_r + \underline{u}^0 a_u = \underline{z}^0 D_z V + \underline{r}^0 D_r V + \underline{u}^0 D_u V$$

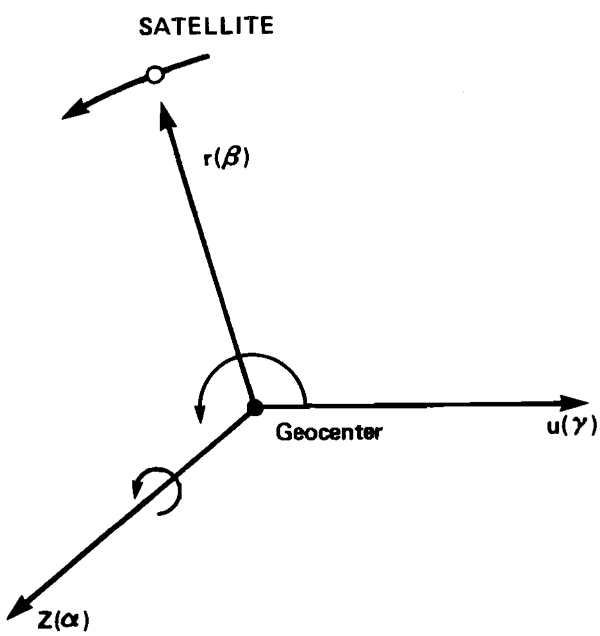


Fig. 1.2.1 The rotating system (z, r, u).

Calling

$$\alpha_k = D_{p_k} z$$

$$\beta_k = D_{p_k} r$$

$$\gamma_k = D_{p_k} u$$

expression (1.3.2) in component form is

$$\ddot{\alpha}_k = D_{p_k} a_z + D_z a_z \alpha_k + D_r a_z \beta_k + D_u a_z \gamma_k \qquad (1.3.3a)$$

$$\ddot{\beta}_k = D_{p_k} a_r + D_z a_r \alpha_k + D_r a_r \beta_k + D_u a_r \gamma_k + \quad n_0^2 \beta_k \quad - \quad 2n_0 \dot{\gamma}_k \qquad (1.3.3b)$$

$$\underbrace{\ddot{\gamma}_k}_{\frac{d^2}{dt^2} D_{p_k} \underline{r}} = \underbrace{D_{p_k} a_u}_{D_{p_k} \underline{a}} + \underbrace{D_z a_u \alpha_k + D_r a_u \beta_k + D_u a_u \gamma_k}_{D_{\underline{r}} \underline{a} D_{p_k} \underline{r}} + \quad \underbrace{n_0^2 \gamma_k}_{-\underline{N}\times(\underline{N}\times D_{p_k} \underline{r})} \quad + \quad \underbrace{2n_0 \dot{\beta}_k}_{-2\underline{N}\times\frac{d}{dt} D_{p_k} \underline{r}} \qquad (1.3.3c)$$

where $n_0 = |\underline{N}|$. For a planet like the Earth it is true that, to an accuracy of one part per thousand of $GM|\underline{r}|^{-3}$, and provided $\underline{r}$ and $\underline{r}^0$ are closely aligned,

$$D_z a_z \approx D_u a_u \approx - \frac{GM}{|\underline{r}|^3} \; ; \; D_r a_r \approx \frac{2GM}{|\underline{r}|^3} \; ;$$

$$D_z a_u \approx D_r a_z \approx D_r a_u \approx D_u a_z \approx D_u a_r \approx D_z a_r \approx 0 \; ,$$

so the variationals can be written, to this order of accuracy, as

$$\boxed{\begin{aligned} \ddot{\alpha}_k &= D_{p_k} a_z - \frac{GM}{|\underline{r}|^3} \alpha_k && (1.3.4) \\ \ddot{\beta}_k &= D_{p_k} a_r + \left(n_0^2 + \frac{2GM}{|\underline{r}|^3}\right) \beta_k - 2n_0 \dot{\gamma}_k && (1.3.5) \\ \ddot{\gamma}_k &= D_{p_k} a_u + \left(n_0^2 - \frac{GM}{|\underline{r}|^3}\right) \gamma_k + 2n_0 \dot{\beta}_k && (1.3.6) \end{aligned}}$$

In so far as these equations depend on $|\underline{r}|$, which is a function of time, their analytical solution can be very complicated. However, if the orbit is close to circular, $|\underline{r}|$ is almost constant. For the simplified expressions (1.3.4-6) to be valid, $\underline{r}$ in the reference orbit and $\underline{r}^0$ must stay closely aligned and, if $|\underline{r}|$ is almost constant, this can be ensured by choosing

$$n_0 = GM\,\bar{r}^{-3}$$

where $\bar{r}$ is the mean value of $\underline{r}$ along the reference orbit. This means taking $n_0$ equal to the mean angular velocity of the satellite. Neglecting the difference between the instantaneous and the average geocentric distance, which for a near-circular orbit around the Earth introduces an error of the order of the flattening (a few parts per thousand), results in the approximate, time-invariant variationals (sometimes called Hill's equations, as explained in the Introduction)

$$\boxed{\ddot{\alpha}_k = D_{p_k} a_z - n_0^2 \alpha_k} \tag{1.3.7}$$

$$\boxed{\ddot{\beta}_k = D_{p_k} a_r + 3n_0^2 \beta_k - 2n_0 \dot{\gamma}_k} \tag{1.3.8}$$

$$\boxed{\ddot{\gamma}_k = D_{p_k} a_u + 2n_0 \dot{\beta}_k} \tag{1.3.9}$$

The first equation is independent from the other two. This helps greatly, as it turns out that $\alpha_k$ and $\dot{\alpha}_k$ are not needed for the linearized model. The solution of the first equation is important, however, for understanding the precession of the orbital plane that occurs when the orbit is not polar. As far as I know, the first to suggest using Hill's equations for studying the satellite-to satellite tracking problem was Dr. Peter Bender, in a circular letter to some of his colleagues, in 1982. In the present derivation $\underline{N}$ has been chosen as independent of the satellite motion, whereas the usual derivation (see Kaplan, Ch. 3, 1976) assumes that $\underline{r}^0$ is aligned with $\underline{r}$. Both approaches lead to the same equations, after small terms are neglected. The idea of a constant rotation rate $n_0$ makes the application of these equations to the two-satellite problem treated in this work easier to understand. The choice of $n_0$ means that the system of coordinates turns with the angular velocity of a "satellite" in a circular orbit of radius $\bar{r}$ around a point of mass M at the origin. In this system, such a "satellite" appears stationary at a distance $\bar{r}$ from the origin, while the actual spacecraft hovers around the stationary one, responding to irregularities in the gravity field. Using a circular reference orbit would simplify greatly the mathematical treatment that follows; unfortunately the actual path of a low satellite of the Earth, for example, will depart by nearly ten kilometers from a circle, however carefully the satellite is put in orbit. This is mostly due to the larger terms in the expansion of the potential, which are the low degree zonals, notably the second and third, or the "oblatness" and "pear-shape" terms. Choosing a circle would mean large perturbations and an eventual breakdown in the validity of the linearized model (1.1.5). This is likely to happen in the part of the model corresponding to high degree potential coefficients, whose spherical harmonics change with a perturbation $\Delta r$ in the radial direction, for instance, by a factor of $\left(\frac{\bar{r} + \Delta r}{\bar{r}}\right)^{n+1}$, where n is the harmonic degree. Therefore the forcing terms $D_{p_k} a_z$, $D_{p_k} a_r$, $D_{p_k} a_u$ should be computed along an orbit that includes the effects of the first N zonals, so the perturbations are small when the problem is linearized along it. The determination of such an orbit is explained in paragraph (2.1); the low zonals that are needed are known very well already, after decades of patient work developing Earth models.

The approximate variationals (1.3.7-8) were obtained from the exact vector equations (1.3.2) by making two changes in the terms corresponding to $D_{\underline{r}}\underline{a}D_{p_k}\underline{r}$: (a) an "spherical" approximation in $D_{\underline{r}}\underline{a}$ and the assumption that $\underline{r}$ and $\underline{r}^0$ are nearly aligned at all times leading to (1.3.4-6); (b) a "circular" approximation, replacing $|\underline{r}|$ with $\bar{r}$. These two steps introduce an error of a few parts per thousand in $D_{\underline{r}}\underline{a}D_{p_k}\underline{r}$, in addition to which there will be a much smaller error (for near-polar reference orbits) due to the dropping of the "linear acceleration" term, which is actually zero for polar orbits; the other terms are all exact.

If the orbit is not polar, the orbital plane precesses about the Earth's axis. This happens because the forcing terms normal to the orbital plane ($D_{p_k} a_z$) corresponding to *even* zonals contain sinewaves of the same period as the orbit[(1)]. Expression (1.3.7) corresponds to a harmonic oscillator with a natural angular frequency $n_0$ that is the same as that

[(1)] See paragraph (AII.2) in Appendix II.

of the orbit. As a result of this the forced response contains a resonant term that grows steadily in amplitude while it oscillates in step with the turns of the satellite. So, every time that the spacecraft crosses the equator, the node will be displaced from the position it had in the initial orbital plane by a distance $\alpha(t)$ that is proportional to the number of turns elapsed. This means that the instantaneous orbital plane rotates about the z axis at a constant rate. As the vector $\underline{N}$ was chosen perpendicular to the instantaneous plane of the reference orbit when obtaining the scalar form of the variationals for the first time (expressions (1.3.3a-c)), $\underline{N}$ must precess at the same rate as that plane. This rate is, to a close approximation for near circular orbits, given by the formula

$$\dot{\Omega} \simeq \frac{3}{2} n_0 \bar{C}^0_{20} \left(\frac{a}{\bar{r}}\right) \cos i \sqrt{5} \tag{1.3.10}$$

where $i$ is the inclination of the orbital plane. Clearly, this rate is zero for a polar orbit, so whatever effects this precession brings about, they vanish for an inclination of $90^0$ and must be negligible for orbits close enough to this inclination. The reason for considering the non-polar case is the lack of accuracy in the "injection" of a satellite into orbit. Ideally a polar orbit will be selected, as it gives the most complete coverage of the gravity field. Errors in the initial steering of the spacecraft will cause the actual orbit to have a small obliquity. The error may be as much as a few degrees in inclination, and it could be too expensive in terms of fuel to carry out a trimming manoeuvre to correct the orbital plane. The main effect of a precessing plane is the introduction in the equations of motion of the *linear acceleration term*, which has the form (expression (1.2.1))

$$a_{linear} = \dot{\underline{N}} \times \underline{r} ,$$

where $|\dot{\underline{N}}| = \dot{\Omega} n_0 \cos i$ and $\dot{\underline{N}} \, . \, \underline{N} = 0$ always, as the length of $\underline{N}$ is constantly $n_0$. The inclusion of this extra term causes the variational equations to be fully coupled and also time-dependent, which does make the search for analytical solutions problematic. It turns out that this term is less than $10^{-3}$ times the size of the other ones in the dynamic part of the variationals when the error in the inclination is only two degrees or less, which is an accuracy quite within present capabilities. For this reason the dynamic part is kept as for a circular polar orbit, and thus unchanged from (1.3.7-9).

## 1.4 The linearized model.

As seen in the first paragraph, the relationship between the measured quantity, the relative line of sight velocity of the two satellites, and the unknown parameters, the corrections to the potential coefficients and to the initial conditions, is a nonlinear one. In order to estimate those unknowns in a relatively simple way it is necessary to obtain an approximate linear relationship, the linearized model. This has the differential form

$$ds = \sum_k D_{p_k} s \, dp_k \simeq \sum_k D_{p_k} s \, (p_k - p_{k0}) \tag{1.4.1}$$

where s is the signal in the measurement (noise will be ignored for the time being) and $p_{k0}$ is the approximate value of the parameter $p_k$, whose correction $\Delta p_k = p_k - p_{k0}$ is one of the unknowns in the problem. If, as it is the case with most potential coefficients, $p_{k0} = 0$, then the correction equals the parameter $p_k$ itself. The partial derivatives $D_{p_k} s$ are taken along a *reference orbit* determined by the $p_{k0}$'s, the known (but probably incorrect) values of the coefficients and of the initial conditions of the two spacecraft. The details regarding this orbit are discussed in section 2; for the present only a few general characteristics have to be considered. In the first place, the potential used to compute this reference orbit include only zonals up to degree N. Second, the initial conditions of both satellites are assumed to be on the same orbit plane (defined by the velocity and the geocentric position vectors of each spacecraft) and to be such that, from the point of view of an observer fixed to this common plane, the second satellite will move until, eventually, its position and velocity coincide with the initial state of the first satellite. Since the field determining the orbits is purely zonal, and so time-invariant in spite of Earth rotation (other time variations being negligible), both satellites must continue to share the same orbital plane indefinitely, and to the observer fixed to this plane they should appear to follow exactly the same orbit passing through the same point always $\Delta t$ seconds apart (if fluctuations in inclination are very small). In the special case where both orbits are polar, they will coincide also in inertial space. In what follows, both orbits will be regarded as one, "the" reference orbit. "Unperturbed" quantities associated with this orbit shall bear the subscript "o". In addition, the first and the second satellite will be distinguished by a subscript "i" taking values "1" and "2", respectively. In reality, the field will contain infinitely many terms, and the first N zonals will be somewhat different from the values adopted for the reference orbit, while the initial states will never fall exactly (or even very close) on this orbit. Such differences are the unknown corrections $\Delta p_k$. According to expression (1.1.2), when $p_k = p_0$ for all k,

$$s_0 = (\dot{\underline{r}}_{10} - \dot{\underline{r}}_{20}) \cdot (\underline{r}_{10} - \underline{r}_{20})\rho_0^{-1} \tag{1.4.2}$$

A small change $\Delta p_k$ in one of the parameters away from its reference value $p_0$ causes a corresponding change $\Delta s_k$ in s:

$$\begin{aligned} s_0 + \Delta s_k &= (\dot{\underline{r}}_{10}+\Delta\dot{\underline{r}}_{1k}-(\dot{\underline{r}}_{20}+\Delta\dot{\underline{r}}_{2k}))\cdot(\underline{r}_{10}+\Delta\underline{r}_{1k}-(\underline{r}_{20}+\Delta\underline{r}_{2k}))(\rho_0+\Delta\rho_k)^{-1} \\ &= [s_0\rho_0+(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20})\cdot(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k})+(\Delta\dot{\underline{r}}_{1k}-\Delta\dot{\underline{r}}_{2k})\cdot(\underline{r}_{10}-\underline{r}_{20}) \\ &\quad +(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k})(\Delta\dot{\underline{r}}_{1k}-\Delta\dot{\underline{r}}_{2k})\ldots]\times(\rho_0^{-1}-\rho_0^{-2}\Delta\rho+\rho_0^{-3}\Delta\rho^2-\ldots) \end{aligned} \tag{1.4.3}$$

Substracting the reference value of the relative line of sight velocity from both members, and neglecting all terms of order higher than one in the perturbations:

$$\begin{aligned} \Delta s_k \simeq \rho_0^{-1} [&(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20})\cdot(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k})+(\underline{r}_{10}-\underline{r}_{20})\cdot \\ &\cdot(\Delta\dot{\underline{r}}_{1k}-\Delta\dot{\underline{r}}_{2k})-(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20})\cdot(\underline{r}_{10}-\underline{r}_{20})\rho_0^{-1}\Delta\rho_k] \end{aligned} \tag{1.4.4}$$

Since $\rho = |\underline{r}_1-\underline{r}_2| = [(\underline{r}_1-\underline{r}_2).(\underline{r}_1-\underline{r}_2)]^{\frac{1}{2}}$
it follows that

$$\Delta\rho_k \simeq D_{p_k}\rho\Delta p_k = \rho_0^{-1}(\underline{r}_{10}-\underline{r}_{20}).D_{p_k}(\underline{r}_1-\underline{r}_2)\Delta p_k \simeq$$

$$\simeq \rho_0^{-1}(\underline{r}_{10}-\underline{r}_{20}).(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k})$$

and so

$$\Delta s_k \simeq \rho_0^{-1}[(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20}).(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k})+(\underline{r}_{10}-\underline{r}_{20}).(\Delta\dot{\underline{r}}_{1k}-\Delta\dot{\underline{r}}_{2k})-$$

$$-(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20}).(\underline{r}_{10}-\underline{r}_{20})\rho_0^{-1}(\underline{r}_{10}-\underline{r}_{20})\rho_0^{-1}.$$

$$.(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k})] \qquad (1.4.5)$$

According to (1.1.3), $\underline{e}^0 = (\underline{r}_1-\underline{r}_2)\rho^{-1}$ is the unit vector along the instantaneous line of sight. Let $\underline{e}_0^0$ be such a vector for the reference orbit, and $\hat{\underline{e}}_0^0$ a unit vector perpendicular to $\underline{e}_0^0$ at all times, and pointing away from the geocenter. Decomposing $(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20})$ and $(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k})$ into components paralel to $\underline{e}_0^0$ and $\hat{e}_0^0$:

$$(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20}).(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k}) = (\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20}).\underline{e}_0^0(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k}).\underline{e}_0^0 +$$

$$+ (\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20}).\hat{\underline{e}}_0^0(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k}).\hat{\underline{e}}_0^0 \qquad (1.4.6)$$

Replacing (1.4.6) in (1.4.5)

$$\Delta s_k \simeq (\Delta\dot{\underline{r}}_{1k}-\Delta\dot{\underline{r}}_{2k}).\underline{e}_0^0+\rho_0^{-1}(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20}).\hat{\underline{e}}_0^0(\Delta\underline{r}_{1k}-\Delta\underline{r}_{2k}).\hat{\underline{e}}_0^0 \qquad (1.4.7)$$

because $\underline{e}_0^0 = (\underline{r}_{10}-\underline{r}_{20})\rho_0^{-1}$.
Dividing both sides by $\Delta p_k$ and taking limits for $\Delta p_k \to 0$, as in the previous paragraph,

$$D_{p_k}s = (D_{p_k}\dot{\underline{r}}_1-D_{p_k}\dot{\underline{r}}_2).\underline{e}_0^0+\rho_0^{-1}(\dot{\underline{r}}_{10}-\dot{\underline{r}}_{20}).\hat{\underline{e}}_0^0(D_{p_k}\underline{r}_1-D_{p_k}\underline{r}_2).\hat{\underline{e}}_0^0 \quad (1.4.8)$$

This expression is valid in *any* system of coordinates, because the vector notation used so far is completely general. To use the perturbation theory developed in this work, it is necessary to state the problem in the rotating Cartesian coordinates in which the variationals have been derived. As it was shown in Fig. 1.3.1, the "r" and "u" axes of this system lie in the orbital plane, which now is common to both satellites, and the "z" axis is constantly perpendicular to this plane. Imagine that at $t = 0$ the "r" axis is normal to the initial direction of the line of sight. From then on, it moves with constant angular speed $n_0$ from the point of view of an observer fixed to the orbital plane. Let $\Delta t$ be the time interval between the passage of the first and the second satellite through the same point of the common reference orbit, as seen by that observer. Consider the unit vector $\underline{r}^0_{10}$, rotated forwards from the axis $\vec{r}$ by $\frac{1}{2}n_0\Delta t$ radians, and the unit vector $\underline{r}^0_{20}$, rotated backwards from $\vec{r}$ by the same angle. As the system turns, both vectors, which appear fixed in the moving coordinates, form a constant angle of $n_0\Delta t$ radians, and remain more or less aligned with the position vectors of the satellites, as indicated in Fig. 1.4.1. Let $\underline{u}^0_{10}$ and $\underline{u}^0_{20}$ be unit vectors normal to $\underline{r}^0_{10}$ and $\underline{r}^0_{20}$, respectively, both in the orbit plane and pointing against the motion of the respective spacecraft. Let also $\underline{r}_{i0}$ and $\dot{\underline{r}}_{i0}$, be the position and the velocity vectors of the unperturbed satellites *as seen in the rotating coordinates*. Decomposing the vector of partial derivatives $D_{p_k}\underline{r}_i$ along $\vec{r}$, $\vec{u}$ and $\vec{z}$:

$$D_{p_k}\underline{r}_i = \underline{z}^0_0 D_{p_k} z_i + \underline{r}^0_{i0} D_{p_k} r_i + \underline{u}^0_{i0} D_{p_k} u_i, \text{ where } \underline{z}^0_0 \text{ points along } \vec{z}$$

and $z_i$, $r_i$, $u_i$ are the components of $\underline{r}_i$ in the $(z,r,u)$ system (with $i = 1$ or $2$, depending on the satellite), then, in the notation of paragraph (1.3),

$$D_{p_k}\underline{r}_i = \underline{z}^0_0\,\alpha_{ik} + \underline{r}^0_{i0}\beta_{ik} + \underline{u}^0_{i0}\gamma_{ik} \tag{1.4.9a}$$

and

$$D_{p_k}\dot{\underline{r}}_i = \underline{z}^0_0\,\dot{\alpha}_{ik} + \underline{r}^0_{i0}\dot{\beta}_{ik} + \underline{u}^0_{i0}\dot{\gamma}_{ik} \tag{1.4.9b}$$

Since $\underline{z}^0_0$ is normal to the common orbital plane, $\underline{z}^0_0 \cdot \underline{e}^0_0 = \underline{z}^0_0 \cdot \hat{\underline{e}}^0_0 = 0$, so

$$D_{p_k}\underline{r}_i \cdot \underline{e}^0_0 = \beta_{ik}\underline{r}^0_{i0} \cdot \underline{e}^0_0 + \gamma_{ik}\underline{u}^0_{i0} \cdot \underline{e}^0_0 \tag{1.4.10a}$$

$$D_{p_k}\underline{r}_i \cdot \hat{\underline{e}}^0_0 = \beta_{ik}\underline{r}^0_{i0} \cdot \hat{\underline{e}}^0_0 + \gamma_{ik}\underline{u}^0_{i0} \cdot \hat{\underline{e}}^0_0 \tag{1.4.10b}$$

and similarly for $D_{p_k}\dot{\underline{r}}_i \cdot \underline{e}^0_0$ and $D_{p_k}\dot{\underline{r}}_i \cdot \hat{\underline{e}}^0_0$, so at this point, $\alpha_{ik}$ and $\dot{\alpha}_{ik}$

"exit" from the discussion.
Calling $\eta_{i0}$ to the angle formed by $\underline{u}_{i0}$ and $\underline{e}_0^0$, and $\delta_{i0}$ to the angle between $\dot{\underline{r}}_{i0}$ and $\hat{\underline{e}}_0^0$,

$$\underline{r}_{i0}^0 \cdot \underline{e}_0^0 = \sin \eta_{i0} \qquad (1.4.10c)$$

$$\underline{r}_{i0}^0 \cdot \hat{\underline{e}}_0^0 = \cos \eta_{i0} \qquad (1.4.10d)$$

and

$$\underline{u}_{i0} \cdot \underline{e}_0^0 = \cos \eta_{i0} \qquad (1.4.10e)$$

$$\underline{u}_{i0} \cdot \hat{\underline{e}}_0^0 = -\sin \eta_{i0} \qquad (1.4.10f)$$

while

$$\dot{\underline{r}}_{i0} \cdot \hat{\underline{e}}_0^0 = |\dot{\underline{r}}_{i0}| \cos \delta_{i0} \qquad (1.4.10g)$$

Replacing (1.4.10a-g) in (1.4.8)

$$D_{p_k} s = \dot{\gamma}_{1k} \cos \eta_{10} - \dot{\gamma}_{2k} \cos \eta_{20} + \dot{\beta}_{1k} \sin \eta_{10} - \dot{\beta}_{2k} \sin \eta_{20}$$
$$+ \rho_0^{-1} (|\dot{\underline{r}}_{10}| \cos \delta_{10} - |\dot{\underline{r}}_{20}| \cos \delta_{20})(\beta_{1k} \cos \eta_{10} - \beta_{2k} \cos \eta_{20} -$$
$$- \gamma_{1k} \sin \eta_{10} + \gamma_{2k} \sin \eta_{20})$$

where $\eta_{10}$, $\eta_{20}$, $\delta_{10}$, $\delta_{20}$, $|\dot{\underline{r}}_{10}|$, $|\dot{\underline{r}}_{20}|$, and $\rho_0$ are all functions of time. To obtain the complete differential of s (i.e., the linearized model) one now must add all the increments $\Delta s_k \simeq \sum_k D_{p_k} s \, \Delta p_k$. Then, for *infinitesimal* variations in the parameters,

$$\boxed{\begin{aligned} \Delta s = \sum_k [&\dot{\gamma}_{1k} \cos \eta_{10}(t) - \dot{\gamma}_{2k} \cos \eta_{20}(t) + \dot{\beta}_{1k} \sin \eta_{10}(t) - \\ &- \dot{\beta}_{2k} \sin \eta_{20}(t) + \rho_0(t)^{-1} (|\dot{\underline{r}}_{10}(t)| \cos \delta_{10}(t) - |\dot{\underline{r}}_{20}(t)| \cos \delta_{20}(t)) \\ &\times (\beta_{1k} \cos \eta_{10}(t) - \beta_{2k} \cos \eta_{20}(t) - \gamma_{1k} \sin \eta_{10}(t) + \\ &+ \gamma_{2k} \sin \eta_{20}(t))] \Delta p_k \qquad (1.4.11) \end{aligned}}$$

This is the full expression of the linearized model, with some of the time-dependencies made explicit. As shown by the computer studies of section 4, only the first four terms are needed for an adequate representation of the model; also it is possible to make the further approximation

$$\cos \eta_{10} \simeq \cos \eta_{20} \simeq \overline{\cos \eta_{i0}} \quad \text{(the overbar indicates the time-average over one orbit)}$$

without appreciable loss of accuracy, because $\eta_{i0}$ is small, so its cosine changes very slightly if the angle is perturbed, unless this perturbation is very large. Ignoring the last part of (1.4.11) eliminates the need to evaluate the complicated time-varying products in the neglected terms. Such terms do have an influence on the signal, but this has basically the form A t sin $n_0 t$ + B t cos $n_0 t$ which, as explained in section 3, can be taken care of in a very simple way without bringing in the whole formula for the differential of s. In consequence, sections 3 and 4 will consider only the simplified model:

$$\boxed{\Delta s = \sum_k [(\dot{\gamma}_{1k}-\dot{\gamma}_{2k})\overline{\cos \eta_{10}}+\dot{\beta}_{1k} \sin \eta_{10}(t)-\dot{\beta}_{2k} \sin \eta_{20}(t)]\Delta p_k} \qquad (1.4.12)$$

It is possible to neglect the last part of (1.4.11) and to have (1.4.12) instead, because the orbits are near-circular and $\rho_0$ is large when compared to the quantities it divides (order of $10^5$ meters). These quantities cancel out along a perfectly circular reference orbit, as then $|\dot{\underline{r}}_{10}| = |\dot{\underline{r}}_{20}|$ and $\eta_{10} = \eta_{20}$ for all t, so they are bound to be small when the orbit is almost circular.

The assumption that both satellites move in coplanar (or near-coplanar) orbits eliminates from the model the perturbations perpendicular to the common plane. When the planes are different, $\alpha_k$ and $\dot{\alpha}_k$ must be incorporated into the model. This generalization can be done without much difficulty, although the numerical computations then become more laborious. Because only the coplanar case has been the subject of serious discussion and research until now, and it is also simpler, the more general problem is not considered here. Some recent calculations by Breakwell (reported to me by P. Bender) indicate that having slightly different planes may improve the estimation of low and medium degree potential coefficients. This could be due to the reinforcement of the signal by the additional terms in $\alpha_k$ and $\dot{\alpha}_k$. This question shall be discussed further in paragraph (3.10).

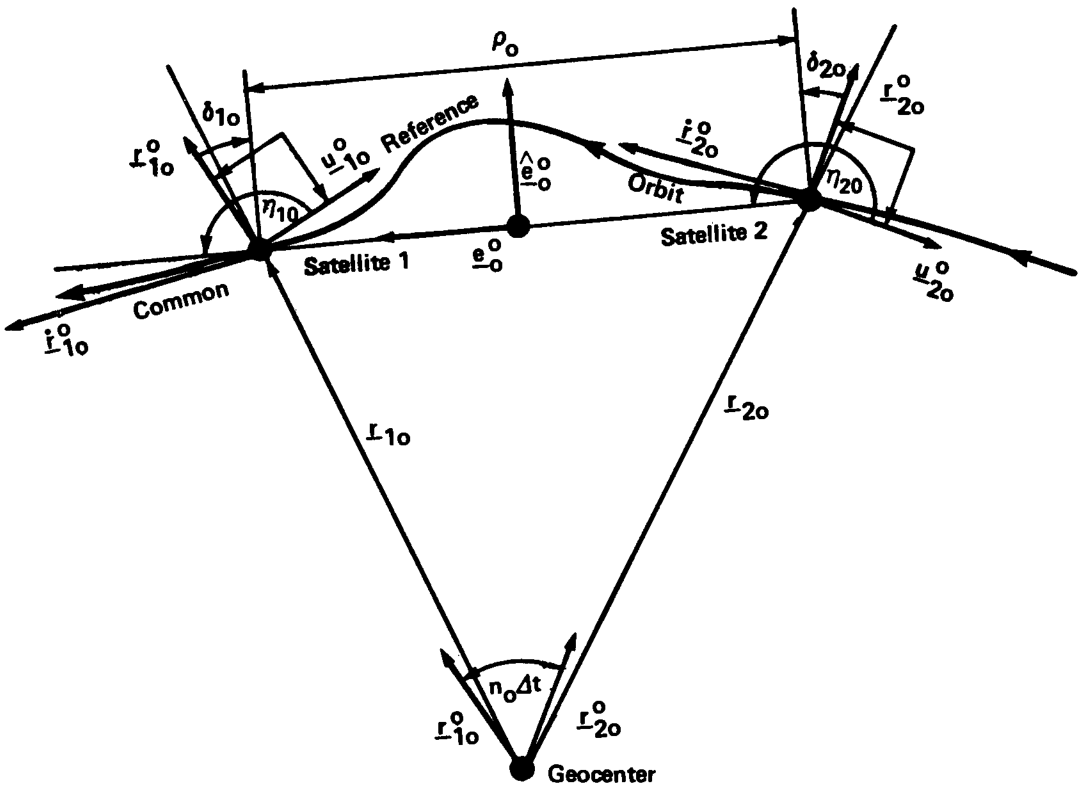


Fig. 1.4.1 The geometry of the common reference orbit.

Summary.

The relationship between the signal (relative line-of-sight velocity) and the parameters to be estimated (potential coefficients, components of the initial state of each spacecraft) is a nonlinear one. To evaluate the parameters by such methods as least squares adjustment, it is necessary to linearize this relationship. The resulting mathematical model includes the derivatives of the states of the satellites at the time of the measurement with respect to the parameters. These derivatives are found by solving the variationals, which are linear differential equations obtained by differentiating the nonlinear equations of motion. To exploit its geometrical symmetries, the problem is defined in a system of Cartesian coordinates, two of whose axes rotate in the plane of the orbit with a constant angular velocity equal to the mean angular velocity of the satellite, and the model is linearized along a special reference orbit described in further detail in the next section. The choice of coordinates results in variationals that, after some simplifications valid for near-circular, near-polar orbits, are relatively easy to solve analytically; they are known as Hill's equations. In the problem studied here, the two satellites follow each other along much the same trajectory, so they can have a common reference orbit and reference orbital plane.
In this special case, perturbations perpendicular to such plane do not appear in the linearized model. After some further approximations, validated by the numerical studies of section 4, only variations in the radial and along-track velocities of each spacecraft remain in the model.

# 2. THE ANALYTICAL SOLUTION OF THE VARIATIONAL EQUATIONS.

## Introductory remarks and overview.

As shown in this section, the variational equations have analytical solutions in the form of sums of sines and cosines of angles that grow linearly with time. These sums take strongly symmetric as well as simple forms when the reference orbit along which the problem is linearized appears, to an observed fixed with respect to the instantaneous orbital plane $\Gamma$ (defined by the instantaneous position and velocity vectors $\underline{r}$ and $\dot{\underline{r}}$), to be a closed path repeating itself periodically with the angular frequency $n_0$. In addition, the orbit should appear to the same observer as symmetrical with respect to the intersection of $\Gamma$ and the meridian plane perpendicular to $\Gamma$. By solving the variationals analytically, the coefficients of the unknown $p_k$ in the linearized signal equation(1.4.12) can be obtained without laborious numerical integrations which are subject to cumulative rounding and truncation errors. The Fourier series form of the analytical solutions (terminated at a sufficiently high frequency) permits the simultaneous adjustment of the hundreds of thousands of potential coefficients that describe the gravitational field at satellite altitude because, partly due to this form, the huge normal equations have a very strong structure. This reduces the work needed to set up and solve these equations to the point that the adjustment becomes feasible.

This section begins with theoretical considerations on the existence of closed, symmetrical orbits in purely zonal fields, followed by a description of a method for obtaining them by refining the initial conditions of the classical "frozen orbit", whose mean Keplerian elements are constant except for a secular precession of the line of nodes. Then comes the development of the forcing terms of the variationals along the reference orbit in the form of Fourier series with time for independent variable. Next, the analytical solutions of the variationals are discussed (the mathematical derivations are in Appendix I) including both forced and free responses. All the formulas needed for computing the Fourier coefficients of the forced response are given in this section.

Finally, the theory is applied to the case where the reference orbit is perfectly circular, because this clarifies some properties of the periodic reference orbit (which can be regarded as a perturbed circular one).

## 2.1 The periodic reference orbit.

To investigate the existence of closed, periodical, symmetrical orbits in a reference field of zonals from the point of view of an observer in the moving orbital plane, it is convenient to start by looking at the problem in "fixed" inertial space. Let (x,y,z) be an inertial system with origin at the geocenter. Assume that the gravitational field has rotational symmetry about the $\vec{z}$ azis, so its expansion contains only zonals. Two particles, or satellites, A and A', start from the same point P, moving perpendicularly to the meridian plane at P with the same velocity and in opposite directions. This situation is illustrated in figure 2.1.1, where the plane xz is also the meridian at P. Due to the rotational symmetry of the field, the choice of meridian plane is irrelevant to this argument. The free-fall trajectories of the particles must be mirror images of each other, because the accelerations in a zonal field are mirror-symmetrical respect to any meridian plane, xz included. Imagine now that A collides with a rigid plane normal to its motion at a point Q, bouncing back in a perfectly elastic fashion. An instant after the impact the position of A is virtually the same as before it, but its velocity vector has reversed its direction while keeping its magnitude. Since the satellite is drag-free, and thus subject only to the forces of a conservative field which depend on position alone, the particle will retrace the arc PQ and return to P with the same velocity as before, but moving in the opposite direction. Then it will have the same state as A' at the start of its trajectory, so now A will follow the arc PQ' described by that particle. In this way A will have moved, from the point of collision, along the symmetrical path QPQ'. This leads to the conclusion that, in a zonal field, any trajectory that is perpendicular at some point to a meridian plane must be, in inertial space, also mirror-symmetrical with respect to that plane.

Clearly, there is an infinite number of such trajectories. According to the argument so far, such conclusion is valid in *inertial* space. Consider now the case of an observer on the instantaneous orbital plane $\Gamma$, whose reference frame is the moving system of polar coordinates (r,F') illustrated in figure 2.1.2. In this system r is the geocentric distance and the origin 0 is the geocenter. This choice is possible because $\Gamma$ is the plane spanned by $\underline{r}$ and $\dot{\underline{r}}$, so it always contains $\underline{r}$ and 0. F' is the angle formed by $\underline{r}$ with the axis $\vec{d}$, which lies along the intersection of $\Gamma$ with the meridian plane perpendicular to $\Gamma$, and whose positive sense is towards the northern hemisphere. F' increases from ascending to descending node, following the movement of the satellite.

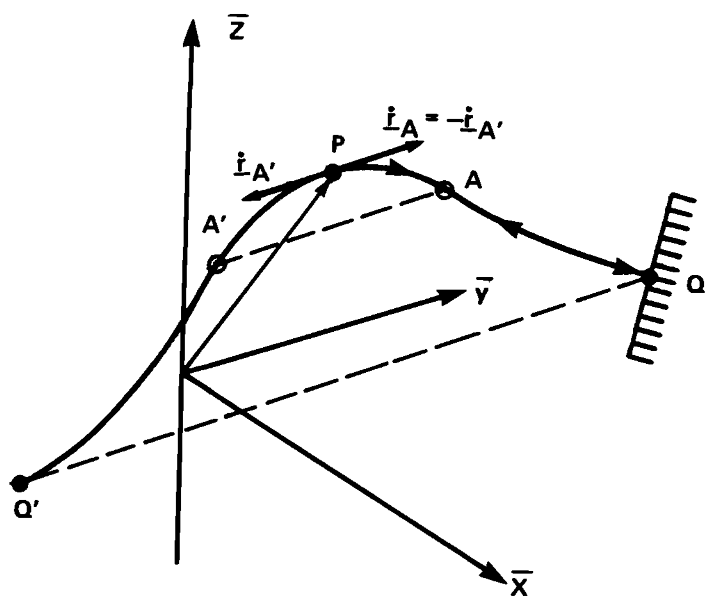


Fig. 2.1.1 Mirror-symmetrical orbit in a zonal field in inertial space (P is in the xz plane).

In the new reference system, two symmetrical points B and B' have coordinates $r_B = r_{B'}$ and $F'_B = -F'_{B'}$. The symmetrical orbit must fulfill the equations of motion in the moving system (r, F') so, the accelerations being always finite, both position and velocity must be continuous and differentiable functions of time. For continuity to be compatible with symmetry about the $\vec{d}$ axis, the velocity vector in the moving system must be normal to $\vec{d}$ at two points where this is crossed by the orbit. This means that the radial component of the velocity must be zero at those points.

This component is the relative velocity between the satellite and the geocenter O. Relative velocities are coordinate-independent, so the radial velocity $\dot{r}$ in the inertial system (x,y,z), whose origin is also the geocenter, must be the same as that in the moving system and, in particular, $\dot{r} = 0$ at the time of a crossing of the $\vec{d}$ axis (another way of explaining this is to notice that r is the same at all times in both systems, so its time derivative has the same value in both). This means that, in inertial space, the velocity vector $\underline{\dot{r}}$ must be perpendicular to the position vector $\underline{r}$ whenever this happens to their counterparts in the moving system. If the zero harmonic is much larger than the rest, so the orbit differs only slightly from a keplerian ellipse, there will be two intersection points P and P' on $\vec{d}$ at opposite sides of the origin, as in the case of an ellipse.

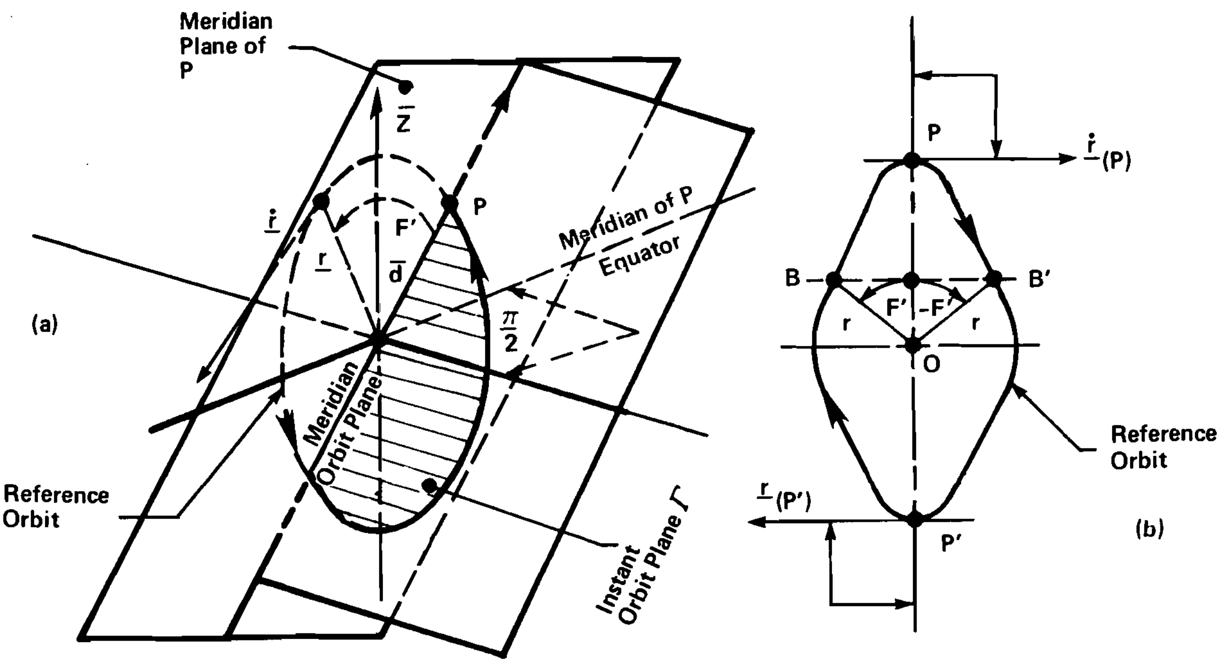


Fig. 2.1.2 The system (r,F') in the instantaneous orbital plane spanned by $\underline{r}$ and $\underline{\dot{r}}$. (a) In inertial space - (b) As seen by observer on Γ.

At either intersection the satellite is on the meridian plane normal to $\Gamma$, according to the definition of $\vec{d}$, and the velocity vector is normal to $\underline{r}$, which is then aligned with $\vec{d}$, and lies (as it does at all times) in $\Gamma$. This means that the velocity vector must be normal to the meridian plane of the satellite at each intersection. Consider the "northern" crossing, where $F' = 0$, and assume that the satellite's meridian at that point is identical to the xz plane, which is a choice we are free to make. This is, once more, the situation shown in Fig. 2.1.1., so the orbit must be symmetrical respect to xz in inertial space. At symmetrical points B and B', the orbital planes are also symmetrical, and so are the instantaneous directions of $\vec{d}$ as seen in inertial space.

This reasoning can be repeated for the "southern" crossing, where $F' = \pi$, with the same conclusion. So $\dot{r} = 0$ when $F' = 0$ or $F' = \pi$ is a *necessary* condition for the orbit to be symmetric, in order that both position and velocity be differentiable, as explained earlier, but it is also a *sufficient* condition for symmetry, as just shown. Furthermore, the differentiability of the orbit makes $\dot{r} = 0$ at both crossings a *necessary and sufficient* condition for any orbit that is symmetric in $\Gamma$ with respect to $\vec{d}$ to be also closed. This means that $\dot{r} = 0$ when $F' = 0$ and $F' = \pi$ is a necessary and sufficient condition for an orbit to be symmetrical and closed in the $(r,F')$ system on the instantaneous orbital plane.

Because $\dot{r}$ is the same in the inertial and in the moving system, the last statement implies that an orbit is symmetric and closed in the moving system if it crosses perpendicularly two meridian planes in inertial space.

The time interval between two such crossings is half the orbital period $T_0$, and the orbital angular frequency is $n_0 = 2\pi/T_0$.

It should be remembered that these conclusions are valid for near keplerian orbits, with only two crossings of $\vec{d}$ per revolution. This is a reasonable assumption for the Earth, other planets and major moons in the solar system, particularly when only the low degree zonals of their fields are involved.

The previous reasoning has shown the existence of symmetrical orbits, but not of orbits that are closed as well. Because of its great complexity, the question of their existence will not be considered further. Instead, making the assumption that such orbits exist, the ideas obtained so far will be used to develop a method for computing near-circular orbits that can be sufficiently symmetrical and periodical for practical purposes. This method involves making successive corrections to the initial state of a trajectory that is, from the start, symmetrical and almost closed. It is an iterative search whose beginning is chosen by taking advantage of a property of the *mean ellipse* of a symmetrical and closed orbit. This is the ellipse that fits best the orbit over a period of time sufficiently large for fluctuations of frequency $n_0$ and higher to average out. When the orbit is periodical, the mean ellipse will have the same shape when averaged over any whole number of revolutions and, in inertial space, the only change will be a steady precession of its plane about the $\vec{z}$ axis. This precession occurs because, in general, the inertial position and velocity vectors at the end of one revolution (when the satellite crosses a meridian plane perpendicularly and north of the equator for the second time) are the same as those at the beginning of that revolution rotated about $\vec{z}$ by a common angle. Because the field is zonal, the next turn of the spacecraft is identical to the previous one rotated by the same angle. As a result, the orbital plane rotates about z by a constant amount at every revolution. As each revolution takes the same time, the average motion is a steady precession. In the case where the orbit is polar, all forces lie in the orbital plane, which does not change, so there is no precession: the closed orbit is a flat curve. For other inclinations the initial and the final state differ, in general, by some rotation, so, as explained above, precession ensues. The rate of precession, for planets with a substantial equatorial bulge, like the Earth, is dictated mostly by the second zonal, as shown in formula (1.3.10). The explanation given here is only valid for periodical orbits; a more general one, based on resonance in the solution of the variational equation (1.3.7), was outlined in paragraph (1.3) and is discussed further in Appendix II.

From the previous discussion follows that the Keplerian elements of the mean ellipse, or *mean elements*, must be all constant for a periodical orbit, with the exception of the longitude of the ascending node, $\Omega$, which changes steadily due to the precession. For all near-circular orbits whose mean inclination is not too close to the "critical" value $i = \sin^{-1}(4/5)^{\frac{1}{2}} \simeq 63.43^{0}$, G.E. Cook (1966) has found relationships linking the mean eccentricity $e$, the mean argument of perigee $\omega$ (the angle between the trace of the orbital plane on the equator, or line of nodes, and the semimajor axis, positive from ascending node to perigee) and the potential coefficients of the dominant zonal terms, given the mean length of the semimajor axis $a$ and $i$. Generally $e$ and $\omega$ are slow-varying functions of time, except when they satisfy simultaneously the conditions

$$\omega = \frac{\pi}{2} \tag{2.1.1}$$

$$e = \text{a certain function of the } \bar{C}^{0}_{n0}.$$

Choosing in this way the initial "trial orbit" means taking

$a = \bar{r}$, the desired mean radius;
$i$ = given value, assumed to be not too close to the critical inclination, as it is the case with near polar orbits.

$$e = C/K \tag{2.1.2}$$

where

$$K = 3\sqrt{5}\left(\frac{GM}{a^3}\right)^{\frac{1}{2}} \bar{C}^{0}_{2\ 0}\left(\frac{a}{a}\right)^2\left(1 - \frac{5}{4}(\sin i)^2\right) \tag{2.1.3}$$

and

$$C = \frac{1}{2}\left(\frac{GM}{a^3}\right)^{\frac{1}{2}} \sum_{n=3}^{N} \bar{C}_{n\ 0}\left(\frac{a}{a}\right)^n (n-1)(2n+1)^{\frac{1}{2}}\bar{P}_{n\ 1}(0)\bar{P}_{n\ 1}(\cos i) \tag{2.1.4}$$

(Cook wrote in his paper equivalent expressions using *unnormalized* potential coefficients and Legendre functions). In (2.1.3) all even zonals above degree n = 2 have been neglected, and in (2.1.4) N is the highest degree included in the zonal expansion, while the summation is over odd values of n only. From these Keplerian elements the Cartesian components of the initial state can be calculated as follows:

$$\dot{x}_0 = \dot{z}_0 = 0 \qquad (2.1.5a)$$

$$\dot{y}_0 = -[\frac{GM}{a}\ (\frac{1+e}{1-e})]^{\frac{1}{2}} \qquad (2.1.5b)$$

$$y_0 = 0 \qquad (2.1.5c)$$

$$z_0 = \hat{r} \sin i \qquad (2.1.5d)$$

$$x_0 = (\hat{r}^2 - z_0^2)^{\frac{1}{2}} \qquad (2.1.5e)$$

where the "o" subscript indicates, simultaneously, that this is the initial state and also the state corresponding to F' = 0 in the moving system, and where $\hat{r} = (1-e)a$.
Cook's formulas are valid as long as certain terms proportional to the square of the eccentricity can be neglected, i.e., for near circular orbits, which are those of interest here, except when $i$ is close to the critical inclination. Away from this problem zone the eccentricities given by (2.1.2) are, for the Earth, of the order of $10^{-3}$, so the use of these expressions is justified. Choosing the initial state of an orbit so that the *instantaneous* Keplerian elements at the start satisfy (2.1.1-2) does not guarantee that the orbit will close, because Cook's theory applies to *mean* elements only. However, one can reasonably expect a near-periodical result, which can be improved by small corrections to the initial state. Since the argument of perigee is $90^0$, the orbit starts perpendicular to the initial meridian plane, so its symmetry is ensured.

In general, Cook's orbit is not perfectly closed, which means that $\dot{r} \neq 0$ when $F' = \pi$.
In such a case the radial velocity "misclosure" $\dot{r}_\pi$ (where the subscript "$\pi$" indicates that this is the value of $\dot{r}$ when $F' = \pi$) has to be eliminated by modifying the initial state. Assuming that $i$ is given and cannot be changed (modifying the inclination of a satellite orbit is expensive

because it requires a good deal of fuel), then, since the initial velocity must remain perpendicular to the initial meridian, and since changing this meridian has no effect on $\dot{r}_\pi$ (rotating the initial state about $\vec{z}$ in a zonal field rotates the orbit as a whole, without modifying its shape), the only elements of the initial conditions that can be modified to reduce $\dot{r}_\pi$ are $r_0$ and $\dot{y}_0$. Calling $\Delta r_0$ and $\Delta\dot{y}_0$ to the corresponding changes, the variation in $\dot{r}_\pi$ due to them is, to first order, $\Delta\dot{r}_\pi = D_{r_0}\dot{r}_\pi\, \Delta r_0 + D_{\dot{y}_0}\dot{r}_\pi\, \Delta\dot{y}_0$. To correct the "misclosure" $\dot{r}_\pi$, $\Delta r_0$ and $\Delta\dot{y}_0$ must bring about a variation $\Delta\dot{r}_\pi = -\dot{r}_\pi$. Therefore, the required changes should satisfy the equation

$$D_{r_0}\dot{r}_\pi\, \Delta r_0 + D_{\dot{y}_0}\dot{r}_\pi\, \Delta\dot{y}_0 = -\dot{r}_\pi \qquad (2.1.6)$$

Two cases are possible:

(a) $D_{r_0}\dot{r}_\pi = D_{\dot{y}_0}\dot{r}_\pi = 0$; here there is no solution to (2.1.6). This means that the misclosure cannot be eliminated by small changes $\Delta r_0$ and $\Delta\dot{y}_0$, so a new starting orbit has to be found, perhaps by varying $\bar{r}$ by several kilometers and using (2.1.1-5) once more to determine the initial conditions. This may not be always necessary. For example, in a central force field (zero zonal only) a change in $r_0$ or $\dot{y}_0$ has no effect on $\dot{r}_\pi$, but $\dot{r}_\pi$ is always zero because the orbits are always closed!

(b) At least one of the partial derivatives is not zero and there are, in general, infinitely many solutions.

In the second case, which is the one of interest here, the solution is made unique by the introduction of an additional constraint. The constraint chosen, for reasons that will become clearer later, is that the mean value of the orbit should remain constant. From the analytical solution of Hill's variationals (given in paragraph (2.4), at the end of this section), for changes in the initial conditions of the orbit $\Delta r_0^R$ and $\Delta\dot{y}_0^R$ (where the superscript R indicates that the coordinates correspond to the rotating frame introduced in section 1) the mean value of the same orbit varies by

$$\Delta\bar{r} = -2\Delta\dot{y}_0^R n_0^{-1} + 4\Delta r_0^R$$

according to expression (2.4.2)[1]. Using the superscript I to indicate inertial coordinates, the relation between both frames is

$$\Delta r_0^R = \Delta r_0^I = \Delta r_0 \qquad (2.1.7a)$$

$$\Delta \dot{y}_0^R = \Delta \dot{y}_0^I + n_0 \Delta r_0^I \qquad (2.1.7b)$$

and from this follows the $\Delta \bar{r} = 0$ constraint for $\Delta r_0^I$ and $\Delta \dot{y}_0^I$:

$$\Delta r_0^I - \Delta \dot{y}_0^I n_0^{-1} = 0 \qquad (2.1.8)$$

so

$$\Delta r_0^I = \Delta \dot{y}_0^I n_0^{-1} \qquad (2.1.9)$$

and expression (2.1.6) becomes (dropping the superscript I, inertial space is now used exclusively)

$$(D_{r_0} \dot{r}_\pi n_0^{-1} + D_{\dot{y}_0} \dot{r}_\pi)\Delta \dot{y}_0 \equiv \frac{d\dot{r}_\pi}{d\dot{y}_0} \Delta \dot{y}_0 = -\dot{r}_\pi$$

The solution is

$$\Delta r_0 = \Delta \dot{y}_0 n_0^{-1} \qquad (2.1.10a)$$

$$\Delta \dot{y}_0 = -\dot{r}_\pi \left(\frac{d\dot{r}_\pi}{d\dot{y}_0}\right)^{-1} \qquad (2.1.10b)$$

The derivative $\frac{d\dot{r}_\pi}{d\dot{y}_0} = (D_r \dot{r}_\pi n_0^{-1} + D_{\dot{y}_0} \dot{r}_\pi)$ cannot be obtained analytically, because Hill's equations, which are only approximately valid, give here singular first order solutions. In fact this singularity does not show up, at least in the cases that I have studied, if the problem is solved by numerical differentiation. First, one computes a trajectory from the initial conditions given by Cook's equations (2.1.3) to (2.1.5e). Then $\dot{y}_0$ is changed by a certain amount (1 m sec$^{-1}$ gives good results) and the trajectory is computed again. In each case the time when $F' = \pi$ and the corresponding value $\dot{r}_\pi$ are found by linear numerical interpolation between the last computed point where $F' < \pi$ and the next one (numerical integration only gives discrete points at regular intervals

[1] Here $\Delta \dot{y}_0^R$ is equivalent to $\Delta \dot{u}(t_0)$ in paragraph (2.4).

along the orbit). If $\Delta\dot{r}_\pi$ is the change in $\dot{r}_\pi$ corresponding to a change $\Delta\dot{y}_0$ in $\dot{y}_0$ when the constraint (2.1.8) is enforced, then

$$\frac{d\dot{r}_\pi}{d\dot{y}_0} \simeq \frac{\Delta\dot{r}_\pi}{\Delta\dot{y}_0} \tag{2.1.11}$$

After the correction $\Delta\dot{y}_0$ is made to the initial conditions, a new orbit is computed and the time and coordinates of the point where $F' = 2\pi$ are obtained, again by interpolation between the two closest computed points. The difference between these coordinates and the initial conditions constitute the misclosure of the orbit. If this is not satisfactory, the procedure is repeated as many times as necessary, until the misclosure becomes negligible. Two iterations have been sufficient, in my experience, to reduce the discrepancies between starting and end points to about 10 microns ($10^{-5}$ m) in position and below $10^{-1}$ microns per second in velocity. It is important to determine accurately both the $F' = \pi$ and the $F' = 2\pi$ crossings. For this purpose I found it very effective to stop the numerical integration at the point immediately after the crossing in question, and to re-start it from this point with the integration step reversed in sign and reduced to a tenth of its previous size, repeating this operation several times (running the orbit backwards and forwards through the crossing with ever decreasing point spacings) until the value of the linearly interpolated crossing time did not change its 10th significant figure from the previous result.

The resulting closed orbit is not very different in ellipticity from the "frozen" orbit that provided the start for the procedure just described. As $\dot{r}_\pi = 0$, expression (2.1.6) indicates that the orbit will stay closed if small perturbations to the initial conditions satisfying

$$D_{r_0}\dot{r}_\pi \, \Delta r_0 + D_{\dot{y}_0}\dot{r}_\pi \, \Delta\dot{y}_0 = 0$$

and, therefore,

$$\Delta r_0 = - \frac{D_{\dot{y}_0}\dot{r}_\pi}{D_{r_0}\dot{r}_\pi} \Delta\dot{y}_0 \tag{2.1.12}$$

are applied. This means that there may be infinitely many closed orbits, and that it is possible to change their shape continuously by modifying

$\Delta r_0$ and $\Delta\dot{y}_0$. There is the possibility that some of these closed orbits may be more circular than the one just obtained, and also that their mean radius may be nearer to the desired one (Cook's formulas used as explained here may give a mean radius that differs from the one chosen by several kilometers). This suggests an iterative search for a closed orbit that minimizes the rms difference between the instantaneous radius and $\bar{r}$ in which, at each iteration, an attempt is made to decrease the functional

$$\Phi = \sum_{j=0}^{J-1} (r(t_j) - \bar{r})^2 \qquad \text{(where } J\Delta t = \text{orbital period and } \Delta t = t_{j+1} - t_j)$$

by linearizing the equations of motion along the orbit to be corrected, or "circularized", and solving the linearized problem, which is a least squares fit of the orbit to a circle of radius $\bar{r}$, to obtain $\Delta r_0$ and $\Delta\dot{y}_0$. A simple way of doing this is to use the analytical solution to Hill's equation for the radial component (expression (2.4.1)) to obtain the partial derivatives for the linearized equation of the residual

$$\bar{r}-r(t_j) = \tfrac{1}{2}[D_{r_0} r(t_j)\Delta r_0 + D_{\dot{y}_0} r(t_j)\Delta\dot{y}_0] \tag{2.1.13}$$

under the constraint that the orbit remains closed (expression (2.1.12)). Using (2.1.7) to formulate the problem in an inertial frame, the solution to the least squares fit is

$$\Delta\dot{y}_0 = -\frac{1}{J}[(n_0^{-1}+\tfrac{3}{2}K)^2+(2n_0^{-1}+4K)^2]^{-1}[(-2n_0^{-1}-3K) \times$$

$$\times \sum_{j=0}^{J-1} \Delta r \cos n_0 j\Delta t + (2n_0^{-1}+4K)\sum_{j=0}^{J-1} \Delta r] \tag{2.1.14a}$$

$$\Delta r_0 = K\Delta\dot{y}_0 \tag{2.1.14b}$$

$$K = D_{\dot{y}_0}\dot{r}_\pi / D_{r_0}\dot{r}_\pi [1-n_0(D_{\dot{y}_0}\dot{r}_\pi / D_{r_0} r_\pi)]^{-1} \tag{2.1.14c}$$

In practice there is little further improvement after two iterations, the result being considerably more circular than the initial orbit. Because the constraint (2.1.12) is a first order approximation, the circularized orbit will not be as well closed as one might wish, so the "closure" procedure already described may be applied once more to ensure a satisfactory result. Because the circularized orbit has, in practice, a mean value very near to $\bar{r}$ (a few meters' difference), it is important that, in closing it once more, this mean value does not drift away. This is why the constancy of the mean radius was chosen as a constraint for this operation earlier on (expression (2.1.8)).

Summing up: the calculation of a reference orbit that in the instantaneous orbital plane appears symmetrical, closed, and near-circular, comprises four main stages:

(a) Given $i$, $\bar{r}$, and the potential coefficients for the main zonals up to degree N, use Cook's formulas and (2.1.5a-e) to find the initial conditions of the first approximate solution;

(b) With (2.1.10) find the corrections $\Delta r_0$ and $\Delta \dot{y}_0$ to the initial conditions that reduce or eliminate $\dot{r}_\pi$. Iterate until some "closedness" criterion is satisfied;

(c) Use the result of (b) in (2.1.14a-c) to find additional corrections to the initial state that make the orbit more circular, iterating if necessary;

(d) Check the misclosure in the result of (c) and, if too large, repeat (b).

The use of Cook's formulas for choosing the reference orbit (without further corrections) has been proposed by Wagner and Gould (1982) as part of their approach to processing satellite-to-satellite tracking data based on the classical first order analytical perturbation theory of the keplerian elements.

As it will be explained in paragraph (2.6), in a field consisting only of zonals the near-circular, near-polar reference orbit can be regarded as a circle of radius $\bar{r}$ plus a perturbation whose radial component is of the form

$$\Delta r = r - \bar{r} = \sum_j K_j \cos j\, n_0 t$$

$$= \sum_j K_j \cos jF' = \Delta r(F')$$

because $F' = n_0 t$ for the unperturbed circular orbit. Moreover, for even degree zonals j takes only even values, and for odd degree zonals only odd ones.

Thus, if all the zonals are *even*:

$$\Delta r(\tfrac{\pi}{2} + \alpha) = \Delta r(\tfrac{\pi}{2} - \alpha)$$

and

$$\Delta r(3\tfrac{\pi}{2} + \alpha) = \Delta r(3\tfrac{\pi}{2} - \alpha),$$

so the orbit is symmetrical with respect to the equator, where $F' = \frac{\pi}{2}$, and $F' = 3\frac{\pi}{2}$. In the case of the Earth, both even and odd zonals must be considered, so the reference orbit has no equatorial symmetry, but due to the near circularity of the orbit obtained by the procedure described here, such a symmetry is not far away. This situation could be exploited to reduce quite considerably the computing effort when estimating the potential coefficients from satellite-to-satellite tracking data, provided that these coefficients are not of very high degree, as explained in paragraph (3.9).

Once the closed reference orbit has been found, the orbital period $T_0$ becomes known and, along with it, the value of the fundamental angular frequency $n_0 = 2\pi/T_0$, which appears in many equations throughout this work.

## 2.2 The Fourier expansion of the forcing terms.

The forcing terms of the variationals (1.3.7-9) are the derivatives of the corresponding terms in the equations of motion (the gravitational accelerations or first gradients of the potential V) with respect to some component $p_k$ of the parameter vector $\underline{p}$. According to (1.2.3), V is only a function of those components of $\underline{p}$ that are potential coefficients $\bar{C}^{\alpha}_{nm}$, so the forcing terms $D_{p_k}a_z$, $D_{p_k}a_r$ and $D_{p_k}a_u$ are zero for all the other $p_k$ and only those corresponding to the $\bar{C}^{\alpha}_{nm}$ have to be considered here. Also, as explained in paragraph (1.4), only the last two variationals are of relevance to this work, so $D_{p_k}a_z$ shall not be studied here. The objective of the argument that follows is to explain how the forcing terms can be expressed as functions of time in the form of Fourier series which, because of the choice of reference orbit, are of a simple kind.

As a consequence, the complete solutions of the variationals must contain "steady state" components, or particular integrals, that are also Fourier series. This happens because the variationals under consideration are linear, time-invariant differential equations. These analytical Fourier-series solutions are essential to the simultaneous adjustment of hundreds of thousands of unknown potential coefficients, as explained in section 3.

Figure (2.2.1) shows the instantaneous plane of the reference orbit in inertial space. The plane forms an angle $i$, the *inclination angle*, with the equatorial plane. The point Q is the *ascending node*, whose longitude L increases gradually due to precession, as already explained, $\lambda$ is the longitude of the satellite, $\varphi$ is its latitude. The origin of L and $\lambda$ is not necessarily the meridian of Greenwich, and will be defined later.

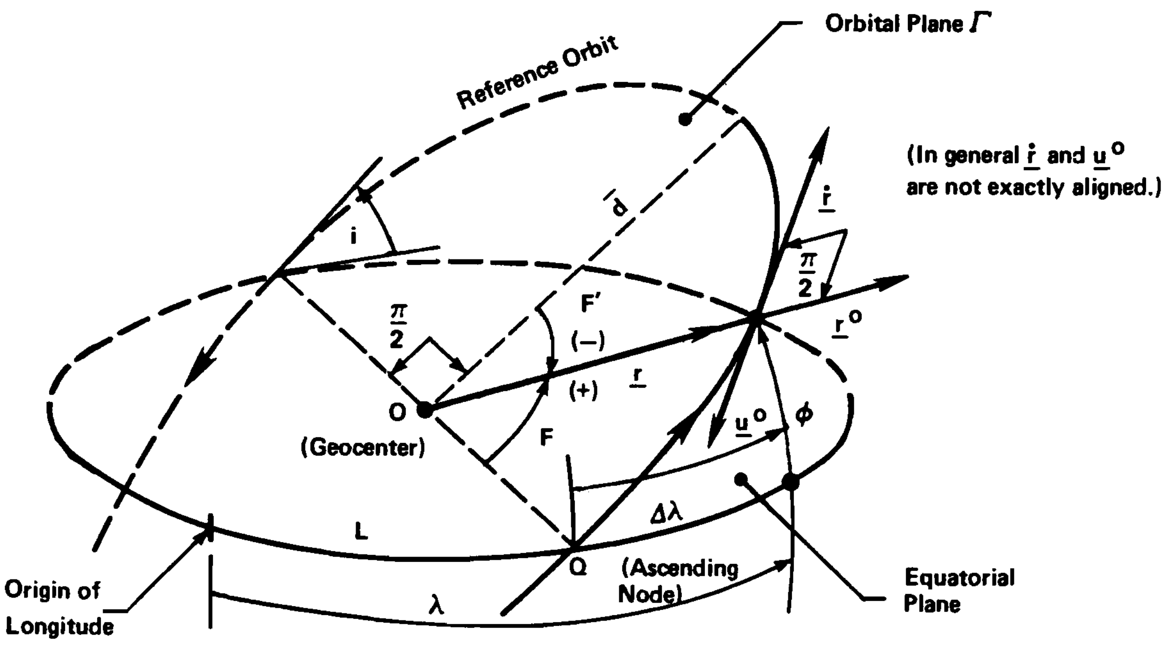


Fig. 2.2.1 Orbital geometry in inertial space.

F' is the same angle as in the previous paragraph (Fig. (2.2.2)), and F is the angle between the ascending node and the satellite. From the definition of F' follows that $F = \frac{\pi}{2} + F'$. The relationships between the geocentric angles F, L and $\varphi$, $\lambda$ are given by the formulas of spherical trigonometry

$$\varphi = \text{arc sin}(\sin i \sin F) \tag{2.2.1}$$

$$\lambda = \text{arc cos}(\cos i \sin F/\cos \varphi) + L \tag{2.2.2}$$

where $0 \le \varphi \le \frac{\pi}{2}$ if $0 \le F \le \pi$, and $-\frac{\pi}{2} \le \varphi \le 0$ if $\pi \le F \le 2\pi$, and where $\lambda - L$ is always in the same quadrant as F.

The inclination of the instantaneous plane fluctuates periodically, but these variations are so small that $i$ shall be considered to be a constant throughout this discussion. As all the orbital variables mentioned in this paragraph correspond to the reference orbit, the "o" subscript introduced in section 1 to single out such variables shall not be used. The geocentric distance $|\underline{r}|$ shall be designated here by "r".

Reasoning as in (Kaula, 1966, Ch. 3), but using F and L instead of the equivalent astronomical angles $\omega + f$ and $\Omega - \theta$, it can be shown that the spherical harmonic expansion of V becomes, after replacing $\varphi$ and $\lambda$ according to (2.2.1-2) into (1.2.3-4),

$$V(r,F,L,i) = \frac{GM}{a} \sum_{n=0}^{N_{max}} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} \sum_{p=0}^{n} \bar{F}_{nmp}(i) \begin{Bmatrix} \bar{C}^0_{nm} \\ \bar{C}^1_{nm} \end{Bmatrix} \cos((n-2p)F+mL) +$$

$$+ \begin{Bmatrix} \bar{C}^1_{nm} \\ \bar{C}^0_{nm} \end{Bmatrix} \sin((n-2p)F+mL) \tag{2.2.3}$$

where a is the mean planetary radius and the expansion has been truncated at $n = N_{max}$, as explained in the Note at the end of paragraph (1.1), while the top of the curly brackets { } corresponds to the case n-m *even* and the bottom to n-m *odd*. The $\bar{F}_{nmp}(i)$ are fully normalized *inclination functions*. Their efficient computation by means of the Fast Fourier Transform is explained in (Wagner, 1979). The last expression already resembles a trigonometric expansion in sines (or cosines) of the potential.

According to paragraph (1.3), the forcing terms of the variationals are the derivatives with respect to $p_k = \bar{C}^{\alpha}_{nm}$ of the components of the gravitational acceleration in the direction of the moving unit vectors $\underline{r}^0$ and $\underline{u}^0$ of an *uniformly* rotating system of coordinates, whose "r" axis is always closely aligned with the instantaneous position vector of the reference orbit, $\underline{r}_0$, while the "u" axis points against the direction of motion. Calling $\Delta F'$ the small misalignment angle between $\underline{r}^0$ and $\underline{r}_0$, the following relationship exists between the spatial derivatives of V in the directions of the $\vec{r}$ and $\vec{u}$ axes and those along the $\vec{r}'$, $\vec{u}'$ axes of the local geocentric frame of the satellite in the reference orbit:

$$a_r = D_r V = D_{r'} V \cos \Delta F' + D_{u'} V \sin \Delta F' \qquad (2.2.4a)$$

$$a_u = D_u V = -D_{r'} V \sin \Delta F' + D_{u'} V \cos \Delta F' \qquad (2.2.4b)$$

The forcing terms of the variationals are

$$D_{\bar{C}^{\alpha}_{nm}} a_r = D_{\bar{C}^{\alpha}_{nm}} D_r V = D_{\bar{C}^{\alpha}_{nm}} D_{r'} V \cos \Delta F' + D_{\bar{C}^{\alpha}_{nm}} D_{u'} V \sin \Delta F'$$

$$= D_{\bar{C}^{\alpha}_{nm}} a_{r'} \cos \Delta F' + D_{\bar{C}^{\alpha}_{nm}} a_{u'} \sin \Delta F' \qquad (2.2.5a)$$

and, similarly,

$$D_{\bar{C}^{\alpha}_{nm}} a_u = -D_{\bar{C}^{\alpha}_{nm}} a_{r'} \sin \Delta F' + D_{\bar{C}^{\alpha}_{nm}} a_{u'} \cos \Delta F' \qquad (2.2.5b)$$

Calling $G(F,L,i)_{nm\alpha}$ to the sum over p in (2.2.3),

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'} = D_{\bar{C}^{\alpha}_{nm}} D_{r'} V = \frac{(n+1)}{r} \left(\frac{a}{r}\right)^{n+1} G(F,L,i)_{nm\alpha} \qquad (2.2.6a)$$

$$D_{\bar{C}^{\alpha}_{nm}} a_{u'} = D_{\bar{C}^{\alpha}_{nm}} D_{u'} V = -\frac{1}{r} \left(\frac{a}{r}\right)^{n+1} D_F G(F,L,i)_{nm\alpha} \qquad (2.2.6b)$$

It is easier to develop first $D_{\bar{C}^{\alpha}_{nm}} a_{r'}$ and $D_{\bar{C}^{\alpha}_{nm}} a_{u'}$ into time Fourier series and then use (2.2.5) above to arrive at the desired expansion of the forcing terms $D_{\bar{C}^{\alpha}_{nm}} a_r$, $D_{\bar{C}^{\alpha}_{nm}} a_u$.

Replacing $G(F,L,i)_{nm\alpha}$ with its full expression according to (2.2.3):

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'} = r^{-(n+2)} \sum_{p=0}^{n} \tilde{\alpha}_{rnmp}[(1-\alpha)\{{\cos \atop \sin}\}((n-2p)F+mL) + \alpha\{{\sin \atop -\cos}\}((n-2p)F+mL)] \tag{2.2.7}$$

$$D_{\bar{C}^{\alpha}_{nm}} a_{u'} = r^{-(n+2)} \sum_{p=0}^{n} \tilde{\alpha}_{unmp}[(1-\alpha)\{{-\sin \atop \cos}\}((n-2p)F+mL) + \alpha\{{\cos \atop \sin}\}((n-2p)F+mL)] \tag{2.2.8}$$

where

$$\tilde{\alpha}_{rnmp} = -GM\ a^{n}(n+1)\bar{F}_{nmp}(i) \tag{2.2.9}$$

$$\tilde{\alpha}_{unmp} = -GM\ a^{n}(n-2p)\bar{F}_{nmp}(i) \tag{2.2.10}$$

so

$$\tilde{\alpha}_{unmp} = \tilde{\alpha}_{rnmp}(n-2p)(n+1)^{-1} \tag{2.2.11}$$

The forcing term $D_{\bar{C}^{\alpha}_{nm}} a_z$ is of no interest, as the solution to the first variational (1.3.7) does not appear in the linearized signal equation (1.4.11). The "$\alpha$" symbol in the expressions above and in all which follow is the superscript of the coefficient $\bar{C}^{\alpha}_{nm}$, unless it is expressly noted otherwise. Comparing (2.2.7) and (2.2.8) above it is easy to see that, but for the different coefficients $\tilde{\alpha}_{rnmp}$ and $\tilde{\alpha}_{unmp}$, the second expression can be obtained from the first by adding $90^{o}$ to the arguments of the trigonometric functions. This relationship holds throughout the following discussion, so it is enough to go through the derivation of the Fourier time series expression for $D_{\bar{C}^{\alpha}_{nm}} a_r$ to understand how that for $D_{\bar{C}^{\alpha}_{nm}} a_u$ can be obtained. Starting the reasoning with (2.2.7), quick examination of this formula shows that it can be written more compactly as

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'} = r^{-(n+2)} \sum_{p=0}^{n} \tilde{\alpha}_{rnmp}\{{\cos \atop \sin}\}((n-2p)F+mL+\phi_{\alpha}) \tag{2.2.12}$$

where

$$\phi_\alpha = \begin{cases} 0 & \text{if } \alpha = 0 \\ -\frac{\pi}{2} & \text{if } \alpha = 1 \end{cases} \qquad (2.2.13)$$

As the field for the reference orbit consists of zonals only, the shape of the orbit is not affected by the rotation of the corresponding "zonal Earth". In the instantaneous orbital plane the satellite is seen to follow the same closed path again and again. If the central angle F between the satellite and the instantaneous nodal point Q is given, then r is uniquely defined, regardless of the time t. So r is a function of F alone. Moreover, as the orbit (in the instantaneous plane $\Gamma$) is symmetrical with respect to the line from the geocenter along which $F = \frac{1}{2}\pi$, r is an *even function* of

$$F' = F - \tfrac{1}{2}\pi \qquad (2.2.14)$$

and it can be written as a sum of cosines

$$r(F') \simeq \bar{r} + \sum_{q=1}^{N} r_q \cos qF' \qquad (2.2.15)$$

because, as explained in paragraph (2.7), if the reference orbit is a slightly perturbed circular orbit, only the first N frequencies in the Fourier expansion of r correspond to terms that are large enough to be considered, so the series can be truncated at q = N, where N is the highest degree in the expansion of the zonal field that shapes this orbit. Since r is even, so is $r^{-(n+2)}$, which has also a cosine expansion. The question is where to truncate this expansion. If the perturbations are small compared to the mean radius of the orbit, so $r_q \ll \bar{r}$, then $r^{-(n+2)}$ in (2.2.12) can be expanded in a Taylor series about $\bar{r}$ as follows. Calling

$$r = \bar{r} + \Delta r$$

where

$$\Delta r = \sum_{q=1}^{N} r_q \cos qF'$$

then

$$r^{-(n+2)} = \bar{r}^{-(n+2)} - (n+2)\bar{r}^{-(n+3)}\Delta r + (n+2)(n+3)\bar{r}^{-(n+4)}\frac{\Delta r^2}{2} - \ldots$$

Therefore, to a first order approximation in $\Delta r$:

$$r^{-(n+2)} \simeq \sum_{q=0}^{N} h_{nq} \cos qF' \qquad (2.2.16a)$$

and the Fourier series can be truncated at $q = N$ like that of r. Also to first order in $r_q$:

$$h_{n0} = \bar{r}^{-(n+2)} \qquad (2.2.16b)$$

$$h_{nq} = -(n+2)\bar{r}^{-(n+3)} r_q \qquad (2.2.16c)$$

Expression (2.2.16) can be used as an alternative to a numerical Fourier analysis of $r^{-(n+2)}$ sampled at regular intervals along the orbit. Due to the nonlinearity of $r^{-(n+2)}$, a truncation at $q = N$ may not be high enough for large values of n, in which case terms with $q > N$ should be included. However, to simplify the notation, the upper limit N is assumed in what follows. Replacing (2.2.16a) in (2.2.12):

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'} = \sum_{p=0}^{n} \sum_{q=0}^{N} \tilde{\alpha}_{rnmp} h_{nq} \cos q(F - \tfrac{\pi}{2}) \left\{ {\cos \atop \sin} \right\} ((n-2p)F + mL + \phi_\alpha) \qquad (2.2.17)$$

From the trigonometric identities

$$2 \cos q(F - \tfrac{\pi}{2}) \cos((n-2p)F + mL + \phi_\alpha) = \cos((n-2p+q)F + mL - \tfrac{q\pi}{2} + \phi_\alpha) + \cos((n-2p-q)F + mL + \tfrac{q\pi}{2} + \phi_\alpha)$$

and

$$2 \cos q(F - \tfrac{\pi}{2}) \sin((n-2p)F + mL + \phi_\alpha) = \sin((n-2p+q)F + mL - \tfrac{q\pi}{2} + \phi_\alpha) + \sin((n-2p-q)F + mL + \tfrac{q\pi}{2} + \phi_\alpha)$$

follows that (2.2.17) can be written

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'} = \sum_{p=0}^{n} \sum_{q=0}^{N} a_{rnmpq}[\{{}^{\cos}_{\sin}\}((n-2p+q)F+mL - \tfrac{q\pi}{2} + \phi_\alpha)$$

$$+\{{}^{\cos}_{\sin}\}((n-2p-q)F+mL + \tfrac{q\pi}{2} + \phi_\alpha)] \qquad (2.2.18)$$

where

$$a_{rnmpq} = \tfrac{1}{2}\,\tilde{a}_{rnmp}h_{nq} \qquad (2.2.19)$$

Remembering that $F = F' + \frac{1}{2}\pi$:

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'} = \sum_{p=0}^{n} \sum_{q=0}^{N} a_{rnmpq}[\{{}^{\cos}_{\sin}\}((n-2p+q)F'+mL+(n-2p)\tfrac{\pi}{2} + \phi_\alpha)$$

$$+\{{}^{\cos}_{\sin}\}((n-2p-q)F'+mL+(n-2p)\tfrac{\pi}{2} + \phi_\alpha)] \qquad (2.2.20)$$

Moreover

$$\cos((n-2p\pm q)F'+mL+(n-2p)\tfrac{\pi}{2} + \phi_\alpha) = \cos(n-2p)\tfrac{\pi}{2}\cos((n-2p\pm q)F'+mL+\phi_\alpha)$$

$$-\sin(n-2p)\tfrac{\pi}{2}\sin((n-2p\pm q)F'+mL+\phi_\alpha)$$

and

$$\sin((n-2p\pm q)F'+mL+(n-2p)\tfrac{\pi}{2} + \phi_\alpha) = \sin(n-2p)\tfrac{\pi}{2}\cos((n-2p\pm q)F'+mL+\phi_\alpha)$$

$$+\cos(n-2p)\tfrac{\pi}{2}\sin((n-2p\pm q)F'+mL+\phi_\alpha)$$

For *n even* $(n-2p)\frac{\pi}{2}$ is an integer multiple of $\pi$, while for *n odd* the same angle is an integer multiple of $\frac{1}{2}\pi$, so

$$\{{}^{\cos}_{\sin}\}((n-2p\pm q)F'+mL+(n-2p)\tfrac{\pi}{2} + \phi_\alpha) =$$

$$= (-1)^{\left(\frac{n-2p}{2}\right)}\{{}^{\cos}_{\sin}\}((n-2p\pm q)F'+mL+\phi_\alpha) \qquad (2.2.21)$$

for n even, and

$$\left\{{\cos \atop \sin}\right\}((n-2p\pm q)F'+mL+(n-2p)\tfrac{\pi}{2} + \phi_\alpha) =$$

$$= (-1)^{\left(\frac{n-2p-1}{2}\right)} \left\{{-\sin \atop \cos}\right\}((n-2p\pm q)F'+mL+\phi_\alpha) \qquad (2.2.22)$$

for n odd. Further inspection of (2.2.20) in the light of (2.2.21-22) shows that the form of this expression depends only on the parity of the harmonic *order* m. Grouping all terms where $k = (n-2p\pm q)$ is the same leads to

$$D_{\bar{C}_{nm}}^{\alpha} a_{r'} = \sum_{k=-(n+N)}^{n+N} \bar{a}_{rnmk} \cos(kF'+mL+\phi_\alpha) \qquad (2.2.23)$$

*for m even*, and to

$$D_{\bar{C}_{nm}}^{\alpha} a_{r'} = \sum_{k=-(n+N)}^{n+N} \bar{a}_{rnmk} \sin(kF'+mL+\phi_\alpha) \qquad (2.2.24)$$

*for m odd*, with

$$\bar{a}_{rnmk} = \sum_{pq} (-1)^{q(n,m,p)} a_{rnmpq}(1+\delta_{q0}) \qquad (2.2.25)$$

Here the sum is over all terms in (2.2.20) where $(n-2p+q) = k$ or $(n-2p-q) = k$. The $(-1)^{q(n,m,p)}$ correspond to the factors multiplying the sines and cosines in the right hand sides of (2.2.21-22)[1]. The two expressions above can be written in a more compact form, as one differs from the other only by a $90^0$ phase shift. Introducing the notation

$$\phi_{m\alpha} = \phi_\alpha - \phi_m \qquad (2.2.26)$$

where

$$\phi_m = \begin{cases} 0 & \text{if m is even} \\ \frac{\pi}{2} & \text{if m is odd} \end{cases}$$

and $\phi_\alpha$ is defined by (2.2.13) leads to

$$D_{\bar{C}_{nm}}^{\alpha} a_{r'} = \sum_{k=-(n+N)}^{n+N} \bar{a}_{rnmk} \cos(kF'+mL+\phi_{m\alpha}) \qquad (2.2.27)$$

This expression is clearly a Fourier series of sines or cosines, but what is needed is an expression where the forcing term appears as a

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[1] $\delta_{q0}$ is the delta Kronecker.

function of *time* (the independent variable in the variationals) and not of the orbital angles F' and L. These angles are themselves functions of time, so a change in variables is possible. Assuming that L = 0 at t = 0 (more will be said about the significance of this choice of time-origin presently)

$$L = \theta' t \qquad (2.2.28)$$

where

$$\theta' = \dot{\Omega} - \dot{\theta} \qquad (2.2.29)$$

$\dot{\theta}$ being the spin rate of the Earth and $\dot{\Omega}$ the precession rate of the orbital plane (expression (1.3.10)), both taken here to be constant. The very small fluctuations in both rates that occur in reality change the actual point at which the forcing terms are calculated by a distance that is always very small compared to the wavelengths of the gravitational features that can be detected at satellite altitude, which are probably longer than 50 km. The total effect of such small changes is negligible, justifying the assumption that the rates are constant. The same reasoning can be used to disregard the effect of polar motion and nutation, while the effect of precession can be ignored during the relatively short duration of the satellite mission, as well as secular changes in the gravitational field. Therefore, this field is considered here as rotating at a constant rate around an axis whose direction is fixed in inertial space, and undergoing no other changes besides this rotation (all tidal perturbations are ignored, see paragraph (1.2)).

Obtaining a relationship between F' and time is more difficult than in the case of L, because the low degree zonals of the field cause the satellite to follow the orbit with a variable angular rate. Approximately, this rate is

$$\dot{F}'(t) \simeq n_0 + \bar{r}^{-1} \sum_{n=1}^{N} \dot{\gamma}_{n\,0} \bar{C}^0_{n\,0} \qquad (2.2.30a)$$

and, as explained in paragraph (2.7),

$$\sum_{n=1}^{N} \dot{\gamma}_{n\,0}\,\bar{C}_{n\,0}^{0} = \sum_{n=1}^{N} \dot{f}_i \cos in_0 t \qquad (2.2.30b)$$

if the reference orbit departs only slightly from a circular one. So

$$F'(t) = \int_0^t \dot{F}'(\tau)d\tau$$

$$= n_0 t + \sum_{i=1}^{N} f_i \sin in_0 t \qquad (2.2.31)$$

where

$$f_i = (\bar{r} i n_0)^{-1} \dot{f}_i$$

It is easy to see that (2.2.31) is in agreement with the fact that $F'(t)$ must be an *odd* function of t, because of the orbital symmetry. From this follows that cos kF' must be an *even* function of t, and sin kF' an *odd* one, so the first has a Fourier expansion of cosines and the second a Fourier expansion of sines. To decide where to truncate these expansions in order to carry out practical computations, one can apply a reasoning like that used for $r^{-(n+2)}$ earlier on. Calling

$$F' = \bar{F}' + \Delta F' \qquad (2.2.32a)$$

where

$$\Delta F' = \sum_{i=1}^{N} f_i \sin in_0 t \qquad (2.2.32b)$$

and

$$\bar{F}' = n_0 t, \qquad (2.2.32c)$$

and assuming that the perturbation $\Delta F'$ is small compared to $\bar{F}'$ (the mean anomaly of the circular orbit of the same period as the reference orbit), one can expand cos kF' and sin kF' in a Taylor series about $\bar{F}'$. To first order in $\Delta F'$:

$$\cos kF' = \cos k(\bar{F}' + \Delta F')$$

$$\simeq \cos k\bar{F}' - \sin k\bar{F}'\, k\Delta F'$$

and

$$\sin kF' \simeq \sin k\bar{F}' + \cos k\bar{F}'\, k\Delta F'$$

so

$$\cos kF'(t) \simeq \cos kn_0 t - \sum_{i=1}^{N} kf_i \sin kn_0 t \sin in_0 t$$
$$= \cos kn_0 t + \sum_{i=1}^{N} \tfrac{1}{2}kf_i[\cos(k+i)n_0 t - \cos(k-i)n_0 t]$$
$$= \sum_{j=k-N}^{k+N} c_{jk} \cos jn_0 t \qquad (2.2.33)$$

where $j = k \pm i$ and

$$c_{jk} = \begin{cases} 1 \text{ if } j = k \\ \tfrac{1}{2}kf_{|j-k|} \text{ sign}(j-k) & \text{otherwise, where } \text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \end{cases}$$

Similarly

$$\sin kF'(t) = \sum_{j=k-N}^{k+N} s_{jk} \sin jn_0 t \qquad (2.2.34)$$

where

$$s_{jk} = \begin{cases} 1 \text{ if } j = k \\ \tfrac{1}{2}kf_{|j-k|} & \text{otherwise} \end{cases}$$

Therefore, for small perturbations, the expansions can be truncated at $|j-k| = N$. This may not be true for the large values of k that go together with very high values of n. In this case, additional cosine or sine terms with $|j-k| > N$ may be needed. For convenience, a truncation at N in all cases is assumed in the reasoning that follows. According to (2.2.33-34),

$$\cos(kF'+mL+\phi_{m\alpha}) = \cos kF'\cos(mL+\phi_{m\alpha}) - \sin kF'\sin(mL+\phi_{m\alpha})$$
$$= \tfrac{1}{2} \sum_{j=k-N}^{k+N} (c_{jk}+s_{jk})\cos((jn_0+m\theta')t+\phi_{m\alpha}) +$$
$$+ (c_{jk}-s_{jk})\cos((jn_0-m\theta')t-\phi_{m\alpha}) \qquad (2.2.35)$$

Replacing (2.2.35) in (2.2.27) and adding together all terms with the same argument $(jn_0 + m\theta')t + \phi_{m\alpha}$ separate from those with the argument $(jn_0 - m\theta')t - \phi_{m\alpha}$:

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'}(t) = \sum_{j=-(n+2N)}^{n+2N} a^{+}_{rnmj} \cos((jn_0+m\theta')t+\phi_{m\alpha})$$

$$+ a^{-}_{rnmj} \cos((jn_0-m\theta')t-\phi_{m\alpha}) \qquad (2.2.36)$$

where

$$a^{+}_{rnmj} = \sum_{k=j-N}^{j+N} \bar{a}_{rnmk} \left(\frac{c_{jk}+s_{jk}}{2}\right) \qquad (2.2.37)$$

$$a^{-}_{rnmj} = \sum_{k=j-N}^{j+N} \bar{a}_{rnmk} \left(\frac{c_{jk}-s_{jk}}{2}\right) \qquad (2.2.38)$$

where $\bar{a}_{rnmk} = 0$ if $k < -(n+N)$ or $k > n+N$.
Going back to (2.2.8) and $D_{\bar{C}^{\alpha}_{nm}} a_{u'}$, which, except for different coefficients and a shift of $90^0$, has the same form as (2.2.7) for $D_{\bar{C}^{\alpha}_{nm}} a_{r'}$, and repeating the same reasoning that lead to (2.2.36) above, one gets

$$D_{\bar{C}^{\alpha}_{nm}} a_{u'}(t) = \sum_{j=-(n+2N)}^{n+2N} a^{+}_{unmj} \cos((jn_0+m\theta')t+\phi_{m\alpha} + \tfrac{\pi}{2}) +$$

$$+ a^{-}_{unmj} \cos((jn_0-m\theta')t-\phi_{m\alpha} - \tfrac{\pi}{2}) \qquad (2.2.39)$$

where $a^{+}_{unmj}$ and $a^{-}_{unmj}$ are Fourier coefficients corresponding to $a^{+}_{rnmj}$ and $a^{-}_{rnmj}$ in (2.2.36). Calling

$$\overset{*}{a}_{rnmj} = a^{+}_{rnmj} + a^{-}_{rnm\,-j}$$

and

$$-\overset{*}{a}_{unmj} = a^{+}_{unmj} + a^{-}_{unm\,-j}$$

expressions (2.2.36-39) can be written as

$$D_{\bar{C}^{\alpha}_{nm}} a_{r'}(t) = \sum_{j=-(n+2N)}^{n+2N} \overset{*}{a}_{rnmj} \cos((jn_0+m\theta')t+\phi_{m\alpha}) \qquad (2.2.40)$$

$$D_{\bar{C}^{\alpha}_{nm}} a_{u'}(t) = \sum_{j=-(n+2N)}^{n+2N} \overset{*}{a}_{unmj} \sin((jn_0+m\theta')t+\phi_{m\alpha}) \qquad (2.2.41)$$

These are the Fourier expansions of $D_{\bar{C}^{\alpha}_{nm}} a_{r'}$ and $D_{\bar{C}^{\alpha}_{nm}} a_{u'}$ as functions of time. From them one can arrive to the desired expansions of the forcing terms of the variationals, $D_{\bar{C}^{\alpha}_{nm}} a_r$ and $D_{\bar{C}^{\alpha}_{nm}} a_u$, by using expressions (2.2.5) to change from the local geocentric frame (z,r',u') at a point in the reference orbit, to the uniformly rotating frame (z,r,u) on which the theory is based. Since the misalignment $\Delta F'$ between both frames is quite small at all times for the near circular orbit (about 10 minutes of arc), the following approximations are acceptable:

$$\Delta F'(t) = \sum_{i=1}^{N} f_i \sin in_0 t \qquad \text{(see expression (2.2.32b))}$$

$$\cos \Delta F'(t) = 1$$

and

$$\sin \Delta F'(t) = \Delta F'(t)$$

Accordingly,

$$\begin{aligned} D_{\bar{C}^{\alpha}_{nm}} a_r(t) &= D_{\bar{C}^{\alpha}_{nm}} a_{r'}(t) \cos \Delta F'(t) + D_{\bar{C}^{\alpha}_{nm}} a_{u'}(t) \sin \Delta F'(t) \\ &= \sum_{j=-(n+2N)}^{n+2N} \overset{*}{a}_{rnmj} \cos((jn_0+m\theta')t+\phi_{m\alpha}) + \\ &\quad + \sum_{j=-(n+2N)}^{n+2N} \overset{*}{a}_{unmj} \sin((jn_0+m\theta')t+\phi_{m\alpha}) \sum_{i=1}^{N} f_i \sin in_0 t \\ &= \sum_{j=-(n+2N)}^{n+2N} \overset{*}{a}_{rnmj} \cos((jn_0+m\theta')t+\phi_{m\alpha}) - \\ &\quad - \sum_{j=-(n+2N)}^{n+2N} \tfrac{1}{2}\overset{*}{a}_{unmj} \sum_{i=1}^{N} f_i [\cos(((j+i)n_0+m\theta')t+\phi_{m\alpha}) - \\ &\quad - \cos(((j-i)n_0+m\theta')t+\phi_{m\alpha})] \end{aligned} \qquad (2.2.42)$$

and, similarly

$$D_{\bar{C}_{nm}}^{\alpha}\, a_u(t) = -D_{\bar{C}_{nm}}^{\alpha}\, a_{r'}(t)\sin\Delta F'(t) + D_{\bar{C}_{nm}}^{\alpha}\, a_{u'}(t)\cos\Delta F'(t)$$

$$= \sum_{j=-(n+2N)}^{n+2N} \overset{*}{a}_{unmj}\sin((jn_0+m\theta')t+\phi_{m\alpha}) -$$

$$- \sum_{j=-(n+2N)}^{n+2N} \tfrac{1}{2}\overset{*}{a}_{rnmj}\sum_{i=1}^{N} f_i[\sin(((j+i)n_0+m\theta')t+\phi_{m\alpha}) -$$

$$-\sin(((j-i)n_0+m\theta')t+\phi_{m\alpha})] \qquad (2.2.43)$$

Calling

$$a_{rnmj} = \overset{*}{a}_{rnmj} - \tfrac{1}{2}\sum_{i=1}^{N} f_i(\overset{*}{a}_{unm(j-i)} - \overset{*}{a}_{unm(j+i)}) \qquad (2.2.44a)$$

and

$$a_{unmj} = \overset{*}{a}_{unmj} - \tfrac{1}{2}\sum_{i=1}^{N} f_i(\overset{*}{a}_{rnm(j-i)} - \overset{*}{a}_{rnm(j+i)}) \qquad (2.2.44b)$$

(keeping in mind that $\overset{*}{a}_{rnmk} = \overset{*}{a}_{unmk} = 0$ if $|k| > n+2N$) expressions (2.2.42-43) can be written as

$$\boxed{\begin{aligned} D_{\bar{C}_{nm}}^{\alpha}\, a_r(t) &= \sum_{j=-(n+3N)}^{n+3N} a_{rnmj}\cos((jn_0+m\theta')t+\phi_{m\alpha}) \\ D_{\bar{C}_{nm}}^{\alpha}\, a_u(t) &= \sum_{j=-(n+3N)}^{n+3N} a_{unmj}\sin((jn_0+m\theta')t+\phi_{m\alpha}) \end{aligned}} \qquad \begin{aligned}(2.2.45)\\(2.2.46)\end{aligned}$$

where

$$\phi_{m\alpha} = -\alpha\,\frac{\pi}{2} - [1-(-1)^m]\frac{\pi}{4} \qquad (2.2.47)$$

and these are trigonometric expansions of the forcing terms as functions of time.

The $|j| > (n+3N)$ limits in the summations above originate in the truncation of the Fourier series of r(t) and F'(t) at the frequency of N cycles per revolution of the satellite in expressions (2.2.16a) and (2.2.33-34). This truncation is based on the argument given in paragraph (2.7) for the perturbations of a circular orbit, which is correct only to first order in $\Delta r$ and $\Delta F'$. For this reason, the truncation may not be valid when the degree n is very high or when the reference orbit departs too much from a circle, so second order and higher terms in $\Delta r$ and $\Delta F'$ become important. In such a situation more frequencies may have to be considered, expanding the range within which j varies. In the case considered in this work, the computer studies of section 4 suggest that the orbit is sufficiently circular and the highest degree considered low enough ($n \simeq 300$) for a *smaller* range of j to be quite adequate. For all practical purposes, restricting the expansions to terms where $|j| \leq (n+5)$ seems to cause no appreciable loss of accuracy.

In deriving (2.2.45-46) it has been assumed that at $t = 0$ the satellite is at its perigee, where $F = \frac{1}{2}\pi$, and that the longitude of the ascending node is zero. In general, neither assumption may be true, but the results obtained so far can be generalized without difficulty to include all cases. For an arbitrary time origin, let $\tau_0$ be the moment when the satellite reaches perigee, and let $L_0$ be the longitude of the node at this time. To take into account non-zero $\tau_0$ and $L_0$ one can replace $\theta' t = L$ with $\theta' t + L_0 = L$ when making the change in variables leading to (2.2.36), and replacing t with $t - \tau_0$ in (2.2.45-46). This leads to the general formulation of the forcing terms

$$\boxed{\begin{aligned} D_{\bar{C}^{\alpha}_{nm}}\, a_r(t) &= \sum_{j=-(n+3N)}^{n+3N} a_{rnmj} \cos((jn_0+m\theta')t+\hat{\phi}_{m\alpha}) \\ D_{\bar{C}^{\alpha}_{nm}}\, a_u(t) &= \sum_{j=-(n+3N)}^{n+3N} a_{unmj} \sin((jn_0+m\theta')t+\hat{\phi}_{m\alpha}) \end{aligned}} \qquad \begin{aligned} (2.2.48) \\ (2.2.49) \end{aligned}$$

where

$$\hat{\phi}_{m\alpha} = \phi_{m\alpha} - (jn_0+m\theta')\tau_0 + mL_0 \qquad (2.2.50)$$

so $\tau_0$ and $L_0$ affect only the phase angle.

## 2.3 Computing the Fourier coefficients of the forcing terms.

The two main reasons for developing the present analytical theory of the motion of a spacecraft have been, first, to understand this motion better and, second, to use this understanding for solving the problem of processing satellite-to satellite tracking data to map the gravity field. The latter involves very extensive numerical computations; a truly vast operation. So it is necessary to find ways of carrying out such computations as efficiently as possible. This can be done by exploiting any symmetries that may exist in the problem, and also by careful programming. One of the most laborious operations is that of finding the Fourier coefficients $D_{\bar{C}^{\alpha}_{nm}} a_r$ and $D_{\bar{C}^{\alpha}_{nm}} a_u$ in expressions (2.2.48-49). Once these are known, the analytical solutions of the variationals can be obtained with relatively few additional operations. In principle it is possible to obtain these coefficients by performing the calculations implied by the long chain of formulas developed in the previous paragraph, all the way from (2.2.7-8) to (2.2.48-50). But while these formulas are needed to understand the nature of the problem, they provide a very awkward way to numerical results. Fortunately, there is a very direct short-cut. Going over the derivation of (2.2.48-50) it becomes clear that the Fourier coefficients are independent of $\theta'$, $\tau_0$ and $L_0$, so they must be the same whatever the value of these parameters, including the case where all three are zero. If so, according to (2.2.45-47), the formulas become

$$D_{\bar{C}^{\alpha}_{nm}} a_r(t) = \sum_j (\alpha_{rnmj}+(-1)^{\alpha+m} \alpha_{rnm\,-j})\cos(jn_0t+\phi_{m\alpha})$$

$$= \sum_j A^{\alpha}_{rnmj} \cos(jn_0t+\phi_{m\alpha}) \qquad (2.3.1)$$

$$D_{\bar{C}^{\alpha}_{nm}} a_u(t) = \sum_j (\alpha_{unmj}-(-1)^{\alpha+m} \alpha_{unm\,-j})\sin(jn_0t+\phi_{m\alpha})$$

$$= \sum_j A^{\alpha}_{unmj} \sin(jn_0t+\phi_{m\alpha}) \qquad (2.3.2)$$

Assuming that $\theta' = 0$ is the same as saying that the spin rate of the Earth has slowed down to the point where it equals that of the nodal precession, which is virtually zero for near-polar orbits (see expression (1.3.10)). With the satellite initially at perigee ($\tau_0 = 0$), and taking an integration step $\Delta t$ that is an exact submultiple of the period $T_0$ of the reference orbit (so $T_0 = K\Delta t$ for some integer K) one can integrate numerically this orbit with initial conditions chosen as explained in paragraph (2.1), and a gravitational force due to the first N zonals and the zero harmonic alone. Then, at each of the computed points one can use the $(r_0,\varphi_0,\lambda_0)$ coordinates to calculate the values of $D_{\bar{C}^\alpha_{nm}}\, a_r$ and $D_{\bar{C}^\alpha_{nm}}\, a_u$, which depend on these coordinates (the details shall be given later). This results in two sequences of values, one for each forcing term, which can now be subject to numerical harmonic analysis to get the coefficients $A^\alpha_{rnmj}$ and $A^\alpha_{unmj}$. In general, it is necessary to do this twice, first for $\alpha = 0$ and then for $\alpha = 1$, to obtain the Fourier coefficients according to the formulas

$$a_{rnmj} = \tfrac{1}{2}(A^{\alpha=0}_{rnmj}+A^{\alpha=1}_{rnmj}) \tag{2.3.3a}$$

$$a_{rnm\ -j} = \frac{(-1)^{\alpha+m}}{2}\,(A^{\alpha=0}_{rnmn}-A^{\alpha=1}_{rnmj}) \tag{2.3.3b}$$

and similarly for $a_{unmj}$, $a_{unm\ -j}$, except that, for $-j$, $(-1)^{\alpha+m}$ is replaced by $-(-1)^{\alpha+m}$. The calculation of the $A^\alpha_{rnmj}$ and $A^\alpha_{unmj}$ is best done with a Fast Fourier Transform algorithm, as this is a very efficient type of procedure. There is no need to use any of the intermediate expressions derived in the last paragraph, and the programming of the operations that are actually performed is quite straightforward, once a Fast Fourier Transform subroutine is available.

To calculate $D_{\bar{C}^\alpha_{nm}}\, a_r$ and $D_{\bar{C}^\alpha_{nm}}\, a_u$ at points along the reference orbit, the formulas listen below are needed. According to (2.2.4a-b), the forcing terms can be found by computing first the components $D_{\bar{C}^\alpha_{nm}}\, a_{r'}$ and $D_{\bar{C}^\alpha_{nm}}\, a_{u'}$ of the vector $D_{\bar{C}^\alpha_{nm}}\, \underline{a}$ in the local geocentric frame (r',u') of the unperturbed satellite, and then converting these results to the uniformly rotating frame (r,u), where $\vec{r}$ goes through the starting point (perigee) at t = 0. This requires finding the angle $\Delta F'$ between $\vec{r}$ and $\vec{r}\,'$, which is a simple calculation once the orbit has been integrated numerically. As for $D_{\bar{C}^\alpha_{nm}}\, a_{r'}$ and $D_{\bar{C}^\alpha_{nm}}\, a_{u'}$, they can be expressed as functions of the geocentric spherical coordinates $(r_0,\varphi_0,\lambda_0)$:

$$\begin{aligned} D_{\bar{C}^\alpha_{nm}}\, a_{r'} &= (n+1)\frac{GM}{a^2}\left(\frac{a}{r_0}\right)^{n+2}\bar{P}_{nm}(\sin\varphi_0)\cos(m\lambda_0-\alpha\,\tfrac{\pi}{2}) \\ &= D_{\bar{C}^\alpha_{nm}}\, D_{r'}V \end{aligned} \tag{2.3.4}$$

because V is a function of $(r,\varphi,\lambda)$ according to (1.2.2-3), and, similarly,

$$D_{\bar{C}^{\alpha}_{nm}}\ a_{u'} = \frac{1}{r_0}[D_{\bar{C}^{\alpha}_{nm}}\ D_{\varphi}V \cos\mu + D_{\bar{C}^{\alpha}_{nm}}\ D_{\lambda}V \cos\varphi_0^{-1}\sin\mu]$$

$$= \frac{GM}{a^2}\left(\frac{a}{r_0}\right)^{n+2}\left[\frac{\partial\bar{P}_{nm}}{\partial\varphi}(\sin\varphi_0)\cos(m\lambda_0-\alpha\ \tfrac{\pi}{2})\cos\mu\right.$$

$$\left. - \frac{m}{\cos\varphi_0}\ \bar{P}_{nm}(\sin\varphi_0)\sin(m\lambda_0-\alpha\ \tfrac{\pi}{2})\sin\mu\right] \quad (2.3.5)$$

and

$$\mu = \cos^{-1}(-\cos\Delta\lambda\ \sin i) \qquad (2.3.6)$$

where $\mu$ is the angle formed in the plane of $\underline{s}_0^0$ and $\underline{t}_0^0$ (pointing to North and East, respectively, and both normal to $\underline{r}_0^{0'}$) by the vectors $\underline{s}_0^0$ and $\underline{u}_0^{0'}$ (remember that $\underline{u}_0^{0'}$ lies in the orbit plane and points against the direction of motion). The various geometrical elements are shown in Fig. 2.3.1.

While the method proposed here is already much simpler and faster than the direct implementation of the formulas leading to (2.2.48-50), there are symmetries in the problem that permit further substantial reductions in computing.

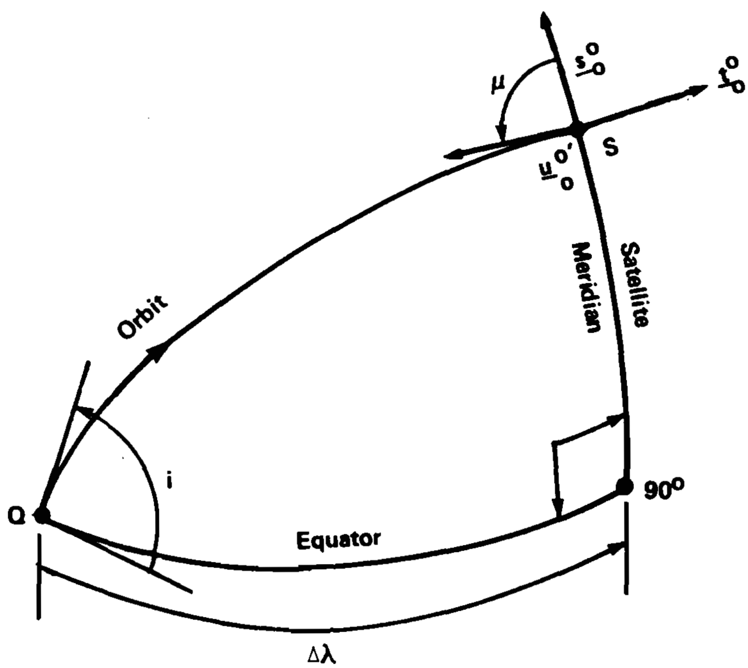


Fig. 2.3.1 Geometry associated with expressions (2.3.4-6).

Even-odd symmetry:

Examination of expressions (2.3.1-2) shows that $D_{\bar{C}_{nm}^{\alpha}} a_r$ and $D_{\bar{C}_{nm}^{\alpha}} a_u$ are even or odd functions of time, and that if one is even the other must be odd, and viceversa, depending on $\alpha$ and m. This helps in two ways: first, it is enough to compute $D_{\bar{C}_{nm}^{\alpha}} a_r$ and $D_{\bar{C}_{nm}} a_u$ over half the orbit, as the values on the other half are the same, except for a possible change in sign; second, analyzing the sum of the functions, instead of each one separately, reduces the number of times the Fast Fourier Transform has to be applied by half while providing the same results, because if $D_{\bar{C}_{nm}^{\alpha}} a_r$ is even and $D_{\bar{C}_{nm}^{\alpha}} a_u$ is odd, then the Fourier coefficients of the cosine terms in the expansion of their sum are also the coefficients of $D_{\bar{C}_{nm}^{\alpha}} a_r$, while those of the sine terms correspond to $D_{\bar{C}_{nm}^{\alpha}} a_u$. The converse is true when $D_{\bar{C}_{nm}^{\alpha}} a_r$ is odd and $D_{\bar{C}_{nm}^{\alpha}} a_u$ is even.

The even-odd symmetry just discussed is quite general, but there are additional symmetries in some special cases, such as m = 0 (zonals), and when the reference orbit is truly polar.

Zonals:

Here there are no harmonic terms in V, $a_r$, and $a_u$, if $\alpha = 1$, so $D_{\bar{C}_{nm}^{1}} a_r$ and $D_{\bar{C}_{nm}^{1}} a_u$ vanish and, with them, their Fourier coefficients in (2.3.1-2), implying that

$$a_{rnmj} = a_{rnm\,-j} = \tfrac{1}{2}\, A_{rnmj}^{\alpha=0} \tag{2.3.7a}$$

$$a_{unmj} = -a_{unm\,-j} = \tfrac{1}{2}\, A_{unmj}^{\alpha=0} \tag{2.3.7b}$$

This means that calculations with $\alpha = 1$ are not needed, and the computing is cut further by half.

Polar reference orbit:

If the true orbit is close enough to polar, the reference orbit can be chosen as truly polar, because the departures between both will be sufficiently small to use the linear theory developed this far. While this may not always be the case in reality, it is important in a pre-mission study like an error analysis, where the orbit can be assumed to be polar because this has little effect on the validity of the results, as long as the planned mission involves a near-polar orbit. As such a reference orbit lies entirely in the meridian plane containing the origin of longitudes, the value of $\lambda$ in (2.3.4-5) is either 0 or $\pi$. In both cases $D_{\bar{C}^{\alpha}_{nm}} a_r$, and $D_{\bar{C}^{\alpha}_{nm}} a_u$, are zero when $\alpha = 1$, and regardless of m ($D_{\bar{C}^{\alpha}_{nm}} D_\lambda V$ disappears from (2.3.5) because $\mu = 0$). So, once more (2.3.7a-b) are true, indicating that the spectra of the forcing terms are symmetric in j and only computations with $\alpha = 0$ are required, thus halving the number of arithmetic operations.

In addition to exploiting symmetry, it is essential to avoid the unnecessary repetition of operations. When computing $D_{\bar{C}^{\alpha}_{nm}} a_u$, to obtain the forcing terms according to (2.3.5), one needs a derivative of the Legendre function $\bar{P}_{nm}(\sin\varphi)$ with respect to latitude. This could be done using special recursive formulas, but the extra operations these formulas require can be avoided by using instead

$$\frac{\partial \bar{P}_{nm}}{\partial \varphi} = -m \operatorname{tg} \varphi \, \bar{P}_{nm}(\sin\varphi) + \bar{P}_{nm+1}(\sin\varphi)\left[\left(1 - \frac{\delta_{0m}}{2}\right)(n-m)(n+m+1)\right] \tag{2.3.8}$$

($\delta_{0m} = 1$ if $m = 0$: $\delta_{0m} = 0$ otherwise)

where $\sin\varphi$ and $\operatorname{tg}\varphi$ are the same for all n, m and need to be computed only once per orbit point, while the two $\bar{P}_{nm}$'s involved have to be calculated anyway to find $D_{\bar{C}^{\alpha}_{nm}} D_r V$ and $D_{\bar{C}^{\alpha}_{nm}} D_\lambda V$. This expression is singular over the poles (and so is (2.3.5)), but this problem can be overcomed completely by changing the latitude by a fraction of a second of arc at those critical points, and using a modern computer whose double precision words are at least 64 bits long. This simple trick has enabled me to carry out without difficulty the calculations reported in section 4. Some simple additional measures, such as computing the powers of $(\frac{a}{r})$ and other factors that depend only on the degree n once per point, and storing them for repeated use in terms with different order m, will ensure the streamlining of the arithmetic operations. Additional savings may be possible by choosing K, the number of integration steps, equal to a power of 2, as this allows the use of the efficient Fast Fourier Transform procedures. On the other hand, K should be larger than twice the highest value of j in (2.3.1-2), or n+5 in practice if n is not much higher than 300. This could result in a power of 2 that is much larger than the number of points needed to satisfy that minimum sampling rate, and an intrinsicly slower mixed-radix algorithm may do the job faster in some cases, because it can work with the smallest possible K, even if this is not a power of 2. This example shows the need for careful planning at all stages of software development.

The direct computation of the Fourier coefficients without using the many formulas in paragraph (2.3) is an idea that can be applied more generally. For example, computing analytical perturbations by the usual method of "variations of constants" (the classical reference for satellite geodesy is Kaula's book, already mentioned) could involve calculating very many inclination and eccentricity functions if one follows the standard long formulas: what is known as the "*literal*" approach. Alternatively, one can choose Gauss' form of the equations of motion (Brower and Clemence, Ch. XI, par. 13, 1961), which makes the calculations of the derivatives of the keplerian elements much easier because it requires finding $D_{\bar{C}^{\alpha}_{nm}} D_{r'}V$, $D_{\bar{C}^{\alpha}_{nm}} D_{u'}V$, and $D_{\bar{C}^{\alpha}_{nm}} D_{z}V$ only, as in the method just explained, and a few simple functions of the mean orbital parameters and time. Computing these derivatives at regular time intervals along the mean ellipse, after having set the rate of the argument of perigee $\dot{\omega} = 0$[1], the same as $\theta'$, $\tau_0$ and $L_0$, results in a series of values for each derivative that can be analyzed by means of the Fast Fourier Transform to obtain the corresponding harmonic coefficients. The Fourier coefficients of the perturbations, which are needed to compute them analytically, are simply those of their derivatives divided by the corresponding angular frequencies. This is an extension of the old "*numerical*" approach, which the astronomers prefer to the "literal" one for certain extensive calculations, made possible by the existence of modern computers and of the Fast Fourier Transform algorithms.

---

(1) If $\dot{\omega} \neq 0$ in reality, the procedure can be extended essentially by repeating the calculations with $\omega = k\Delta\omega$, $k = 0,1,2...k_{max}$ ($k_{max}\Delta\omega = 2\pi$), where $k_{max} = 8$ may be enough.

## 2.4 The analytical solution of the variational equations.

The variationals derived in paragraph (1.3) are linear differential equations with constant coefficients; their solution by the method of the Laplace transform is explained in Appendix I. Of the three equations (1.3.7-9), the first corresponds to perturbations $\alpha_k$ normal to the reference orbital plane and is completely independent of the last two, which describe the fluctuations $\beta_k$, $\gamma_k$ of the orbit in this plane. Moreover, when both satellites in the pair move in the same plane, which is the case considered here, only the in-plane first order perturbations appear in the linearized model, as shown in paragraph (1.4). For this reason, the discussion that follows is limited to the solutions of (1.3.8-9) (though that of the (1.3.7) equations can be found in Appendix II). These solutions, because of the type of equation, consist of *particular integrals* $\tilde{\beta}_k$, $\tilde{\gamma}_k$, which satisfy the equations when the forcing terms are not zero, and of *homogeneous parts* $\overset{0}{\beta}_k$, $\overset{0}{\gamma}_k$, which satisfy them when the forcing terms $D_{p_k} a_r = D_{p_k} a_u = 0$. The *complete solutions* are their sums: $\beta_k = \overset{0}{\beta}_k + \tilde{\beta}_k$, $\gamma_k = \overset{0}{\gamma}_k + \tilde{\gamma}_k$. They must satisfy both the equations and the initial conditions $\beta_k(0)$, $\dot{\beta}_k(0)$, $\gamma_k(0)$, $\dot{\gamma}_k(0)$, and this is ensured by the homogeneous parts alone; the particular integrals depend only on the forcing terms.

The solution $\beta_k$ is the derivative of the radial position of the satellite with respect to the parameter $p_k$, which can be one of the potential coefficients, or else a component of the initial state to be estimated, while $\gamma_k$ is the derivative of the along-track displacement (positive against the motion of the spacecraft) with respect to $p_k$. So, according to the nature of $p_k$, two cases have to be considered:

(1) $p_k$ *is a component of the initial state of the satellite.*
In this case $D_{p_k} a_r = D_{p_k} a_u = 0$, because the gravitational accelerations $a_r$ and $a_u$ are not direct functions of the initial state. As there are no forcing terms, the solutions of the variationals consist of the homogeneous part only. As shown in Appendix I, these solutions have the general form

$$\beta_k(t) = B_{0k} + B_{1k} \cos n_0 t + B_{2k} \sin n_0 t \qquad \text{(AI.17)}$$

and

$$\gamma_k(t) = G_{0k} + G_{1k} \cos n_0 t + G_{2k} \sin n_0 t + G_{3k} t \qquad \text{(AI.18)}$$

where the $B_{ik}$ and $G_{ik}$ constants are functions of the initial conditions of the variationals. These are the values of $\beta_k$, $\dot{\beta}_k$ and $\gamma_k$, $\dot{\gamma}_k$ at $t = t_0$. The table below shows the initial conditions for each component of the initial state $\underline{r}(t_0)$, $\dot{\underline{r}}(t_0)$ in the (z,r,u) coordinates:

| $p_k$ | $\beta_k(t_0)=D_{p_k} r(t_0)$ | $\dot{\beta}_k(t_0)=D_{p_k} \dot{r}(t_0)$ | $\gamma_k(t_0)=D_{p_k} u(t_0)$ | $\dot{\gamma}_k(t_0)=D_{p_k} \dot{u}(t_0)$ |
|---|---|---|---|---|
| $r(t_0)$ | 1 | 0 | 0 | 0 |
| $\dot{r}(t_0)$ | 0 | 1 | 0 | 0 |
| $u(t_0)$ | 0 | 0 | 1 | 0 |
| $\dot{u}(t_0)$ | 0 | 0 | 0 | 1 |

These values are a consequence of the mutual independence of the various components of the initial state. Equating the right hand sides of (AI.17) and (AI.18) and of their first derivatives to the respective initial conditions, it is possible to solve for the values of the $B_{ik}$ and $G_{ik}$. If $\beta_{r(t_0)}(t)$ stands for $D_{r(t_0)} r(t)$, $\gamma_{u(t_0)}(t)$ for $D_{u(t_0)} u(t)$, and so on, then, making the change of variable $t' = t - t_0$ to simplify the results,

$$\boxed{\begin{aligned}
\beta_{r(t_0)} &= -3 \cos n_0 t' + 4 && \text{(2.4.1a)}\\
\beta_{\dot{r}(t_0)} &= n_0^{-1} \sin n_0 t' && \text{(2.4.1b)}\\
\beta_{u(t_0)} &= 0 && \text{(2.4.1c)}\\
\beta_{\dot{u}(t_0)} &= 2n_0^{-1}(\cos n_0 t' - 1) && \text{(2.4.1d)}\\
\gamma_{r(t_0)} &= -6(\sin n_0 t' - n_0 t) && \text{(2.4.1e)}\\
\gamma_{\dot{r}(t_0)} &= -2n_0^{-1}(\cos n_0 t' - 1) && \text{(2.4.1f)}\\
\gamma_{u(t_0)} &= 1 && \text{(2.4.1g)}\\
\gamma_{\dot{u}(t_0)} &= 4n_0^{-1} \sin n_0 t' - 3t' && \text{(2.4.1h)}
\end{aligned}}$$

Multiplying the partial derivatives $\beta_k$, $\gamma_k$ by $\Delta p_k$ and adding over all k to obtain the total radial and along-track variations:

$$\Delta r(t') = \Delta\dot{r}(t_0)n_0^{-1}\sin n_0t' - (3\Delta r(t_0) - 2\Delta\dot{u}(t_0)n_0^{-1})\cos n_0t' +$$

$$+(4\Delta r(t_0) - 2\Delta\dot{u}(t_0)n_0^{-1}) \tag{2.4.2}$$

$$\Delta u(t') = -2\Delta\dot{r}(t_0)n_0^{-1}\cos n_0t' + (4n_0^{-1}\Delta\dot{u}(t_0) - 6\Delta r(t_0))\sin n_0t' +$$

$$+(\Delta u(t_0) + 2\Delta\dot{r}(t_0)n_0^{-1}) + (6n_0\Delta r(t_0) - 3\Delta\dot{u}(t_0))t' \tag{2.4.3}$$

Therefore

$$\Delta\dot{r}(t') = \Delta\dot{r}(t_0)\cos n_0t' + (3n_0\Delta r(t_0) - 2\Delta\dot{u}(t_0))\sin n_0t' \tag{2.4.4}$$

$$\Delta\dot{u}(t') = 2\Delta\dot{r}(t_0)\sin n_0t' + (4\Delta\dot{u}(t_0) - 6n_0\Delta r(t_0))\cos n_0t' +$$

$$+(6n_0\Delta r(t_0) - 3\Delta\dot{u}(t_0)) \tag{2.4.5}$$

These expressions are very useful in understanding the perturbations caused by changes in the initial state. The constant term $4\Delta r(t_0)-2\Delta\dot{u}(t_0)n_0^{-1}$, in particular, is proportional to the change in total energy (kinetic + potential) of the orbit. So the drift $(6n_0\Delta r(t_0)-3\Delta\dot{u}(t_0))t$ is clearly also proportional to this change.

(2) $p_k$ *is a potential coefficient.*
Here the forcing terms are given by (2.3.48-49). The particular integral is a sum of sine and cosine terms with the same frequencies as those of the forcing functions $D_{\bar{C}^{\alpha}_{nm}} a_r$ and $D_{\bar{C}^{\alpha}_{nm}} a_u$. If the *phases* of the forcing terms change, but not their *amplitudes*, then the phases of the corresponding terms in the integral change by the same angle, but their amplitudes stay the same; these are properties of the response of any linear, time-invariant system. As shown in Appendix I, the complete solution is

$$\beta_k(t) \equiv \beta_{nm\alpha}(t)$$

$$= B_{0nm\alpha}+B_{1nm\alpha} \cos n_0t+B_{2nm\alpha} \sin n_0t+ \sum_{j=-(n+3N)}^{n+3N} b_{nmj} \cos((jn_0+m\theta')t+\hat{\phi}_{m\alpha}) \quad (2.4.6)$$

$$\gamma_k(t) \equiv \gamma_{nm\alpha}(t)$$

$$= G_{0nm\alpha}+G_{1nm\alpha} \cos n_0t+G_{2nm\alpha} \sin n_0t+G_{3nm\alpha}t+ \sum_{j=-(n+3N)}^{n+3N} g_{nmj} \sin((jn_0+m\theta')t+\hat{\phi}_{m}) \quad (2.4.7)$$

where

$$b_{nmj} = \frac{-((jn_0+m\theta')a_{rnmj}+2n_0a_{unmj})}{(jn_0+m\theta')((jn_0+m\theta')^2-n_0^2)} \quad (2.4.8)$$

$$g_{nmj} = \frac{-(2(jn_0+m\theta')n_0a_{rnmj}+((jn_0+m\theta')^2+3n_0^2)a_{unmj})}{(jn_0+m\theta')^2((jn_0+m\theta')^2-n_0^2)} \quad (2.4.9)$$

while the constant phase angle $\hat{\phi}_{m\alpha}$ is given by (2.3.50), and *does not* depend on $t_0$. The summations in (2.4.6-7) are the particular integrals; the remaining terms represent the homogeneous response. Expressions (2.4.8-9) show clearly that the $b_{nmj}$ and $g_{nmj}$ are independent of the initial conditions. Because a change in gravitation cannot affect the initial state vector,

$$D_{\bar{C}^{\alpha}_{nm}} r(t_0) = D_{\bar{C}^{\alpha}_{nm}} \dot{r}(t_0) = D_{\bar{C}^{\alpha}_{nm}} u(t_0) = D_{\bar{C}^{\alpha}_{nm}} \dot{u}(t_0) = 0$$

or, in the "$\beta,\gamma$" notation,

$$\beta_{nm\alpha}(t_0) = \dot{\beta}_{nm\alpha}(t_0) = \gamma_{nm\alpha}(t_0) = \dot{\gamma}_{nm\alpha}(t_0) = 0.$$

so all initial conditions are 0 at $t_0$. In general, $t_0 \neq \tau_0$, the time when the satellite first reaches perigee (F' = 0), because the variationals are solved, in practice, along orbital arcs that can begin and end at any time during the mission. To ensure zero initial conditions, the homogeneous part must cancel the particular integral at $t = t_0$, because, in general, this integral is not 0 at this time. So, calling $\beta^0_{nm\alpha}$ to the homogeneous part of $\beta$,

$$\begin{aligned}\beta^0_{nm\alpha}(t_0) &= B_{0nm\alpha}+B_{1nm\alpha} \cos n_0t_0+B_{2nm\alpha} \sin n_0t_0 \\ &= -\tilde{\beta}_{nm\alpha}(t_0) \\ &= \sum_{j=-(n+3N)}^{n+3N} b_{nmj} \cos((jn_0+m\theta')t +\hat{\phi}_{m\alpha})\end{aligned}$$

From this follows that the coefficients $B_{inm\alpha}$ are all functions of $t_0$, the starting time of the arc. Not so the $b_{nmj}$, $g_{nmj}$ and $\hat{\phi}_{m\alpha}$. If one arc ends and another starts at $t_0$ (perhaps with slightly different $\underline{r}$ and $\dot{\underline{r}}$ due to a corrective orbital manoeuvre, as explained later) the particular integrals for the second arc are the analytical continuations of those for the first, but not so the homogeneous parts.
For all t, the particular integrals are

$$\tilde{\beta}_{nm\alpha}(t) = \sum_{j=-(n+3N)}^{n+3N} b_{nmj} \cos((jn_0+m\theta')t+\hat{\phi}_{m\alpha}) \qquad (2.4.10)$$

$$\tilde{\gamma}_{nm\alpha}(t) = \sum_{j=-(n+3N)}^{n+3N} g_{nmj} \sin((jn_0+m\theta')+\hat{\phi}_{m\alpha}) \qquad (2.4.11)$$

regardless of the arc in which the spacecraft happens to be (remember that $\hat{\phi}_{m\alpha}$ depends on $\tau_0$ and not on $t_0$).

## 2.5 Resonance.

The particular integral contains terms of angular frequency smaller than $n_0$, corresponding to *short period* perturbations, and others of frequency larger than $n_0$, corresponding to *long period* perturbations. These terms grow larger as their frequencies approach 0 or $n_0$, where expressions (2.4.8-9) become indefinite. This phenomenon is known as *resonance*.
Oscillations with frequencies very close to either of the critical values can have extremely large amplitudes. When forced exactly at a critical frequency, the linear dynamic system defined by the variationals behaves essentially differently than at any other

frequency. At zero frequency, as shown in Appendix I, the response is like that to perturbations in the initial conditions only, or free response: a constant term in $\beta$, oscillations at the system's natural frequency $n_0$, and a secular drift in $\gamma$. At $n_0$ the response contains all of these, but there are additional terms of the form $A\ t \sin n_0 t$ or $B\ t \cos n_0 t$: secularly increasing oscillations in $\beta$ and in $\gamma$. The theory developed in paragraph (2.2) shows that, in general, for a rotating planet like the Earth, where $\theta'$ can never be 0, because it spins too fast, only the zonals can produce forcing terms with frequencies 0 and $n_0$. For nearly circular orbits, those that contribute most to the zero frequency are the even zonals, while the odd zonals provide most of the $n_0$ oscillations. This is because, as shown in paragraph (2.5) for a given degree n, the more circular the orbit, the smaller those terms in the forcing functions corresponding to odd multiples of $n_0$ if n is even, and to even multiples if n is odd.

As long as the perturbations in the orbit remain small enough to be considered solutions of the linear variationals, any secular drift along-track can be due to the zonals only, so the coefficient $G_{3nm\alpha}$ (except in some especial cases, like geostationary orbits, where $jn_0+m\theta'$ can be zero) corresponding to such a drift in the general expression (2.4.7) must be negligible if $m \neq 0$, i.e. for tesserals and sectorials. In physical terms, any along-track drift implies a change in the mean orbital energy, which must be linked to a non-zero average value, along the whole orbit, for the anomalous potential that creates the perturbations. If this potential is expanded in a Fourier series, like $D_{\bar{C}^{\alpha}_{nm}}\ a_r$ and $D_{\bar{C}^{\alpha}_{nm}}\ a_u$ were in paragraph (2.2), only that part that is due to the zonals contributes to the mean value, except in the case of a planet in which $\theta' = 0$ is possible, or of especial orbits where perfect resonance occurs with $m \neq 0$.

The resonance at $n_0$ produces oscillations that should grow for ever, or until the satellite finally crashes against the ground in one of its downward swings. But such a catastrophic end is merely an extrapolation of a behaviour that is only valid for very small perturbations, so any conclusions on what may happen when they become very large are likely to be wrong. Resonance occurs whenever the satellite encounters the same disturbing gravitational feature at repeated, virtually identical intervals of which the orbit period is an exact multiple. The response to this feature is then gradually reinforced and grows with time. In the case of a closed, periodical orbit, a zonal is precisely this type of feature. The actual orbit, in general, is not periodical. The increasing oscillations that separate it from the reference orbit, at least, are not so. Therefore, once this separation has become sufficiently large, the satellite may be passing through the same disturbance at intervals that are appreciably different from each other, in the end experiencing large (but bounded) oscillations, accompanied by secular changes in the argument of perigee.

Whatever the ultimate fate of drag-free satellites orbiting close to the Earth, those forming the pair used for the mission under study can be kept always sufficiently near their desired trajectories by using the same rocket engines that they carry for compensating drag and other non-gravitational forces. This would require some brief and widely spaced manoeuvres, in addition to their normal operation, to simply turn the drift away from the reference orbit back towards it, without causing any immediate change in position, so fuel consumption can be kept to a minimum.

The familiar first order perturbation theory based on Lagrange's planetary equations, widely used in satellite geodesy, regards the orbit as an ellipse with one focus at the geocenter, whose size, shape and orientation in space are continuously changing under the effect of gravitational anomalies. These changes are expressed as perturbations of the Keplerian elements that define the ellipse; some are secular, like the precession of the line of nodes and of the main axes, and some are periodical. Resonance occurs near the zero frequency, so very slow variations tend to become also large. In the rotating coordinates of the variantionals, changes in the size of the ellipse will show up also as very slow oscillations, but changes in the shape (i.e., in the eccentricity) will be seen as fast oscillations close to $n_0$, modulated by very slow ones close to zero. This is because the perturbed orbit has a

period (the time between consecutive crossings of the perigee) that is slightly different from $n_0$. As the satellite turns, its geocentric distance varies from a maximum at apogee to a minimum at perigee once per revolution. A slow change in eccentricity will displace the focus within the major axis, so the difference between maximum and minimum at each oscillation will change slowly, with the same period as the eccentricity. Such variations in the argument of perigee and in the mean anomaly mostly affect the orbital frequency. To understand the presence of pulsating oscillations in terms of the theory developed so far, consider the forced response of the variationals to a simple sinewave of frequency $n_0 + \Delta n$, where $\Delta n$ is a small difference between the frequency of the perturbation and $n_0$.
According to expression (2.4.10-11) this response must contain terms of the form $A \sin n_0 t + B \sin(n_0+\Delta n)t$ and $C \cos n_0 t + D \cos(n_0+\Delta n)t$; those of frequency $n_0$ belong to the homogeneous part, and the others to the particular integral. Without loss of generality, they can be written as

$$(A-B)\sin(n_0 t+\zeta)+B[\sin((n_0+\tfrac{1}{2}\Delta n)t-\tfrac{1}{2}\Delta nt+\zeta)+\sin((n_0+\tfrac{1}{2}\Delta n)t+\tfrac{1}{2}\Delta nt+\zeta)]$$

where $\zeta = 0$ or $\frac{1}{2}\pi$. This is equivalent to

$$E \sin(n_0 t+\zeta)+F \cos \tfrac{1}{2}\Delta nt \sin((n_0+\tfrac{1}{2}\Delta n)t+\zeta)$$

where $E = A-B$ and $F = 2B$.
Since E and F are always such as to ensure that the expression above is zero at $t_0$, the response begins increasing very gradually from zero, reaches eventually a maximum, and goes back to zero in a long cycle of period $\frac{1}{2}\Delta n$. If $\Delta n$ is very small, a secularly growing oscillation at $n_0$ and a slowly growing pulsation at $n_0+\frac{1}{2}\Delta n$ look much the same at their initial stage, when both are small enough to be explicable in terms of a first order theory. Only after they get quite large do their differences become apparent.

The zonal resonances explained here are similar to those that take place at the "critical inclination" $i = \sin^{-1}\{(\frac{4}{5})^{\frac{1}{2}}\}$. When the orbit is periodical, its mean ellipse is "frozen" in its precessing plane, so the argument of perigee $\omega$ does not change, and $\dot{\omega} = 0$. At the critical inclination, all orbits round a planet closely resembling an oblate ellipsoid, like ours, have a very small $\dot{\omega}$ and are, therefore, "frozen" enough to show resonance (some important formulas of the classical first order theory become singular at this inclination).

To complete the topic of zonal resonances, I shall mention briefly those that perturb an orbit at right angles to its plane. They affect this plane by making it turn slowly about the Earth's axis, as mentioned in earlier paragraphs and shown in Appendix II. For a near-polar orbit, forces normal to the orbit plane are quite weak, so this effect is extremely slow.

Because there are values of the order m for which $m\theta'$ comes close to being a whole multiple of the orbital frequency, the coefficients $\bar{C}^{\alpha}_{nm}$ of such an order will contribute frequencies $(jn_0+m\theta')$ that, for some j, will be very close to either 0 or $n_0$. The resulting steady oscillations in the forced response can be very large. Coefficients related to these near-resonances are usually referred to as *resonant coefficients*. Whether perfectly or nearly resonant, some perturbations may grow so large in the absence of compensatory manoeuvres, that they cannot be treated by a linear, or first order theory any longer. In this case a nonlinear treatment is required; some examples of this kind of approach are given in the book by Kaula already mentioned.

## 2.6 Perturbations of a perfectly circular orbit.

Assuming that the reference orbit corresponds to a central force field where the potential is $V = GMr^{-1}$, so it is perfectly circular, and that the true orbit results from slight perturbations to that circular one introduced by the $\bar{C}^{\alpha}_{nm}$ (this assumption is not valid for the Earth, because of the large "oblate" and "pear-shaped" terms related to $\bar{C}^{0}_{20}$ and $\bar{C}^{0}_{30}$, respectively), then the forcing terms of the variationals given as functions of F and L are:

$$D_{\bar{C}^{\alpha}_{nm}}\, a_r = \sum_{p=0}^{n} \tilde{\alpha}_{rnmp} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} ((n-2p)F+mL+\phi_{\alpha}) \tag{2.6.1}$$

$$D_{\bar{C}_{nm}}^{\alpha} a_u = \sum_{p=0}^{n} \tilde{a}_{unmp} \begin{Bmatrix} -\sin \\ \cos \end{Bmatrix} ((n-2p)F+mL+\phi_\alpha) \tag{2.6.2}$$

where $\phi_\alpha = -\alpha \frac{\pi}{2}$ and

$$\tilde{a}_{rnmp} = -GM\ a^n \bar{r}^{-(n+2)} (n+1) \bar{F}_{nmp}(i) \tag{2.6.3}$$

$$\tilde{a}_{unmp} = -GM\ a^n \bar{r}^{-(n+2)} (n-2p) \bar{F}_{nmp}(i) \tag{2.6.4}$$

These expressions can be derived directly from (2.2.7-8) if r is assumed to be constant and equal to $\bar{r} = \sqrt[3]{GM\ n_0^{-2}}$. Changing from F to $F' = F-\frac{1}{2}\pi$ and reasoning as in paragraph 2.2 (except that now r is a constant) leads to the equivalent of (2.2.27):

$$D_{\bar{C}_{nm}}^{\alpha} a_r = \sum_{p=0}^{n} a_{rnmp} \cos((n-2p)F'+mL+\phi_{m\alpha}) \tag{2.6.5}$$

$$D_{\bar{C}_{nm}}^{\alpha} a_u = \sum_{p=0}^{n} a_{unmp} \sin((n-2p)F'+mL+\phi_{m\alpha}) \tag{2.6.6}$$

where

$$a_{rnmp} = \begin{cases} (-1)^{\frac{(n-2p)}{2}} \tilde{a}_{rnmp} & \text{if n is } \textit{even} \\ (-1)^{\frac{(n-2p+2m-1)}{2}} \tilde{a}_{rnmp} & \text{if n is } \textit{odd} \end{cases} \tag{2.6.7}$$

$$-\ a_{unmp} = \begin{cases} (-1)^{\frac{(n-2p)}{2}} \tilde{a}_{unmp} & \text{if n is } \textit{even} \\ (-1)^{\frac{(n-2p+2m-1)}{2}} \tilde{a}_{unmp} & \text{if n is } \textit{odd} \end{cases} \tag{2.6.8}$$

Choosing $t_0 = \tau_0 = 0$ to simplify matters, and using the relationships $L = m\theta' t$ and $F' = n_0 t$ (the latter is valid, because the reference orbit is here a circle), one obtains

$$D_{\bar{C}_{nm}}^{\alpha} a_r(t) = \sum_{p=0}^{n} a_{rnmp} \cos(((n-2p)n_0+m\theta')t+\phi_{m\alpha}) \tag{2.6.9}$$

$$D_{\bar{C}_{nm}}^{\alpha} a_u(t) = \sum_{p=0}^{n} a_{unmp} \sin(((n-2p)n_0+m\theta')t+\phi_{m\alpha}) \tag{2.6.10}$$

The solutions to the variationals corresponding to perturbations in the initial state have the form given by (2.4.1a-h) in a previous paragraph. The complete solutions for gravitational perturbations affecting the coefficients $\bar{C}^{\alpha}_{nm}$ are going to resemble (2.4.6-7), with same terms in the "homogeneous" response as in (2.4.1a-h), and terms in the particular integral of the same frequency as those in (2.6.9-10):

$$\beta_{nm\alpha}(t) = B_{0nm\alpha}+B_{1nm\alpha} \cos n_0 t+B_{2nm\alpha} \sin n_0 t + \sum_{p=0}^{n} b_{nmp} \cos(((n-2p)+m\theta')t+\phi_{m\alpha}) \qquad (2.6.11)$$

$$\gamma_{nm\alpha}(t) = G_{0nm\alpha}+G_{1nm\alpha} \cos n_0 t+G_{2nm\alpha} \sin n_0 t+G_{3nm\alpha} t + \sum_{p=0}^{n} g_{nmp} \sin(((n-2p)+m\theta')t+\phi_{m\alpha}) \qquad (2.6.12)$$

where $b_{nmp}$ and $g_{nmp}$ are given by (2.4.8-9) after replacing "j" by "(n-2p)". Notice that the terms in the particular integral have frequencies that are even harmonics of $n_0$ if the degree n is even, and odd harmonics if n is odd, "shifted" by $m\theta'$.

## 2.7 The periodical reference orbit seen as a perturbed circular orbit.

The reference orbit can be regarded as the result of perturbing a perfectly circular orbit with small changes in the initial state vector at t = 0, and in the gravity field (by adding zonal terms of degree n between 1 and $N^{(1)}$). The orbit is periodical, so the effect of all secular perturbations and exact resonances must cancel out. The solutions to the variationals, $\beta_k$ and $\gamma_k$, corresponding to changes in the initial state have the same form as before, given by (2.4.1a-h), while those corresponding to $p_k = \bar{C}^0_{n0}$ are given by (2.6.11-12) with $m = \alpha = 0$, so $\phi_{m\alpha} = 0$, and

$$\beta_{n00}(t) = B_{0n00}+B_{1n00} \cos n_0 t+B_{2n00} \sin n_0 t + \sum_{p=0}^{n} b_{n0p} \cos((n-2p)n_0 t) \qquad (2.7.1)$$

---

(1) The non-linear effects associated with $\bar{C}^0_{20}$ and $\bar{C}^0_{30}$ can be ignored here.

$$\gamma_{n00}(t) = G_{0n00}+G_{1n00} \cos n_0 t+B_{2n00} \sin n_0 t+B_{3n00} t$$
$$+ \sum_{p=0}^{n} g_{n0p} \sin((n-2p)n_0 t) \qquad (2.7.2)$$

From the last two expressions and (2.4.1-2) one can see that $\beta_k$ and $\gamma_k$ have the form

$$\beta_k = \sum_{j=-n}^{n} B_{nj} \cos jn_0 t+B_k' \sin n_0 t \qquad (2.7.3)$$
$$\gamma_k = \sum_{j=-n}^{n} G_{nj} \sin jn_0 t+G_k' \cos n_0 t+G_k'' t \qquad (2.7.4)$$

where

$B_{nj} = G_{nj} = 0$ if $p_k$ is a component of the initial state and $j \neq 1$, or $p_k = \bar{C}^{\alpha}_{nm}$ and j and n have different parities.

Calling $\Delta r$ to the radial departure of the reference orbit from the unperturbed circular one, which has the same period, and $\Delta u$ to the along-track departure between both orbits, then, to a first order approximation,

$$\Delta r \simeq \sum_k (D_{p_k} r) \Delta p_k = \sum_k \beta_k \Delta p_k \qquad (2.7.5)$$

$$\Delta u \simeq \sum_k (D_{p_k} u) \Delta p_k = \sum_k \gamma_k \Delta p_k \qquad (2.7.6)$$

where $\Delta p_k = p_k - p_{k0}$, as defined in paragraph (1.4), is the variation in the value of $p_k$ responsible for the perturbation. Also

$$\Delta \dot{u} = \sum_k \dot{\gamma}_u \Delta p_k \qquad (2.7.7)$$

As all $\gamma_k$ and $\beta_k$ contain terms of frequency $n_0$, while some contain terms of frequency 0 (constants) and of frequencies $(n-2p)n_0$, with $-n \leq p \leq n$ and $1 \leq n \leq N$, the sums (2.7.5-7) of such terms multiplied by constants, have the general form (remember that the secular terms cancel out, that the orbit is symmetrical respect to $t = 0$, and that $\beta_k$, $\gamma_k$ have the general form (2.7.3-4))

$$\Delta r(t) = \sum_{j=0}^{N} r_j \cos jn_0 t \tag{2.7.8}$$

and

$$\Delta \dot{u}(t) = \sum_{i=1}^{N} \dot{f}_i \cos in_0 t \tag{2.7.9}$$

Accordingly, the orbital radius is

$$r = \bar{r} + \Delta r \simeq \bar{r} + \sum_{j=1}^{N} r_j \cos jn_0 t \tag{2.7.10}$$

where $\bar{r}$ is the mean value of r, and this is the same formula in paragraph (2.2) that leads to expression (2.2.15). The tangential velocity along the reference orbit is

$$\dot{u} = n_0 \bar{r} + \Delta \dot{u}$$

($n_0$ is the angular velocity along the circular orbit) so, according to (2.7.9),

$$\dot{u}(t) \simeq n_0 \bar{r} + \sum_{i=1}^{N} \dot{f}_i \cos in_0 t$$

and

$$\dot{F}'(t) \simeq \frac{\dot{u}(t)}{\bar{r}} \simeq n_0 + \bar{r}^{-1} \sum_{i=1}^{N} \dot{f}_i \cos in_0 t \tag{2.7.11}$$

which is the same as (2.2.30). This completes the partial justification given in paragraph (2.2) for the "3N" that appears in the upper and the lower limits of the summations in the forcing terms and in the solutions of the variationals. In the case discussed in this paragraph the degree of the field of the "reference" circular orbit is 0, which explains the absence of "3N" in expressions (2.7.1) to (2.7.4). The "N" in (2.7.10-11) corresponds to the zonals that disturb the circular orbit, which in this paragraph plays the role of "reference orbit". So "N" here has the same meaning as "$N_{max}$", the degree at which the spherical harmonic expansion of the potential responsible for the perturbations can be truncated.

Summary.

The equations of motion and the expression of the relative line of sight velocity are linearized along a reference orbit that is closed and periodical in its plane, to introduce symmetries in the mathematical model that can be exploited to make the estimation of the potential coefficients very efficient, as explained in section 3. Insight gained from the discussion of the existence and properties of such orbits in a purely zonal field can be used to find numerical methods for obtaining the initial conditions of trajectories that, when numerically integrated over one revolution, close almost perfectly, returning to their initial state, and have a mean radius very near the one specified for the actual satellite orbit (the method proposed in this chapter is a refinement of Cook's theory of perturbations of near-circular orbits). The forcing terms of the variational equations along the reference orbit can be expressed as functions of time developed in trigonometric Fourier series after a long sequence of mathematical operations, as it is common in celestial mechanics. These operations produce an equally long (and cumbersome) sequence of formulas that *could* be used to obtain the Fourier coefficients of the forcing terms, which are essential to the analytical solution of the variationals. To use such formulas, what astronomers call the "literal" approach (because formulas are written with letters!) would require very lengthy and complex computations. Alternatively, first the forcing terms can be computed at equal time intervals along the reference orbit using the coordinates obtained by numerical integration, and then their values can be subject to Fourier analysis by some efficient algorithm of the Fast Fourier Transform family. This can be seen as an extension, possible in the "computer age", of the "numerical" approach, of which Gauss'method is a well-known example. The idea is feasible because the coefficients are independent of Earth's rotation and orbital precession. Once the Fourier coefficients have been found, expressions (2.4.1a-h) and (2.4.6-9) give the analytical solutions of the variationals corresponding to changes in the initial state and in the potential coefficients, respectively. The detailed derivations are given in Appendix I. These solutions correspond only to perturbations in the orbital plane, which are

all that is needed for this study. To complete the orbit theory, perturbations normal to that plane, and non-periodical reference orbits, are discussed in Appendix II. The analytical solutions show the existence of secular variations in position and in velocity; of oscillations of frequency lower than $n_0$ (the angular frequency of the reference orbit) called *long period* perturbations; of oscillations above $n_0$, or *short period* perturbations; and of secularly increasing oscillations of frequency $n_0$ caused usually by zonal gravitational anomalies. These growing oscillations reveal the resonant character of the dynamic system defined by the linearized equations of motion, or variationals. This system is totally undamped, because the satellite is supposed to be drag-free. Resonance is also shown by the fact that the steady forced oscillations have larger amplitudes as their frequencies aproach 0 or $n_0$ (the peak at $n_0$ is a consequence of the use of a system of coordinates that rotates at this frequency). The zonals can excite growing oscillations because the reference orbit is periodical. Due to all the secular perturbations, the true (or perturbed) orbit moves continuously away from the reference one, until the solutions of the variationals become inapplicable because the perturbations have grown too large to be explained by them. The two satellites to be used for mapping the geopotential must follow reasonably closely the reference orbit, so the model linearized along it remains valid, and the mapping can be done in a single global operation in an efficient way. Their thrusters, used normally for drag-compensation, must also be fired occasionally to nudge each spacecraft back towards its proper course, before it strays too far. The closed reference orbit can be seen as a circular orbit perturbed by the first N zonals, so the properties of disturbed circular orbits must be studied to understand the shape of the former, which modulates the forcing terms of the variationals giving them a richer frequency content.

## 3. THE ADJUSTMENT.

### Introductory remarks and overview.

After having described the linearized model for the signal in satellite-to-satellite tracking, and explained the orbit theory behind that, it is time to get on with the question of how to use these ideas to estimate some $10^5$ unknown corrections to the potential coefficients out of some $10^6$ measurements. Clearly, this requires a very sparse system of normal equations with a helpful structure. Such structure, in this case, is that of an arrow, with a "shaft" and "tip" made up of diagonal blocks, and two "wings" (the right and the bottom edges). The rest of the normal matrix is all zeros. This structure occurs when the sites of the observations, and, thus, the orbits, form a strong geometrical pattern. Imagine the periodical orbit of the previous section, with a period that fits an exact number of times in an interval of a whole number of days. In that interval, the orbit must look to an Earth-fixed observer like a helix wrapped around the nearly-spherical surface swept by the near-circular orbit itself as its plane precesses and the Earth rotates. At the end of this period, the helix must close, "biting its own tail". This makes the whole orbit repeat itself indefinitely, and the measurements along it too, if we ignore the non-periodical component of the signal, due to discontinuities at the start of the various arcs in between orbital manoeuvres, and to increasing oscillations caused by zonal resonances. At every point along this orbit, an observer knowing only where the poles (or the equatorial plane) are, would be able to tell his latitude, but not his longitude,because the shape of the orbit would convey no information to him on this respect: the "tail biting" helix has rotational symmetry about the Earth's axis, and so does the discrete subset of its points where measurements were taken. The periodical part of the signal consists, then, of harmonics of the basic frequency at which the grand sequence of measurements repeats itself. If the coverage is fine enough, and the sampling rate of the instruments high enough, the contributions to the periodic part of the signal coming from

$\Delta\bar{C}^{\alpha}_{nm}$ with a given m and $\alpha$ are completely independent (orthogonal) to those coming from all other $\Delta\bar{C}^{\alpha}_{nm}$, provided that $N_{max}$ is not too high. Assigning so-called "arc parameters" (extra unknowns) to the non-periodic part, as explained in paragraph (3.1), this independence of the components of the periodical part leads to the arrow-structured normal matrix, which makes solving the corresponding equations, if not easy, at least feasible even with present-day machines. *At the heart of the method lies the rotational symmetry of the overal trajectory*, and the unbroken nature of the data stream. Of course, the overall mission can be divided into shorter portions, each one "biting its tail", instead of a single grand cycle from beginning to end. The practical consequences are the same. In each of these sub-intervals, the main parameters of the mission, such as satellites' height and separation, could be chosen to reinforce the estimation of a particular part of the spectrum, or for other reasons. In a real mission there may be, from time to time, interruptions in the stream of measurements caused by malfunctions, ionospheric disturbances, orbital manoeuvres, and so on. Planned interruptions must be brief, so interpolation from measurements adjacent to the break may "seal" the gaps without much harm to the results. Large, unplanned breaks would make the method described here inapplicable. As an alternative, one may consider local maps, or solutions, to recover information wherever there is enough data coverage. A third possibility, sketched in paragraph (3.11), is to ignore the non-periodical part of the signal, and then regard the measurements, separated according to whether they belong to ascending or to descending passes (half-orbits containing the node after which they are named), as "point" measurements of a function of position. Then, the measurements can be processed in the same way as, say, gravimeter or satellite altimeter measurements. By averaging the observations within the blocks of a regular grid laid over the surface swept by the reference orbit, very efficient procedures that exploit the rotational symmetry of this grid become applicable. This is, of course, less satisfactory than the more rigorous adjustment of the uninterrupted data stream, which is the main topic of this section.

Because the model is obtained by linearizing along the periodic reference orbit, which is too rigidly defined to follow the actual trajectories very closely, nonlinear effects will become progressively larger, as true and reference orbits diverge, and they could invalidate very soon the whole idea. Instead, numerically integrated orbits should be fitted to all the data (including tracking from terrestrial stations) using models for all important forces acting on the satellites not included in the calculation of the reference orbit. From the velocities of these so-called *nominal* orbits, a reference line-of-sight velocity should be computed and substracted from the data, to form *residuals*. Eventually, when the separations grow too large, the thrusters in the satellites should be used to bring them back to their proper courses. In this way, nonlinearities and other unwanted effects can be eliminated or greatly reduced, diminishing the bias in the linear estimates of the potential coefficients.

The first paragraph considers the advantage of using residual observations as data, for the reasons outlined above, and introduces the concept of arc parameters. The two paragraphs that follow discuss estimation methods. First, it is ordinary, linear least squares adjustment; then, "conditioning" in general, and least squares collocation in particular. This is followed by a detailed derivation of the observation equations, particularly of the periodical part, which retains the Fourier-series structure of the perturbations of section 2. Starting with the linearized model for the instantaneous line-of-sight velocity, the formulas are modified to cover the case of practical interest: averaged line-of-sight velocity. The next two paragraphs look at the overall periodic structure of the trajectories, and explain how this structure brings about a sparse normal matrix. Detailed formulas for setting up this matrix are given, and a method for solving them and then obtaining formal variances and covariances for the solution, is outlined. Several important details, including aliasing, the idea of iterating the solution, downward continuation of the results to the Earth's surface, and how to compute orbits with a very high degree and order force field, come next. The section closes with an outline of how to treat residual measurements as "point" observations of a function defined in space, rather than in time; and with a discussion of local solutions, and on how they can complement global ones.

3.1 Observations, nominal orbits, and residuals.

In section 1, the model for the perturbation $\Delta s$ of the line-of-sight relative velocity, the difference between the signal s and its reference value $s_0$, was obtained by linearizing the mathematical expression of s along a reference orbit defined by a set of reference parameter values $p_{k0}$. If the $p_k$ are the true values, the perturbations $\Delta s$, an analytic function of the parameters, can be developed in a Taylor series that converges for small differences $(p_k - p_{k0})$, as follows:

$$\Delta s = s - s_0 = \sum_k D_{p_k} s\,(p_k - p_{k0}) + \sum_{k_1} \sum_{k_2} \tfrac{1}{2} D_{p_{k_1} p_{k_2}} s\,(p_{k_1} - p_{k_1 0})$$

$$(p_{k_2} - p_{k_2 0}) + \sum_{k_1} \sum_{k_2} \ldots \sum_{k_h} \frac{1}{h!} D^h_{p_{k_1} \ldots p_{k_h}} s\,(p_{k_1} - p_{k_1 0})$$

$$(p_{k_2} - p_{k_2 0}) \ldots (p_{k_h} - p_{k_h 0}) + \ldots + \Delta\mathcal{s} \qquad (3.1.1)$$

where $\Delta\mathcal{s}$ is that part of $\Delta s$ that is not due to the $\Delta p_k$ (differences in potential coefficients and initial states) but to other causes, such as the attraction of the Sun and the Moon, etc., not accounted for in the reference orbit.

Let $\overset{*}{p}_k$, $\overset{*}{\Delta\mathcal{s}}$ represent parameter values and estimates of $\Delta\mathcal{s}$ obtained a priori, the ones from some already existing model of the gravitational field, the other from tables of ephemerides, tidal models, etc., which should be good enough to ensure that both $(p_k - \overset{*}{p}_k)$ for all $\overset{*}{p}_k$ and $(\Delta s - \Delta\overset{*}{s})$ are quite small. In general, as the maximum degree N in the reference field defined by the $p_{k0}$ is low, one can take the highest degree $\overset{*}{N}$ for the $\overset{*}{p}_k$ to be larger than N, so the $\overset{*}{p}_k$ may include the values of more potential coefficients $\bar{C}^{\alpha}_{nm}$ than do the $p_{k0}$. If $\overset{*}{s}$ is the relative line-of-sight velocity corresponding to the $\overset{*}{p}_k$, then

$$\Delta\overset{*}{s} = \overset{*}{s} - s_0 = \sum_k D_{p_k} s\,(\overset{*}{p}_k - p_{k_0}) + \sum_{k_1} \sum_{k_2} \tfrac{1}{2} D_{p_{k_1} p_{k_2}} s\,(\overset{*}{p}_{k_1} - p_{k_1 0})$$

$$(\overset{*}{p}_{k_2} - p_{k_{20}}) + \ldots + \sum_{k_1} \sum_{k_2} \ldots \sum_{k_h} \frac{1}{h!} D^h_{p_{k_1} p_{k_2} \ldots p_{k_h}} s\,(\overset{*}{p}_{k_1} - p_{10})$$

$$(\overset{*}{p}_{k_2} - p_{k_{20}}) \ldots (\overset{*}{p}_{k_h} - p_{k_{h0}}) + \ldots + \overset{*}{\Delta\mathcal{s}} \qquad (3.1.2)$$

(for both $\Delta s$ and $\Delta \overset{*}{s}$ the partial derivatives are taken at the same points along the reference orbit).
In what follows, it shall be assumed that $\Delta \overset{*}{s}$ corresponds to a pair of orbits that the satellites would follow, in a field whose $\bar{C}^{\alpha}_{nm}$ are equal to the respective $\overset{*}{p}_k$, from initial states adjusted to fit all tracking data available over a long period, or *arc*. These adjusted initial states complete the set of $\overset{*}{p}_k$. The fitted orbits, better approximations to the true ones than the common reference orbit introduced in paragraphs (1.4) and (2.1), shall be called the *nominal* orbits. Substracting the expansions of $\Delta s$ and $\Delta \overset{*}{s}$ term by term (Taylor series converge absolutely)

$$\begin{aligned}
\Delta s - \Delta \overset{*}{s} &= (s-s_0)-(\overset{*}{s}-s_0) \\
&= (s-\overset{*}{s}) \\
&= \delta s \\
&= \sum_k D_{p_k} s\,(p_k-\overset{*}{p}_k)+\sum_{k_1}\sum_{k_2} \tfrac{1}{2} D_{p_{k_1} p_{k_2}} s\,[(p_{k_1}-p_{k_{10}})(p_{k_2}-p_{k_{20}})- \\
&\quad -(\overset{*}{p}_{k_1}-p_{k_{10}})(\overset{*}{p}_{k_2}-p_{k_{20}})]+..+\sum_{k_1}\sum_{k_2}\sum_{k_h} \frac{1}{h!} D^h_{p_k \ldots p_{k_h}} s \\
&\quad [(p_{k_1}-p_{k_{10}})\ldots(p_{k_h}-p_{k_{h0}})-(\overset{*}{p}_{k_1}-p_{k_{10}})\ldots(\overset{*}{p}_{k_h}-p_{k_{h0}})]+.. \\
&\quad \ldots+\Delta s-\Delta \overset{*}{s} \qquad (3.1.3)
\end{aligned}$$

If the $\overset{*}{p}_k$ are close enough to the true values, then those terms

$$\frac{1}{h!} D_{p_{k_1} \ldots p_{k_h}} s\,[(p_{k_1}-p_{k_0})\ldots(p_{k_h}-p_{k_{h0}})-(\overset{*}{p}_{k_1}-p_{k_{10}})\ldots(\overset{*}{p}_{k_h}-p_{k_{h0}})]$$

including only the known parameters (the $\overset{*}{p}_k$ and $p_{k_0}$ for unknown ones are all zero) must be very small. This is true, not only for linear terms, that have the form

$$D_{p_k} s\,(p_k-\overset{*}{p}_k),$$

but also for nonlinear ones involving higher powers and cross-products of the $(p_k-\overset{*}{p}_{k_0})$. Also, if the models used to calculate $\Delta s$ are sufficiently good, the difference $\Delta s-\Delta\overset{*}{s}$ can be neglected, except for slowly growing secular effects improperly accounted for in $\Delta\overset{*}{s}$. As the satellites are prevented from drifting too far away from their common reference orbit by corrective manoeuvres, the buildup of secular effects is kept from becoming large enough to matter. Therefore, with good values $\overset{*}{p}_k$ and good models to approximate $\Delta s$, the relationship

$$\delta s \approx \sum_k D_{p_k} s \, (p_k-\overset{*}{p}_k) \qquad (3.1.4a)$$

could be a closer approximation to $\delta s$ than

$$\Delta s \approx \sum_k D_{p_k} s(p_k-p_{k_0}) \qquad (3.1.4b)$$

is to $\Delta s$. In other words, the use of good nominal orbits may result in residual perturbations $\delta s = s-\overset{*}{s}$ that are freer from nonlinear and other effects than $\Delta s = s-s_0$. The cancellation of nonlinearities by using nominal orbits is very important, as the common reference orbit along which the problem is linearized does not necessarily provide a good fit to the true ones. This is so because the reference orbit is rigidly defined, to give a special structure to the normal equations of the adjustment, as it will be explained in paragraphs (3.5) and (3.6). If the adjustment is iterated, this rigid orbit is not likely to be changed, as it is by nature impervious to any significant improvement. This creates a situation not unusual in the adjustment of geodetic networks, where the same normal matrix is used iteration after iteration to save computing effort and without real ill-effects.
It is, however, not done in satellite geodesy, where the problem is always linearized along the best fitting orbits that can be obtained using the best a priori values of the parameters and as much tracking data as possible. This improves convergence and, as most potential coefficient adjustments are done without iteration, because of the massive computations involved, strong convergence is needed to secure good results in a single step. Here it is necessary to depart from this practice, as the gains in doing so are likely to outweight the

losses. The main gain is that, in this way, some kind of global solution becomes possible at all! By eliminating many of the unwanted effects, so linear terms with the $D_{p_k}$ s *computed on the reference orbit* can account for most of the residual signal $\delta s$, the use of nominal orbits extends the validity of the linearized model of section 1 to perturbations that are much too large to be described as first order variations along the reference orbit.

According to (3.1.4a-b), one can use the same partial derivatives to model $\delta s$ and $\Delta s$; such derivatives are the brackets containing combinations of $\dot{\beta}_k$, $\dot{\gamma}_k$ that appear in expression (1.4.12). The $\dot{\beta}_k$, $\dot{\gamma}_k$ consist of periodical parts $\dot{\tilde{\beta}}_k$, $\dot{\tilde{\gamma}}_k$, corresponding to the particular integrals of the variationals, quite independent from the starting time $t_0$ of any given arc, and of terms of the form $A \sin n_0 t + B \cos n_0 t + C\, t \sin n_0 t + D\, t \cos n_0 t$ where A, B, C, D depend on $t_0$ (see paragraph (2.5) and Appendix I). These latter terms are due both to the particular integral and to the free response. Similarly, the residual signal $\delta s$ can be separated into a periodical part $\delta\tilde{s}$, and an aperiodical part $\delta s_a$ of the form shown above (the changes in A and B from arc to arc destroy the periodicity of the sine and the cosine). This split extends also to the derivatives $D_{\bar{C}^{\alpha}_{nm}}\, s$, as $D_{\bar{C}^{\alpha}_{nm}}\, s = D_{\bar{C}^{\alpha}_{nm}}\, \tilde{s} + D_{\bar{C}^{\alpha}_{nm}}\, s_a$.

Grouping the periodical effects of the $\Delta\bar{C}^{\alpha}_{nm}$ apart from all aperiodic ones, and having the latter accounted for by a few terms with coefficients like A, B and C that change from arc to arc, and are, therefore, called arc *parameters*, the linearized model adopts the equivalent form

$$\delta s \simeq \sum_{nm\alpha} D_{\bar{C}^{\alpha}_{nm}}\, \tilde{s}\, \Delta\bar{C}^{\alpha}_{nm} + \text{aperiodic terms} + \text{unknown and ignored effects} \qquad (3.1.5)$$

where $\Delta s - \Delta\overset{*}{s}$ = unknown and ignored effects, and the $\Delta\bar{C}^{\alpha}_{nm}$ are now the differences between the true $\bar{C}^{\alpha}_{nm}$ and their *nominal* values $\overset{*}{\bar{C}}{}^{\alpha}_{nm}$.

The estimation of the $\Delta\bar{C}^{\alpha}_{nm}$ is based on residual measurements, or *"residuals"* $s_{(\text{observed})}(t_i) - s_{(\text{computed})}(t_i)$. A complete description of these should include the random part of the measurement error, n, the systematic errors, and also the errors in $s_{(\text{computed})}$ caused by the numerical integration of the orbits. Their complete expression or *observation equation*, is

$$\boxed{\begin{aligned} & s_{(\text{observed})} - s_{(\text{computed})} \\ & = \sum_{nm\alpha} D_{\bar{C}^{\alpha}_{nm}}\, s\ \Delta\bar{C}^{\alpha}_{nm} + \text{aperiodic terms (arc parameters)} + n \\ & + \text{systematics} + \text{numerical integrator errors} + \text{unknown and} \\ & \quad \text{ignored effects (including model errors)} \qquad (3.1.6) \end{aligned}}$$

A detailed form of this equation, needed to carry out numerical calculations, shall be given in paragraph (3.4).

The arc parameters have an indirect relationship with the initial state errors, whose place they take in expression (3.1.6). In fact, they depend on the $\Delta\bar{C}^{\alpha}_{nm}$ as well as on the state errors, representing in lumped form the aperiodic effects due to *all* parameters. The reason for replacing state corrections with arc parameters is twofold: (a) it permits exploiting the periodical nature of $\delta\tilde{s}$ to create a special structure in the normal matrix, as explained later; (b) it eliminates ill-conditioning in this matrix due to the virtual non-estimability of the state errors (when both satellites are close to each other) from satellite-to-satellite tracking data alone. What can be estimated are the differences between those errors, but not the errors themselves. One way of seeing why this is so is to notice that, when nearly on the same orbit, all perturbations due to state errors are virtually at the same frequencies (mainly 0 and $n_0$). From now on, the $\Delta\bar{C}^{\alpha}_{nm}$ are the potential coefficients of the difference between the true and the *nominal field* (defined by the $\overset{*}{C}{}^{\alpha}_{nm}$), which shall be called the *disturbing field*: $\Delta\bar{C}^{\alpha}_{nm} = \bar{C}^{\alpha}_{nm} - \overset{*}{C}{}^{\alpha}_{nm}$.

## 3.2 Least squares adjustment.

If everything besides the linear terms in the signal and the random noise can be ignored in a model of the residual measurements, then the set of all observation equations can be written in matrix-vector notation as follows:

$$A\,\underline{x} + \underline{n} = \underline{d} \qquad (3.2.1)$$

where $\underline{x}$ is the vector of unknowns $x_k$ (here, the corrections $\Delta\bar{C}^{\alpha}_{nm}$ and the arc parameters), $\underline{n}$ the vector of random measurement errors, and $\underline{d}$ the vector of residual measurements $s_{(observed)}(t_i)-s_{(computed)}(t_i)$:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_L \end{bmatrix}, \quad \underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ \cdot \\ n_{N_M} \end{bmatrix}, \quad \underline{d} = \begin{bmatrix} s_{(obs)}(t_1)-s_{(comp)}(t_1) \\ s_{(obs)}(t_2)-s_{(comp)}(t_2) \\ \cdot \\ \cdot \\ \cdot \\ s_{(obs)}(t_{N_M})-s_{(comp)}(t_{N_M}) \end{bmatrix}$$

The dimension of $\underline{x}$ is L, the number of unknown parameter corrections, the dimension of $\underline{n}$ and $\underline{d}$ is $N_M$, the number of measurements, and A is a $N_M \times L$ matrix whose elements are the coefficients of the unknowns in the linearized observation equations. In the case considered here $L < N_M$, so the system of equations is overdetermined, or redundant, assuming that A has full rank. In fact, $N_M$ is more than one order of magnitude larger than L. A is usually known as the *design* matrix, or matrix of *partials*, or matrix of the *observation equations*.

If $\underline{n}$ is a series of samples of an stochastic process of zero mean $E\{n\}$ and known covariance $E\{n_i n_j\}$ (where $E\{\ \}$ is the operator corresponding to averaging over all outcomes of the process) and if W is the inverse of the variance-covariance matrix P of $\underline{n}$:

$$\begin{aligned} W &= P^{-1} \\ &= (E\{\underline{n}\,\underline{n}^T\})^{-1} \end{aligned} \tag{3.2.2}$$

($y^T$ indicates the *transpose* of vector or matrix y), then the linear unbiased estimator of the form

$$\underline{\hat{x}} = F\,\underline{d} \tag{3.2.3}$$

(where F is the $L \times N_M$ estimator's matrix) that minimizes the main diagonal elements of the variance-covariance matrix of the *error vector* $\underline{e}_x = (\underline{\hat{x}} - \underline{x})$ in the estimate $\underline{\hat{x}}$ ,

$$E_x = E\{\underline{e}_x \, \underline{e}_x^T\} \qquad (3.2.4)$$

is

$$\underline{\hat{x}} = (A^T W \, A)^{-1} A^T W \, \underline{d} \qquad (3.2.5)$$

so $F = (A^T W \, A)^{-1} A^T W$. The *vector of estimates* $\underline{\hat{x}}$ has the same dimension L as $\underline{x}$, and is also the solution to another variational problem, the minimization of the quadratic form

$$Q = \underline{v}^T W \, \underline{v} \qquad (3.2.6)$$

where

$$\underline{v} = \underline{d} - A \, \underline{\hat{x}} \qquad (3.2.7)$$

is the vector of *misclosures*. In other words: least squares adjustment minimizes the variance of the estimation errors and also gives the best weighted fit to the data. This type of estimator is *unbiased*, because the mean of the error, $E\{\underline{e}_x\}$, is the zero L-vector. Another way of understanding unbiased estimation (which is equivalent to the statistical definition under the assumptions made here) is that, if the estimator is as in (3.2.5), but the data in $\underline{d}$ are actually free of errors, i.e. $\underline{n} = \underline{0}$, then $\underline{\hat{x}} = \underline{x}$. So *perfect data yield perfect estimates*. This can be verified easily by replacing $\underline{d} = A \, \underline{x}$ (because $\underline{n} = \underline{0}$) in (3.2.5). Moreover, if the noise is a normal, or Gaussian, stochastic process, then $\underline{\hat{x}}$ is not only unbiased, but also of *maximum likelihood*. Of course, all these good qualities of the estimator depend on the linear model $\underline{d} = A \, \underline{x} + \underline{n}$ being true. In general, the existence of non-linear terms, systematics, etc., and a more or less non-gaussian $\underline{n}$, robs the estimator of those qualities, at least in a narrow sense. But the reason why least squares adjustment is so widely used, besides its relative simplicity in theory and in practice, is that it usually is sufficiently "robust" to provide estimates which are almost free of bias, of high, if not maximum, likelihood, etc., even when the statistical assumptions on which the method is founded are not exactly fulfilled by the data. Another way to look at it is to see it as a sensible method, which takes more account, through the weighting with W, of good measurements than of bad ones.

In the particular case where $\underline{n}$ comes from a *random stationary process* ($E\{n_i n_j\} = 0$ if $i \neq j$, $E\{n_i^2\} = \sigma^2$ for all i), W becomes

$$W = \sigma^{-2}\ I \tag{3.2.8}$$

where I is the unit matrix of dimension $N_M$, $\sigma^2$ is the *variance* of the noise, while $\sigma$ is its *standard deviation*. In (3.2.5), the matrix

$$G = (A^T W\ A) \tag{3.2.9}$$

of dimension L, is known as the *normal matrix*. Accordingly, obtaining the estimate $\hat{\underline{x}}$ is equivalent to solving the system of *normal equations*

$$G\ \hat{\underline{x}} = \underline{b} \tag{3.2.10}$$

where $\underline{b}$, the *vector of the right-hand sides of the normals*, is

$$\underline{b} = A^T W\ \underline{d} \tag{3.2.11}$$

In the particular adjustment discussed in this section, the unknowns' vector $\hat{\underline{x}}$ shall be partitioned into $\hat{\underline{c}}$, the vector of corrections $\Delta\hat{\bar{C}}^{\alpha}_{nm}$ to the potential coefficients, and $\hat{\underline{a}}$, a vector of auxiliary variables related to the nominal orbital errors and to the increasing oscillations of frequency $n_0$ due to zonal coefficients' resonances (the "^" denotes "the estimate of"). The corrections to the initial states of the nominal orbits will not appear directly in the observation equations, but through some of these auxiliarly parameters. The reason is that these are not fully observable from satellite-to-satellite tracking data alone, and their direct inclusion would cause A, and thus G, to be virtually rank-deficient. As a result of this separation of the unknowns, the system of observation equations is partitioned as follows:

$$\left[\begin{array}{c|c} A_c & A_a \end{array}\right] \left[\begin{array}{c} \underline{c} \\ \hline \underline{a} \end{array}\right] + \underline{n} = \underline{d} \tag{3.2.12}$$

where $A_c$ is a $N_M \times [(N_{max}+1)^2-6]$ matrix[1], $A_a$ a $N_M \times [L-(N_{max}+1)^2+6]$ matrix, and $\underline{\hat{c}}$ and $\underline{\hat{a}}$ are $(N_{max}+1)^2-6$ and $L-(N_{max}+1)^2+6$ dimensional vectors. The normal equations are partitioned accordingly:

$$\left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \left[\begin{array}{c} \underline{\hat{c}} \\ \hline \underline{\hat{a}} \end{array}\right] = \left[\begin{array}{c} \underline{b}_c \\ \hline \underline{b}_a \end{array}\right] \; ; \quad \begin{array}{l} \underline{b}_c = A_c^T W \, \underline{d} \\ \\ \underline{b}_a = A_a^T W \, \underline{d} \end{array} \; ; \quad \begin{array}{l} G_{11} = A_c^T W A_c \\ G_{22} = A_a^T W A_a \\ G_{12} = G_{21} = A_c^T W A_a \end{array} \qquad (3.2.13)$$

Details of the solution of this partitioned system are given in paragraph (3.8).

## 3.3 Conditioning and least squares collocation.

Given the system of observation equations $A\,\underline{\hat{x}} + \underline{n} = \underline{d}$, the estimator that minimizes

$$\phi = \underline{\hat{x}}^T K \underline{\hat{x}} + \underline{v}^T W \underline{v} \qquad (3.3.1)$$

for a given positive matrix K, is

$$\underline{\hat{x}} = (A^T W A + K)^{-1} A^T W \, \underline{d} \qquad (3.3.2)$$

and the corresponding normal equations and normal matrix are

$$(A^T W A + K)\underline{\hat{x}} = A^T W \, \underline{d} \qquad (3.3.3)$$

$$G = A^T W A + K \qquad (3.3.4)$$

This is a *biased* estimator, as can be verified by replacing $\underline{d} = A\underline{x}$ in (3.3.2); because of the presence of K, $\underline{\hat{x}} \neq \underline{x}$, so perfect data do not result in perfect estimates. The difference is the *bias* of the estimator. Bias estimation is used mostly to improve the condition of the normal matrix, so its inversion is numerically stable, and to reduce the variances $E\{(\underline{\hat{x}}-x)^2\}$ of the individual estimates. If the latter purpose is the main one, there are several choices of K corresponding to the various methods for biased estimation now in use (such as *Bayesian*, quite common in satellite geodesy, *ridge regression*, etc.).

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(1) The coefficients of the zero and of all the 1st harmonics, as well as the 2nd and 3rd zonals, are not adjusted (see par. (3.8)).

In the problem at hand, the unknowns have been partitioned into the corrections to potential coefficients, grouped in $\underline{\hat{c}}$, and to the arc parameters, grouped in $\underline{\hat{a}}$, as indicated in expression (3.2.12), which can also be written

$$A_c\underline{c} + A_a\underline{a} + \underline{n} = \underline{d} \tag{3.3.5}$$

Let $C^{-1}$ be the diagonal matrix of the same dimension as $\underline{c}$, and where the diagonal element $c_{nm\alpha}$ corresponding to the column associated with a given $\Delta C^{\alpha}_{nm}$ is

$$\begin{aligned} c_{nm\alpha} &= \left(\frac{1}{2n+1} \sum_{m\alpha} \Delta\bar{C}^{\alpha\,2}_{nm}\right)^{-1} \\ &= (2n + 1)\sigma_n^{-2} \end{aligned} \tag{3.3.6}$$

Here $\sigma_n^{-2}$ is known as the *degree variance* of the *disturbing field* whose potential coefficients are the corrections $\Delta\bar{C}^{\alpha}_{nm}$ to the *nominal field* introduced in paragraph (3.1); $\frac{\sigma_n^2}{2n+1} = \epsilon^{-1}_{nm\alpha}$ is the *global r.m.s. of the average coefficient of degree n.* If K is chosen

$$K = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right] \tag{3.3.7}$$

then, given that the number of potential coefficients defining the disturbing field is that of the elements of $\underline{c}$, and, therefore, *finite*, and if the parameters in $\underline{a}$ can be regarded as independent from the disturbing field, the linear estimator resulting from this choice of K is that of the technique known as *least squares collocation with parameters* (Moritz, 1972). The proof of this has been given by K.-P. Schwarz (1976), who discusses this question at length in his article in a book edited by Moritz and Sünkel (1978). So, collocation, in this particular application, is equivalent to the conditioning of G by weighting the unknown $\Delta\hat{\bar{C}}^{\alpha}_{nm}$ with the extra quadratic term $\underline{\hat{c}}^T$ C $\underline{\hat{c}}$ in the functional to be minimized by the estimates. As already noted, the elements of $\underline{a}$ are related to some extent to the $\Delta\bar{C}^{\alpha}_{nm}$, and, therefore, conditioning as described here is not equivalent to collocation in a rigorous sense. Including the correlations between all the unknowns would result in a full normal matrix, while leaving these correlations out brings about the very convenient sparse structure to be described in paragraph (3.7). Be as it may, I am happy to let the reader decide what to call this form of conditioning. I shall refer to it as "collocation" for convenience, being just one word, and because I like the sound of it. Those interested in the origins of the idea of collocation may read Kaula (1963, 1976), Moritz (1966), and Krarup (1969).

The choice of C as the conditioning matrix must result in estimates that have the main properties typical of collocation. The idea behind this method is to minimize a certain quadratic measure of the estimation error. Perhaps it is easier to explain this in the case when the estimate is that of a single quantity, for example a single potential coefficient. The general linear estimator for a scalar s out of a vector of measurements $\underline{d}$ is

$$\hat{s} = \underline{f}^T \underline{d} \tag{3.3.8}$$

where $\underline{f}$ is the *estimator vector*. The error in the estimate, assuming that $\underline{d}$ has the form $A\underline{c} + \underline{n} = \underline{d}$, is

$$\begin{aligned} e &= \hat{s} - s \\ &= \underline{f}^T(A\underline{c} + \underline{n})-s \end{aligned} \tag{3.3.9}$$

e can be separated in two parts: the *bias*

$$e_b = \underline{f}^T A\underline{c} - s \tag{3.3.10}$$

and the *propagated data error*

$$e_n = \underline{f}^T \underline{n} \tag{3.3.11}$$

Unbiased least squares minimizes the error measure $E\{e_n^2\}$ under the constraint that $e_b = 0$. Collocation minimizes, without constraints,

$$\sigma^2_{col.} = M\{e_b^2\} + E\{e_n^2\} \quad , \tag{3.3.12}$$

which is called the (hybrid) variance of the estimate $\hat{s}$ (looking back at expressions (3.3.1-7) the connection with (3.3.12) is by no means obvious, nevertheless they are equivalent in this case). The operator M{ } indicates the average of $e_b$, over all possible rotations about the geocenter, of the set of points where the measurements are taken. One could think of this as follows: the field of the first N zonals that define the reference orbit stays fixed in space and the disturbing field of the $\Delta\bar{C}^{\alpha}_{nm}$ can rotate. With the disturbing field in one position, the mission is "flown", data is collected, analized, and the results obtained, with their biases $e_b$. Then the disturbing field rotates to a new position, a new mission is flown, with the same initial states, new data and new results with their biases ensue (as shown by (3.3.10), $e_b$ depends on the field through $\underline{c}$). This is repeated time and again, over all possible rotations, and then the term $M\{e_b^2\}$ is the average of all the corresponding values of $e_b^2$. Collocation tries to minimize this average while keeping the more conventional $E\{e_n^2\}$ also small. The idea may sound strange, but it is not stranger than minimizing $e_n^2$ averaged "over all outcomes of a stochastic process", which is the goal of ordinary least squares estimation. The estimates produced by collocation have some convenient properties. In the first place, the propagated error cannot be larger than for least squares adjustment, and it can be smaller (this follows from C being a positive matrix). Moreover, the hybrid standard deviation $\sigma^2_{col}$ cannot exceed 100% of the rms value of s, otherwise the *optimal* estimate of collocation would be worse than that of an estimator that always assigns to $\hat{s}$ the value zero (i.e., where $\underline{f} = \underline{0}$). I have discussed these questions at some length in (Colombo, 1981a, and 1981b). For general information on collocation, the reader can consult the book by Moritz (1980). Conditioning based on collocation has been used in satellite geodesy to obtain field models such as GEM9 (Lerch et al., 1977).

A good property of collocation is that it produces smooth estimates. In the present case this means that the higher degree coefficients, whose signal is weaker because of its attenuation with height (so ordinary least squares may give rather wild estimates, mostly propagated noise) are going to be assigned small values by collocation. This damping down of the higher spatial frequencies results in a smoothed field. While some fine detail may be lost, problems associated with the truncation of the series at $N_{max}$, such as Gibson's phenomenon, are less likely to occur when mapping the field on the Earth's surface using the estimated $\Delta\bar{C}^{\alpha}_{nm}$, particularly in the neighbourhood of strong gravitational anomalies. Another consequence of this smoothing can be seen in the results of my error analysis of a satellite-to-satellite tracking mission (Colombo, 1981a). The standard deviations for unbiased least squares estimates increases suddenly, exceeding 100% of the rms of the coefficients, at degrees associated with the attenuation bands discussed in paragraph (3.5). The errors for least squares collocation are smaller, change more gradually with degree, and never exceed 100%. As for the loss of fine details, it is better to map these by local methods, rather than by global ones involving potential coefficients, as argued in paragraph (3.12).

To implement collocation one needs to know the degree variances $\sigma_n^2$ in order to set up C. In the case of the Earth, this variances have been estimated from gravity measurements and satellite altimetry, as well as from models of the geopotential obtained from ordinary satellite tracking data. Perhaps the most up-to-date models for the $\sigma_n^2$ as functions of n are those proposed by Rapp (1979). Wagner and Colombo (1979) have used a method for estimating $\sigma_n^2$ from altimetry passes that can be generalized to planetary probes' tracking data (Wagner, 1979) and also to satellite-to-satellite tracking. In this method the $\sigma_n^2$ are derived from the autocovariance function of the stream of data, treated as a stationary process, after correcting for white measurement noise. The $\bar{C}^{\alpha}_{nm}$ with known a priori values $\overset{*}{C}{}^{\alpha}_{nm}$ have $\sigma_n^2 = \sum_{m\alpha} \delta^2_{nm\alpha}$, where $\delta_{nm\alpha}$ is the a priori standard deviation of $\overset{*}{C}{}^{\alpha}_{nm}$ (see Colombo 1981a).

## 3.4 The observation equation.

The signal is the average over a fixed time-interval $\Delta h$ of the residual relative line-of-sight velocity $\delta s$. To arrive to the corresponding equation, I shall first consider the instantaneous value of $\delta s$ according to the approximate linearized model defined by expression (1.4.12), and also to (3.1.4a),

$$\delta s(t) = \sum_k \{(\dot{\gamma}_{k1}(t)-\dot{\gamma}_{k2}(t))\overline{\cos\eta_{10}}+\dot{\beta}_{k1}(t)\sin\eta_{10}(t)-$$

$$-\dot{\beta}_{k2}(t)\sin\eta_{20}(t)\}\delta p_k$$

where now the $\delta p_k$ are the corrections, to be estimated, to the known values $\overset{*}{p}_k$ of the potential coefficients and of the initial state components of the nominal orbits (or, rather, to some linear combinations of the components, as these are not fully estimable from relative velocity measurements). When developing the orbit theory, time was counted from $\tau_0$, the moment when the satellite along the unperturbed reference orbit reached perigee ($F' = 0$) for the first time. Here a small change is convenient: $\tau_0$ shall be now the instant when the unperturbed positions of both satellites do stand, for the first time, at symmetrical points with respect to the perigee. Since the reference positions ran backwards along the same path when the velocities are reversed, and since the reference orbit is symmetrical, in its plane, with respect to the line where $F' = 0,\pi$, it follows that

$$\eta_{10}(t') = -\eta_{20}(-t') \qquad (3.4.1)$$

where $t' = t-\tau_0$. The angles $\eta_{10}(t')$, $\eta_{20}(t')$ and their sines are periodical of period $T_0 = 2\pi\, n_0^{-1}$, the same as the orbit, and being continuous and bounded functions of time, can be expanded in Fourier series which, because of (3.4.1) above, have the form

$$\sin\eta_{10}(t') = \overline{\sin\eta_{10}}+\sum_{k=1}^{\infty} a_k \cos kn_0t'+b_k \sin kn_0t' \qquad (3.4.2a)$$

and

$$\sin\eta_{20}(t') = -\overline{\sin\eta_{10}}+\sum_{k=1}^{\infty} -a_k \cos kn_0t'+b_k \sin kn_0t' \qquad (3.4.2b)$$

where $\overline{\sin \eta_{10}}$ indicates the time-average of $\sin \eta_{10}(t')$ over one orbital period $T_0$. As the numerical studies of section 4 indicate, the sizes of the coefficients $a_k$ and $b_k$ decrease very rapidly with $k$, because $\sin \eta_{10}(t')$ is a very smooth function for a nearly-circular orbit. As a result, only two terms in each formula must be retained to achieve satisfactory accuracy, as shown below,

$$\sin \eta_{10}(t') = \overline{\sin \eta_{10}} + b_1 \sin n_0 t' \qquad (3.4.3)$$

and

$$\sin \eta_{20}(t') = \overline{-\sin \eta_{10}} + b_1 \sin n_0 t' \qquad (3.4.4)$$

As shown in paragraph (2.4), the particular integrals in the solutions of the variationals, when $\delta p_k$ is a correction to a potential coefficient $\overset{*}{\bar{C}}{}^{\alpha}_{nm}$, consist of trigonometric sums for all frequencies $\omega_{mj} = jn_0 + m\theta'$, except 0 and $n_0$, which are replaced by aperiodic, secular effects. Let $\tilde{\beta}_{nm\alpha}(t')$ and $\tilde{\gamma}_{nm\alpha}(t')$ be the periodic parts of $\beta_{nm\alpha}(t')$ and $\gamma_{nm\alpha}(t')$. The leading satellite, satellite 1, is initially ahead of *perigee* by an angle

$$n_0 c \simeq \sin^{-1}(\tfrac{1}{2}\bar{\rho}\ \bar{r}^{-1}) \qquad (3.4.5)$$

where $\bar{\rho}$, $\bar{r}$ are the average intersatellite distance and orbital radius, respectively, while c is the time it takes the satellite to move between perigee and its initial position. Satellite 2 is *behind* perigee by the same angle and time interval, so the Fourier expansions of $\tilde{\beta}$ and $\tilde{\gamma}$, replacing t with $t' = t - \tau_0$, can be obtained by a simple change in the phase angles in (2.4.10-11) to give (assuming $L_0 = 0$ in (2.2.50))

$$\tilde{\beta}_{nm\alpha i}(t') = \sum_{j=-(n+3N)}^{n+3N} b_{nmj} \cos(\omega_{mj}(t' - (-1)^i c) + \phi_{m\alpha}) \qquad (3.4.6a)$$

$$\tilde{\gamma}_{nm\alpha i}(t') = \sum_{j=-(n+3N)}^{n+3N} g_{nmj} \sin(\omega_{mj}(t' - (-1)^i c) + \phi_{m\alpha}) \qquad (3.4.6b)$$

where $i = 1, 2$, and the coefficients $b_{nmj}$ and $g_{nmj}$ are as shown in expressions (2.4.8-9), while $\phi_{m\alpha}$, because of the change from t to t', is as in (2.2.47). Differentiating with respect to time,

$$\dot{\tilde{\beta}}_{nm\alpha i}(t') = -\sum_{j=-(n+3N)}^{n+3N} \omega_{mj}\, b_{nmj} \sin(\omega_{mj}t'+\phi_{m\alpha}-(-1)^{i}\omega_{mj}c) \quad (3.4.7a)$$

and

$$\dot{\tilde{\gamma}}_{nm\alpha i}(t') = \sum_{j=-(n+3N)}^{n+3N} \omega_{mj}\, b_{nmj} \cos(\omega_{mj}t'+\phi_{m\alpha}-(-1)^{i}\omega_{mj}c) \quad (3.4.7b)$$

Replacing $\dot{\tilde{\beta}}_{nm\alpha i}$, $\dot{\tilde{\gamma}}_{nm\alpha i}$ and $\sin \eta_{10}(t)$, $\sin \eta_{20}(t)$ in the linearized model (1.4.12) according to (3.4.3 -4) and (3.4.7a-b), and applying the trigonometric identities for the sine and cosine of the sum and difference of two angles a few times, leads to

$$\boxed{\begin{aligned}\tilde{\delta}s(t') = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \Delta\bar{C}^{\alpha}_{nm} \sum_{j=-(n+3N+1)}^{n+3N+1} &\{2\omega_{mj}[\sin n_0c\, b_{nmj} \\ &\cos \omega_{mj}c - \cos n_0c\, g_{nmj} \sin \omega_{mj}c] + b_1[\omega_{m(j-1)}b_{nm(j-1)} \\ &\sin \omega_{m(j-1)}c - \omega_{m(j+1)}b_{nm(j+1)} \sin \omega_{m(j+1)}c]\} \\ &\times \sin(\omega_{mj}t'+\phi_{m\alpha}) \qquad (3.4.8)\end{aligned}}$$

where $\tilde{\delta}s$ denotes the periodical part of $\delta s$, excluding terms of frequency 0 or $n_0$. The term in brackets multiplying $b_1$, as well as the new sum limits $\pm(n+3N+1)$, result from the modulation of $\dot{\tilde{\beta}}_{nm\alpha i}$, $\dot{\tilde{\gamma}}_{nm\alpha i}$ by $\sin \eta_{10}(t')$, $\sin \eta_{20}(t')$. Changes in the initial conditions of the satellites, as well as zero frequency terms in the forcing functions of the variationals, produce components of frequencies 0 and $n_0$ in $\dot{\beta}_k$ and $\dot{\gamma}_k$ (see expressions (2.4.1a-h)) which, in turn, are reflected by extra terms of the same frequencies in the relative line-of-sight signal. The components of the free response are modulated, like those of the particular integral by $\sin \eta_{10}(t')$ and $\sin \eta_{20}(t')$. Since the initial state errors can be relatively large, producing strong perturbations at $n_0$ (the natural frequency of the system), their modulation by the components of $\sin \eta_{10}(t')$ and $\sin \eta_{20}(t')$ at frequencies higher than $n_0$, neglected in the development so far, cannot be ignored altogether. In particular, they can produce second and third harmonic components in $\delta s$ that are comparable to those in the "periodic" part $\tilde{\delta}s$. To all this is added the contribution, at frequency $n_0$, from the homogeneous parts of the forced responses, as indicated by (2.4.6-7), that are also modulated by sin $\eta_{10}$ and sin $\eta_{20}$, contributing to $2n_0$ and $3n_0$. Also, there are the secular perturbations due to the zonals, contributing to an overall term of the form At cos $n_0t$. Moreover, there is a term of the form Bt sin $n_0t$ caused by perturbations in the initial states affecting $r_i$ or $\dot{u}_i$. This last effect comes from those terms in the expression of the full linearized model, (1.4.11), that were neglected to obtain (1.4.12). Perturbations in r and $\dot{u}$ produce drift in $\gamma$, according to (2.4.1e) and (2.4.1h), and the result is a quantity At, coming from $\pm\gamma_{ik}(t)$ sin $\eta_{i0}(t)$, multiplied by sin $n_0t$, mostly from $|\dot{\underline{r}}_{10}|\cos\delta_{10}-|\dot{\underline{r}}_{20}|\cos\delta_{20}$ (this can be verified, to the order of the eccentricity, by regarding the orbit as slightly elliptical and using the corresponding formulas for $\dot{r}(t+c)-\dot{r}(t-c)$, when the satellites are not very far from each other). Lumping together the various contributions to the free and the secular perturbations, the complete expression for the residual relative line-of-sight *instantaneous* velocity is

$$\delta s(t') = \sum_{k=0}^{3} C_k \cos kn_0t'+S_k \sin kn_0t'+At \cos n_0t+$$

$$+Bt \sin n_0t'+\tilde{\delta}s(t') \qquad (3.4.9)$$

Notice that, when $m = \alpha = 0$, i.e. the case of the zonals, both the zero frequency term and the Bt sin $n_0t$ disappear, unless they are caused by perturbations in the initial state (the contributions of the zonals to the zero frequency are equal in both satellites, and cancel out when their velocities are substracted). Since $\phi_{m0} = 0$ as well, according to (2.2.47), $\delta s(t')$ due to zonals is an odd function of time (see (3.4.8-9)). This agrees with what one would expect from the symmetry of the orbit in a purely zonal field, when the initial states of the satellites are symmetrical respect to a meridian plane, as in the present case. Changing from t' back to t, modifies the values of the coefficients $C_k$, $S_k$, A, B, but not the form of their terms, and turns $\phi_{m\alpha}$ into $\hat{\phi}_{m\alpha}$ according to (2.2.50), so

$$\delta s(t) = \sum_{k=0}^{3} \hat{C}_k \cos kn_0t+\hat{S}_k \sin kn_0t+\hat{A}t \cos n_0t+\hat{B}t \sin n_0t$$

$$\sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \Delta\bar{C}^{\alpha}_{nm} \sum_{j=-(n+3N+1)}^{n+3N+1} \{2\omega_{mj}[\sin n_0c\ b_{nmj}$$

$$\cos \omega_{mj}c-\cos n_0c\ g_{nmj} \sin \omega_{mj}c]+b_1[\omega_{m(j-1)}b_{nm(j-1)}$$

$$\sin \omega_{m(j-1)}c-\omega_{m(j+1)}b_{nm(j+1)}\sin \omega_{m(j+1)}c]\}$$

$$\sin(\omega_{mj}t+\hat{\phi}_{m\alpha}) \qquad (3.4.10)$$

The signal in the actual residual measurements is time-averaged:

$$\overline{\delta s}(t) = \frac{1}{\Delta h} \int_{t-\Delta h}^{t} \delta s(\hat{t})d\hat{t} \qquad (3.4.11)$$

(the "^" distinguishes the integration variable from the integration limit) where $\Delta h$ is of the order of a few seconds, so the averaging operation has very little effect in the *shape* of the secular and "free response" terms in (3.4.10), and they can be assumed to transform into terms of the same form, with negligible error, because of their much longer periods (thousands of seconds):

$$\overline{\delta s}(t) = \sum_{k=0}^{3} \bar{C}_k \cos kn_0t+S_k \sin kn_0t+\bar{A}t \cos n_0t+\bar{B}t \sin n_0t$$

$$+ \frac{1}{\Delta h} \int_{t-\Delta h}^{t} \tilde{\delta}s(\hat{t})d\hat{t} \qquad (3.4.12)$$

To find $\frac{1}{\Delta h} \int_{t-\Delta h}^{t} \tilde{\delta}s\ d\hat{t}$, consider the time-average of a single term of the form $F_{mj} \sin(\omega_{mj}t+\hat{\phi}_{m\alpha})$:

$$\frac{1}{\Delta h} \int_{t-\Delta h}^{t} F_{mj} \sin \omega_{mj}\hat{t}+\hat{\phi}_{m\alpha})d\hat{t} = \frac{1}{\Delta h}\left[\frac{-\cos(\omega_{mj}t+\hat{\phi}_{m\alpha})}{\omega_{mj}}\right]_{t-\Delta h}^{t}$$

$$= d_{mj}(\Delta h)\sin(\omega_{mj}t+\bar{\phi}_{m\alpha}) \qquad (3.4.13)$$

where

$$d_{mj}(\Delta h) = \frac{1}{\Delta h\omega_{mj}} [(\cos \omega_{mj}\Delta h-1)^2+\sin^2 \omega_{mj}\Delta h]^{\frac{1}{2}} \qquad (3.4.14)$$

and

$$\bar{\phi}_{m\alpha} = \hat{\phi}_{m\alpha}+\tan^{-1}\left(\frac{\cos \omega_{mj}\Delta h-1}{\sin \omega_{mj}\Delta h}\right) \tag{3.4.15}$$

Replacing this last result in (3.4.1)

$$\overline{\delta s}(t) = \sum_{k=0}^{3} \bar{C}_k \cos kn_0t+\bar{S}_k \sin kn_0t+\bar{A}t \cos n_0t+\bar{B}t \sin n_0t$$
$$+\sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \Delta\bar{C}^{\alpha}_{nm} \sum_{j=-(n+3N+1)}^{n+3N+1} \bar{s}_{nmj} \sin(\omega_{mj}t+\bar{\phi}_{m\alpha}) \tag{3.4.16}$$

where

$$\boxed{\begin{aligned}\bar{s}_{nmj} = {} & \frac{1}{\Delta h}[(\cos \omega_{mj}\Delta h-1)^2+\sin^2 \omega_{mj}\Delta h]^{\frac{1}{2}}\{2[\sin n_0c\, b_{nmj} \\ & \cos \omega_{mj}c-\cos n_0c\, g_{nmj} \sin \omega_{mj}c]+\omega_{mj}^{-1}b_1[\omega_{m(j-1)} \\ & b_{nm(j-1)}\sin \omega_{m(j-1)}c-\omega_{m(j+1)}b_{nm(j+1)}\sin \omega_{m(j+1)}c]\}\end{aligned}} \tag{3.4.17}$$

which is the complete expression of the time-averaged signal. To write the observation equation from this, the measurement noise n(t) must be added; t itself has to be replaced by $t_i$, the actual instant in which the ith observation takes place; moreover, the coefficients $\bar{C}_k$, $S_k$, $\bar{A}$, $\bar{B}$ must be written using the notation $\bar{C}^w_k$, $S^w_k$, $\bar{A}^w$, $\bar{B}^w$, where the superscript w is a positive integer that indicates in which *orbital arc* the observation is taken. The orbital arcs start just after an orbital manoeuvre to bring the satellites closer to the reference orbit, or at the beginning of a period of time over which a nominal orbit is fitted to the data to obtain the residuals; also after both events, if they coincide (which is desirable, to reduce the number of unknowns in the problem). There are, therefore, nine unknown *arc parameters* per arc, in addition to the potential coefficients. These arc variables will depend on the starting time of their arc, $t^w_0$. Initial state errors in r and $\dot{u}$ will produce a term of the form $a(t-t^w_0)\sin n_0(t-t^w_0) =$ $a[-t^w_0 \sin n_0(t-t^w_0)+t(\sin n_0t^w_0 \cos n_0t+\cos n_0t^w_0 \sin n_0t)]$, so $\bar{A}^w$ will not be determined by the particular integral exclusively, but will depend on the orbit errors at $t_0^W$ as well through a t sin $n_0 t_0^W$ cos $n_0 t$, and the same applies to $\bar{B}^W$. According to all this, the observation equation is (the lower limit for n shall be explained shortly)

$$\boxed{\begin{aligned}&\bar{s}_{(observed)}(t_i)-\bar{s}_{(nominal)}(t_i) = \sum_{k=0}^{3} \bar{C}_k^W \cos kn_0(t_i-t_0^W)+\\ &+S_k^W \sin kn_0(t_i-t_0^W)+A^W(t_i-t_0^W)\cos n_0(t_i-t_0^W)+\\ &+\bar{B}^W(t_i-t_0^W)\sin n_0(t_i-t_0^W)+\sum_{n=2}^{N_{max}} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \Delta\bar{C}_{nm} \sum_{j=-(n+3N+1)}^{n+3N+1}\\ &\bar{s}_{nmj} \sin(\omega_{mj}t_i+\bar{\phi}_{m\alpha})+n_i+ \text{systematics + numeric errors}\\ &+ \text{linear model errors + nonlinear perturbations +}\\ &\text{unknown and neglected effects (tides, etc.)}\end{aligned}}$$

(3.4.18)

where $\bar{\phi}_{m\alpha}$ is given by (3.4.15), $\bar{s}_{nmj}$ by (3.4.17), and $\omega_{mj} = jn_0+m\theta'$.

While the arc parameters convey information on the corrections to the initial states of the nominal orbits of the satellites, they also contain information about the potential coefficients and both cannot be unscrambled. Even if they could, the perturbations in the relative velocity of the satellite contain only information on the in-plane states'errors, and even those cannot be observed fully, because of the differential nature of the measurement; only differences in errors can be estimated properly and, according to (2.4.1a-h), this is not even true for along-track errors, which have no effect in $\dot{\beta}$, $\dot{\gamma}$ and, thus, in $\delta$s. *The only purpose of the arc parameters is to absorbe the influence of errors in the initial conditions of the nominal orbits, not to estimate these errors.* The latter can be done once the potential coefficients have been adjusted, by using the corrected coefficients to calculate better nominal orbits fitted to all available tracking data, including that from tracking stations. Because of the auxiliary role of the arc parameters, the computing effort should be directed to estimating the $\Delta\bar{C}_{nm}^{\alpha}$ in the first place, if possible avoiding the determination of the other unknowns altogether.

The extra terms $\sum_{k=0}^{3} \bar{C}_k^w \cos kn_0t+\bar{S}_k^w \sin kn_0t$ contain practically the same frequencies as those in the periodical part of $\delta s$ corresponding to $\Delta\bar{C}_{n0}^0$ for $0 \leq n \leq 3$ (for near circular orbits and low degrees, the limits $\pm(n+3N+1)$ can be replaced by $\pm n$ without appreciable error), so the respective columns in the matrix A of the observation equations will be almost linearly related, making this matrix virtually rank-defficient. To avoid this problem, the equation can be modified by "dropping" $\Delta\bar{C}_{10}^0$, $\Delta\bar{C}_{20}^0$ and $\Delta\bar{C}_{30}^0$ as unknowns, thus assuming that the second and third zonals are known perfectly. This is a reasonable assumption, because they are, even today, very well known quantities. Improvements to these zonals could be attempted separately, by analyzing the residuals of the adjustment of the other coefficients, for example. The same can be done to extract information about certain phenomena, such as ocean tides, which may have a small but detectable effect on the measurements over extended periods of time, different from the mostly "high" frequency perturbations considered here. In addition, $\bar{C}_{1m}^\alpha = 0$ in a geocentric system, so the corresponding unknowns $\Delta C_{1m}^\alpha$ are also "dropped", which explains the limit $n = 2$ in the sum in (3.4.18).

If an arc is sufficiently long to contain nearly overlapping passes, then the difference of the residual measurements along such passes will consist primarily of the differences between the secularly increasing oscillations in $\delta s$, because two points corresponding to different passes and close to each other (by comparison to the shortest spatial wave-length clearly detectable in the signal) will correspond to epochs $t_1$ and $t_2$ differing by a whole number of revolutions, so $t_1-t_2 = k\ T_0$, where k is an integer, $T_0$ the orbit period, and $\cos n_0t_1 = \cos n_0t_2$, $\sin n_0t_1 = \sin n_0t_2$. Therefore

$$\delta\bar{s}_{12} = (\bar{s}_{(observed)}(t_1)-\bar{s}_{(nominal)}(t_1))-(\bar{s}_{(observed)}(t_2)-\bar{s}_{(nominal)}(t_2)) = \bar{A}^w kT_0 \cos n_0t_1+\bar{B}^w kT_0 \sin n_0t_1 \tag{3.4.19}$$

and $\bar{A}^w$, $\bar{B}^w$ could be estimated by linear regression from successive values of $\delta\bar{s}_{12}$. If this is possible, then $\bar{A}^w$ and $\bar{B}^w$ can be eliminated from the adjustment simply by substracting $\hat{\bar{A}}^w t \cos n_0 t + \hat{\bar{B}}^w t \sin n_0 t$ from the residual observations, where $\hat{\bar{A}}^w$ and $\hat{\bar{B}}^w$ are the estimated values of $\bar{A}^w$ and $\bar{B}^w$.

## 3.5 Attenuation bands.

For the type of gravity mapping mission considered here, $\bar{r}\omega_{mj}\Delta h$ should be small, compared to the shortest spatial wavelength to be resolved, while the angular separation $n_0 c$ between the satellites should be a small fraction of a full revolution, and $b_1$ a small number, compared to $\cos n_0 c$. Under these assumptions, the size and sign of $\bar{s}_{nmj}$ is largely dictated by those of the term $-2 \cos n_0 c\ g_{nmj} \sin \omega_{mj} c$ in (3.4.17). Accordingly, $\bar{s}_{nmj}$ should be very small for $\omega_{mj}$ in the neighbourhood of a frequency at which, for a given value of c, that term is zero. This happens whenever $n_0 c$ is an exact multiple of the wavelength (in radians) characteristic of $\omega_{mj}$, that is to say, when c = k x period of $\omega_{mj}$, or

$$c = \frac{2\pi k}{\omega_{mj}} = \frac{2\pi k}{jn_0 + m\theta'} \text{ , } k = 1, 2, 3, \ldots \tag{3.5.1}$$

Since the frequencies $\omega_{mj}$ depend on the rates $\theta'$ and $n_0$, while c is given and these three quantities are independent of each other, it is not likely that $\bar{s}_{nmj}$ will be exactly zero for any of the frequencies actually present. However, for each $\omega_{mj}$ nearly satisfying (3.5.1), there will be a band of frequencies more or less centered on this value where the spectral components of the signal will be considerably attenuated. In the case where $\bar{\rho} \simeq \bar{r} n_0 c$ = 300 km, for example, the first band is centered at about 134 cycles/revolution, and the others at multiples of this one. For 100 km separation the first band moves to a higher frequency, near 400 cycles/rev. The simulation studies in section 4 show these bands very clearly. As the field is limited to terms of degree $n \leq N_{max}$ = 300, and the highest significant frequency is about 306 cycles/rev., there are two such bands visible in the spectrum of the signal when $\bar{\rho}$ = 300 km, centered at about 134 cycles/rev. and at 270 cycles/rev., and non when $\bar{\rho}$ = 100 km.

The attenuation of the Fourier spectrum of the signal may affect the accuracy of the adjusted coefficients $\bar{C}^{\alpha}_{nm}$ for n close to

$$n = \frac{2\pi k}{n_0 c} \quad , \; k = 1, 2, 3, \ldots \tag{3.5.2}$$

according to several error analysis conducted in the past, including one by myself (Colombo, 1981a), and more recently by C. Wagner (1983, private communication). Clearly, choosing a separation such that the first attenuation band is above the highest significant frequency in the signal must result in a more homogeneous quality of the estimated coefficients, as the center of such a band corresponds to a peak in the inaccuracy of the $\hat{\bar{C}}^{\alpha}_{nm}$ at $n \simeq 2\pi k n_0^{-1} c^{-1}$. To some extent, these fluctuations in the results can be ironed-out by appropriate use of conditioning of the normal matrix, as indicated in (Colombo, 1981a), based, for instance, on least squares collocation; but it is better not to have to deal with these attenuation bands in the first place. On the other hand, the smaller the separation between the satellites, the weaker the signal, as both velocities become increasingly similar, their difference consequently smaller, and the average accuracy of the recovered coefficients deteriorates. To get the best of both worlds, P. Bender, among others, has suggested having both satellites at an average separation of 100 km for part of the mission, and then moving them apart to 300 km, keeping them thus for the remainder. To process such data in the efficient way described in the next paragraphs, two separate adjustments must be done, one using the "100 km data" and the other, the "300 km data" (assuming that each subset constitutes an unbroken stream of data). The two sets of estimated potential coefficients can be combined, afterwards, according to expression (3.8.8).

## 3.6 Linking the orbital period and the length of the data stream to the rotation of the planet.

Here "length of the data stream" means the duration $T_d$ of an *unbroken* stream of data (except for very minor interruptions) sufficient to estimate all potential coefficients of interest. There may be several such intervals within one mission. "Rotation of the planet" must be understood in a node-fixed system, where its apparent rate is $\theta'$. The "orbital period" is $T_0 = 2\pi n_0^{-1}$, the period of the *reference* orbit.

The question to be considered now is: how to choose $T_0$ (or $n_0$), and $T_d$, given $\theta'$, so the normal matrix has many zero elements, arranged in such a way that the solution of the normal equations can be calculated in a reasonable time with existing computers.
Let

$$N_M = \frac{T_d}{\Delta t} \tag{3.6.1}$$

be an *integer* equal to the number of measurements comprising the data stream to be analyzed, where $\Delta t$ (usually equal or larger than the averaging interval $\Delta h$) is the *sampling interval*, the separation between consecutive measurements. If $T_0$ and $T_d$ are chosen so that both $n_0$ and $\theta'$ are exact multiples of the *fundamental frequency of the data stream* $\omega_d = 2\pi T_d^-$ , then all the angular frequencies $jn_0+m$ present in the data will be harmonics of this fundamental (for a polar orbit, this is the same as saying that $T_d$ equals an exact number of *sidereal days*, during which the satellites perform a whole number of revolutions). Let $n_0$ be such that, for different values m and m' of the harmonic order, there are *no* two integers j and j' such that

$$jn_0+m\theta' = j'n_0+m'\theta'$$

while $\omega_{mj}$ is in the range of *significant frequencies* $|\omega_{mj}| \le (N_{max}+3N+1)n_0+N_{max}\theta'$. Finally, assume that the sampling interval $\Delta t$ is such that the *sampling frequency*

$$\omega_s = \frac{2\pi}{\Delta t}$$

is higher than twice the highest significant frequency. Under these suppositions, the following trigonometric formulas are valid (they are the basis of all numerical Fourier analysis methods with equally spaced data):

$$\begin{aligned}&\sum_{i=1}^{N_M} \cos(\omega_{mj}+\Gamma_{mj})\cos(\omega_{m'j'}+\Gamma_{mj})\\ &= \sum_{i=1}^{N_M} \sin(\omega_{mj}+\Gamma_{mj})\sin(\omega_{m'j'}+\Gamma_{mj}) = \begin{cases} 0 & \text{if } m \ne m' \\ \tfrac{1}{2}N_M & \text{if } m = m' \end{cases}\end{aligned} \tag{3.6.2}$$

and

$$\sum_{i=1}^{N_M} \cos(\omega_{mj}+\Gamma_{mj})\sin(\omega_{m'j'}+\Gamma_{mj}) = 0 \quad \text{always} \qquad (3.6.3)$$

regardless of the phase angle $\Gamma_{mj}$. The elements of the *normal matrix* G consist of the scalar products of columns of the matrix of observation equations A, weighted by the matrix W, according to (3.2.9). To simplify the argument, assume that W is the identity matrix times a constant $\sigma^{-2}$ as in the case of uncorrelated, homogeneous observations. The $a_{nm\alpha i}$ elements of the column of A corresponding to $\Delta\bar{C}^{\alpha}_{nm}$ are the coefficients of this unknown in the observation equations, and can be regarded as equispaced samples of a Fourier series of the form

$$a_{nm\alpha i} = \sum_{j=-(n+3N+1)}^{n+3N+1} \bar{s}_{nmj} \sin\left(\omega_{mj}t_i - \frac{\pi}{2}\left(\alpha + \frac{(-1)^m-1}{2}\right)+\Gamma_{mj}\right)$$

according to (3.4.15-16) and (2.2.50), so $\Gamma_{mj}=-\omega_{mj}\tau_0+\tan^{-1}\left(\frac{\cos\omega_{mj}\Delta h-1}{\sin\omega_{mj}\Delta h}\right)+mL_0$ is a phase shift independent of i. The element of $G_{11}$ (the submatrix of G corresponding to the potential coefficients) related to $\Delta\bar{C}^{\alpha}_{nm}$ and $\Delta\bar{C}^{\alpha'}_{n'm'}$ is

$$g^{n'm'\alpha'}_{nm\alpha} = \sigma^{-2} \sum_{i=1}^{N_n} a_{nm\alpha i}\, a_{n'm'\alpha' i'} \qquad (3.6.4)$$

If the assumptions made at the beginning hold, so expressions (3.6.2-3) are applicable, (3.6.4) becomes

$$g^{n'm'\alpha'}_{nm\alpha} = \begin{cases} \frac{\sigma^{-2}}{2} N_m \sum_{j=-(n+H)}^{n+H} \bar{s}_{nmj}\, \bar{s}_{n'm'j'} & \text{if } m = m' \text{ and } \alpha = \alpha' \\ 0 \quad \text{otherwise} \end{cases} \qquad (3.6.5)$$

where H is the smaller of n+3N+1 and n'+3N+1. This means that many elements in $G_{11}$ (which occupies most of G) are zeroes. In fact, *if the* $\Delta\bar{C}^{\alpha}_{nm}$ *are separated in groups by order* m, *and each one of these groups is partitioned in two according to* $\alpha$, *then* $G_{11}$ *becomes block-diagonal.* The largest non-zero block (corresponding to the zonals) has dimension $N_{max}$-3 (the zero to third zonals are not among the unknowns), which is about $N_{max}$ times smaller than the dimension of $G_{11}$. As the size of the blocks decrease with increasing m (dimension = $N_{max}$-m+1), the ratio of the number of elements in the blocks to those outside, which are all zero, is of the order of $N_{max}$, so it may be close to 300. Therefore, $G_{11}$ is mostly zeroes. Its block-diagonal structure greatly facilitates both the setting up of the normals and their solution, as explained in paragraphs (3.7) and (3.8).

The assumption that $j\ n_0 + m\theta' \neq j'\ n_0 + m'\theta'$ for all values of j, j', m, and m' within their respective ranges, as long as $m \neq m'$, can be ensured by choosing the reference orbit so $n_0$ and $T_d$ (the length of the data stream) are such that $N_r$, the total number of revolutions, and $N_d$, the total of *nodal* days in $T_d$, are *mutually prime* integers, or relative primes. This means that they have no common divisor other than the unity. That this is a sufficient condition is demonstrated in (Colombo, paragraph (2.6), 1981a). A *nodal day* is the period $T_n = 2\pi/\theta'$ after which the node of the orbit returns to the same place on the equator, as seen by an Earth-fixed observer. It is slightly different from the sidereal day because of the precession of the orbital plane.

$T_0$ and $T_d$ are commonsurable, so, neglecting secular perturbations, the orbits of both spacecraft "bite their own tails", in Earth-fixed coordinates, and their ground-tracks do the same. If the mission lasts for an additional interval $T_d$, both spacecraft merely repeat the path the have already followed: the mission is periodical, with period $T_d$. The requirement that $N_r$ and $N_d$ must be relative primes ensures that there is no smaller number of nodal days, within the grand period $T_d$, after which the ground-tracks repeat themselves exactly (cycles within cycles), resulting in a coarser coverage of the Earth. With such a coarse coverage, some orders would become coupled inside $G_{11}$, and some unknown $\Delta\bar{C}^{\alpha}_{nm}$ could have columns in A that are linearly related, causing rank-deficiency, which is the equivalent of aliasing in ordinary Fourier analysis. Finally, this choice of $n_0$ and $T_d$ means that $j\ n_0 + m\theta' \neq 0$, other than in the trivial case $j = m = 0$. According to paragraph (2.5), this must exclude perfect resonances for all orders in the permitted interval $0 \leq m \leq N_{max}$, except for the zonals. This is so because, as long as $N_r$ and $N_d$ are relative primes, $j\ n_0 + m\theta'$ can be zero (zonals excluded) only when m is a multiple of $N_r$, so the lowest resonant order is $m = N_r$. By choosing $T_d$ so long that $N_{max}$ = maximum order < $N_r$, there cannot be detectable perfect resonances. Of course, there may be frequencies in the signal quite close to 0 and to $n_0$, causing considerable oscillations. But secular effects, the product of perfect resonance, can only come from the zonals (the reader can check that, if no zero frequency terms are possible for $0 < m \leq N_{max}$, then $n_0$ terms must be absent, as well). Within the grand period $T_d$ there are no overlapping arcs, so the coverage of the Earth and its gravitational field is the finest and most even that can be achieved in this interval.

To choose $n_0$ and $T_d$ one must proceed by successive approximations. After launch, the orbits are trimmed to get them as close to circular and polar as possible, within practical limits set by the fuel available. After this, the inclination of the orbital plane is "given" (changing it, to make the orbit closer to polar is costly in fuel), and any further manoeuvre to adjust $n_0$ must change mostly the mean radius of the orbit. Starting with the specified mean radius (say, 160 km), the number of revolutions in $N_d$ nodal days (virtually the same as the sidereal day for near-polar orbits) is $N_r = T_d/T_0 = N_d(GM\,\bar{r}^{-3})/\theta'$. One way to ensure that $N_r$ and $N_d$ are relative primes is to choose $N_d$ first, which, given $\bar{r}$ and $\theta'_0$, determines $T_d$ and $T_d/T_0$. In general, $T_d/T_0$ will not be an integer. Next, check wether the integer part of $T_d/T_0$ and $N_d$ are relative primes. If not, change $N_d$ by one, plus or minus, and try again. When both numbers are relative primes, call the integer part of $T_d/T_0$ "$N_r$". There will be still a fraction of revolution to go, after $N_r$ turns, in $N_d$ days. This can be corrected by varying the mean radius $\bar{r}$ to adjust $T_0$ until the discrepancy disappears. There are about 16 revolutions per day for low satellites, so if $N_d$ is of the order of months, a change in $T_0$ of a few parts per thousand, requiring a variation in $\bar{r}$ of the same order, or no more than a few kilometers, can bring the number of revolutions in $T_d$ to be exactly $N_r$. Once this adjusted value of $\bar{r}$ is found, a closed periodic orbit with the given inclination and $\bar{r}$ must be found by a procedure like the one described in paragraph (2.1). After this is done, the precise period $T_0$ becomes known. In general, this will not correspond exactly to a whole number of turns in $N_d$ days, so a further trimming of the orbital radius will be needed, followed by another application of the "closing" and "circularizing" procedures, and so on. Fortunately a perfect return of the satellites to the starting point after $N_d$ days is not necessary, and in any case, it is unlikely that the sampling interval $\Delta t$ will divide $T_d$ exactly, so perfect periodicity in the mission is not possible. *What is required is a return to positions that differ very little from the initial ones when compared to the smallest wavelength to be resolved.* In any forseable application, this means that the satellites must go back to points that are no more than a few kilometers away from where they started. The whole discussion actually refers to the positions along the reference orbit. Those along the actual orbits may be within a kilometer each of the former, being brought periodically closer, as already explained, by means of the drag-compensating rockets. The period $T_d$ may be the whole length of the mission, or a sub-interval in which the orbital parameters, distance between space-craft, etc., may be different from those in other subintervals. The important thing is that *the stream of data should be unbroken*, or that any gaps should be short enough to be filled adequately by interpolation. The data of each subinterval $T_d$ must be used in a separate adjustment of the $\bar{C}^{\alpha}_{nm}$, the various solutions then being combined as explained in paragraph (3.8).

*Summarising*: the coefficients should be recovered from a practically unbroken data stream lasting almost exactly $N_d$ nodal days, and the reference orbit should be chosen so that, its mean radius being as close as possible to the one originally specified, it has a period that results in an exact number of revolutions $N_r$ in $N_d$ days. The sampling rate $\frac{1}{\Delta t}$ and $N_r$ must be larger than $2N_{max}$ and $N_{max}$, respectively, to avoid aliasing. To prevent secular effects due to perfect tesseral or sectorial resonances, while ensuring the finest coverage possible in $N_d$ days, $N_r$ and $N_d$ must be relative primes. The choice of a reference orbit and $N_d$ satisfying these conditions can be done by successive approximations, involving repeated small changes of the mean orbital radius.

Note: In addition to the "frozen orbit" where $\dot{\omega} = 0$ (fixed argument of perigee, closed reference orbit) one could choose $\dot{\omega} = k\ 2\pi/T_d$ with $k = 0,1,2,\ldots$ and obtain a rotational-symmetrical data structure leading to a sparse normal matrix (sparser than that in next paragraph, which corresponds to $k = 0$, because the even and odd degrees within the same order become decoupled). The problem is that this requires a more eccentric orbit, unless $T_d$ is very long, and the use of a less accurate orbit theory based on a processing ellipse.

## 3.7 The normal equations arrow structure and their direct setting up.

As shown in the previous paragraph, when the unknown $\Delta\bar{C}^{\alpha}_{nm}$ are grouped, first by order and then according to $\alpha$, the partition $G_{11}$ of the normal matrix is block-diagonal, the size of the blocks decreasing with m according to the formula: block dimension = $(N_{max}-m+1)$, except for orders 0 and 1, where this dimension is less. Moreover, as shown by (3.6.5), the value of the element $g^{n'm'\alpha'}_{nm\alpha}$ of $G_{11}$ does not depend on $\alpha$, so both blocks corresponding to the same m *are identical*, except for the zonals, where there is only the block for $\alpha = 0$. The overall dimension of $G_{11}$ is $(N_{max}+1)^2-6$, or the number of all coefficients through degree $N_{max}$, minus those of degree zero and one, and the second and third zonals, all of which are "dropped" from the adjustment for various reasons given in (3.4). All the blocks in $G_{11}$ are symmetric.

If the "arc parameters" of each arc are grouped together, the submatrix $G_{22}$, corresponding to all those unknowns, is also block-diagonal, the blocks being all of dimension 9; there are as many blocks as arcs, and they are all symmetrical. Naturally enough, the dimension of $G_{22}$ is $9 \times$ number of arcs.

Matrix $G_{12}$ and its transpose, $G_{21}$, both are of dimension $((N_{max}+1)^2-6) \times (9 \times \text{number of arcs})$, and have no especial structure. The elements of $G_{12}$ and $G_{21}$ are the scalar products of columns of the A matrix related to the $\Delta\bar{C}^{\alpha}_{nm}$ with those that correspond to the "arc parameters".

With the unknowns grouped as explained above, the normal matrix G has an overall *arrow* pattern of non-zero elements, the diagonal blocks of $G_{11}$ forming the "shaft", those of $G_{22}$ the "tip" while $G_{12}$ and $G_{21}$ are the "cutting edges", as shown in figure (3.7.1). As explained in paragraph (3.8), solving a system of equations with an arrow matrix is much easier than solving another system with the same number of unknowns and a full matrix.

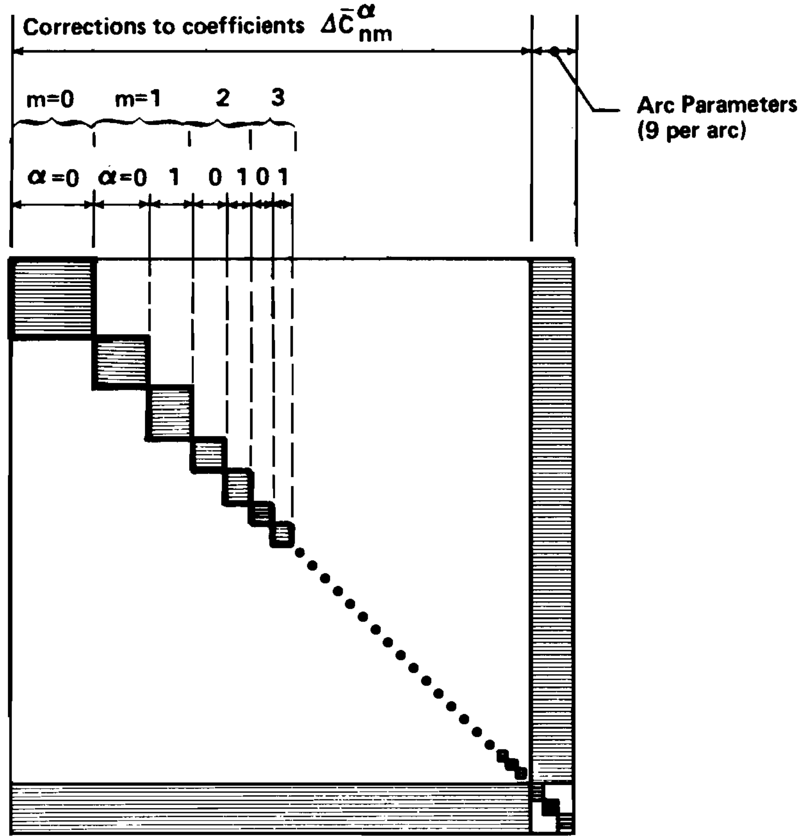


Fig. 3.7.1 The arrow pattern of the normal matrix.

A very important property of G, from the point of view of carrying out the adjustment, is that its elements can be found directly, without having to compute and store first the whole matrix A (which is truly gigantic in the case under study) in order to obtain the scalar products of its columns. Furthermore, calculating the elements of G in this way involves less operations than when forming scalar products, all of which brings about not only a great saving in computing costs, but also greater accuracy, as the arithmetic rounding errors decrease with the number of operations. This direct determination of G shall be treated now in some detail.

Before considering the setting up of the various partitions of G, it is better to generalize the problem somewhat to include *correlated observations*. These may arise, for example, when the average interval $\Delta h$ is longer than the sampling interval $\Delta t$, so there is an overlap of contiguous measurements, and their errors. Assuming that the random part $n_i$ of the error constitutes a stationary stochastic process, so the correlation between a measurement at $t_i$ and another one at $t_{i+k}$ depends only on k, and assuming also that the correlations are symmetrical about $t_i$ in time and become practically zero after some small number of sampling intervals K before and after $t_i$, then the variance-covariance matrix of the noise, or P, has a very strong structure, being symmetrical and banded, with bandwidth 2K+1, and also Toeplitz, i.e. the elements along any of the principal diagonals are all equal, provided that the observations are ordered in the same sequence as they were taken and the data stream is unbroken. The key to simplifying the adjustment is the overall periodicity of the data stream, resulting from the choice of reference orbit explained in paragraph (3.6). This periodicity is disrupted by the existence of correlations between the errors; since the last measurement is not correlated with the first K ones, the data stream does not "bite its tail" anymore, as in the uncorrelated case. This "tail biting" simplifies matters so much that, assuming that K is a very small number compared to that of the measurements $T_M$ (perhaps 1 or 2, versus several millions), it is preferable to improve slightly on reality and suppose that the few first and last measurements *are* , in fact, correlated. If we accept this little lie (which only applies to correlated measurements), then the W matrix becomes *Toeplitz circulant*, i.e., successive rows are cyclical permutations of the first one, in addition to being symmetrical and banded as before. Furthermore, the elements of the first row are equivalent to samples of an *even* function whose period equals the length of the data stream. All this properties are retained by the inverse of P, or weight matrix W. An extremely useful characteristic of W, when it has this structure, is that, because the first row has the form

$$W_i = \sum_{k=0}^{\frac{1}{2}(N_M-1)} A_k \cos k \frac{2\pi}{N_M} (i-1) \qquad (3.7.1)$$

(assuming $N_M$ is even, for simplicity) where $\frac{2\pi}{N_M \Delta t} = \omega_d$ is the fundamental frequency of the data stream, then the eigenvectors of W come in pairs $\underline{w}_c^k$, $\underline{w}_s^k$, with components

$$w^k_{c_i} = A_k \lambda_k^{-1} \cos k \frac{2\pi}{N_M} (i-1) \qquad (3.7.2a)$$

$$w^k_{s_i} = A_k \lambda_k^{-1} \sin k \frac{2\pi}{N_M} (i-1), \quad 1 \le i \le N_M \qquad (3.7.2b)$$

associated to the common eigenvalue

$$\lambda_k = [\frac{(1+\delta_{0k})}{2} N_M]^{\frac{1}{2}} A_k, \text{ where } \delta_k = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ otherwise} \end{cases} \qquad (3.7.2c)$$

If $\underline{v}$ and $\underline{u}$ are of the type

$$v_i = V \sin(\omega_{mj} t_i + \bar{\phi}_{m\alpha}) \quad , \quad \text{where } \omega_{mj} = jn_0 + m\theta' = \frac{k_{(mj)} 2\pi}{N_M \Delta t}$$

(because of the congruences among $n_0$, $\theta'$ and $\frac{2\pi}{N_M \Delta t} = \omega_d$) and, furthermore, $\omega_{mj} \neq 0$, $\omega_{mj} < \pi \Delta t^{-1} = \frac{1}{2}\omega_s$ (or half the sampling frequency), then

$$\underline{v}^T W \underline{u} = VU(\tfrac{1}{2}N_M)^2 A_{k(mj)} \qquad (3.7.3)$$

From all this follows that, when the observations are correlated, and we accept a slight distortion of facts regarding the first and last measurements, then expression (3.6.5) can be generalized quite easily to

$$\boxed{g^{n'm'\alpha'}_{nm\alpha} = \begin{cases} (\tfrac{1}{2}N_M)^2 \sum\limits_{j=-(n+H)}^{n+H} A_{k(mj)} \bar{S}_{nm\ j} \bar{S}_{n'm'\ j'} & \text{if } m=m' \text{ and } \alpha=\alpha' \\ 0 & \text{otherwise,} \end{cases} \qquad (3.7.4)}$$

where $H = \text{Min}(n+3N+1, n'+3N+1)$. If $W = \sigma^{-2} I$, then $A_k = \sigma^{-2} 2/N_M$ for all $k \le \frac{1}{2}(N_M-1)$, and (3.7.4) reverts to (3.6.5).
H can be reduced further; starting with $H = 1$, it can be increased gradually up to $H = 4$ for $\text{Min}(n,n') = 300$. Expression (3.7.4) indicates how to set up $G_{11}$ directly. The number of products and sums is $2(n+H)+2$ and $2(n+H)$, respectively. If the elements of $G_{11}$ were formed in the usual way, as scalar products between the columns of A, the number of operations would equal that of the measurements, which is much larger. Conditioning, as explained in paragraph (3.3), can be introduced by adding suitable positive quantities to the diagonal elements of $G_{11}$.

Finding the elements in the 9 x 9 diagonal blocks of $G_{22}$, corresponding to the arc parameters, is better done by the usual scalar products instead of using Fourier coefficients, as for $G_{11}$. The reason for this is that the elements of a column of A corresponding to an arc parameter are zero everywhere except on the rows of A associated with observations in that arc, so its Fourier coefficients, $C^{wk}_{\omega_{mj}}$, $S^{wk}_{\omega_{mj}}$, are not zero at any frequency, because of the discontinuous nature of the arcs, and their number equals the largest dimension of A.

The elements of $G_{12}$ (and of its transpose, $G_{21}$) are best obtained by multiplying the Fourier coefficients $\bar{s}_{nmj}$ by the corresponding coefficients of the columns associated with arc parameters, $C^{wk}_{\omega_{mj}}$ and $S^{wk}_{\omega_{mj}}$. The reason for using here the Fourier coefficients of those columns, is that the number of operations is determined by the number of coefficients $\bar{s}_{nmj}$, which is always much less that the number of observations, or largest dimension of A. The formula for the elements of $G_{12}$ is, therefore,

$$g^{wk}_{nm\alpha} = (\tfrac{1}{2}N_M)^2 \sum_j A_{k(mj)}\bar{s}_{nmj}(C^{wk}_{\omega_{mj}} \sin \bar{\phi}_{m\alpha} + S^{wk}_{\omega_{mj}} \cos \bar{\phi}_{m\alpha}) \quad (3.7.5)$$

where w identifies the arc, and k = 1,2,...q, the particular arc variable. To finish setting up the normals, one must find the right hand side vector $A^T W \underline{d} = \underline{b}$. The components of $\underline{b}$ corresponding to the $\Delta\bar{C}^{\alpha}_{nm}$ are

$$b_{nm\alpha} = (\tfrac{1}{2}N_M)^2 \sum_j A_k(mj)\bar{s}_{nmj}(C_{\omega_{mj}} \sin \bar{\phi}_{m\alpha} + S_{\omega_{mj}} \cos \bar{\phi}_{m\alpha}) \quad (3.7.6)$$

where $C_{\omega_{mj}}$ and $S_{\omega_{mj}}$ are the Fourier coefficients of the data stream, corresponding to frequency $\omega_{mj} = jn + m\,'$. The $\underline{b}$ componens for the arc parameters, $b_{wk}$, are best found using conventional matrix vector multiplication. To obtain the coefficients $C_{\omega_{mj}}$ and $S_{\omega_{mj}}$ one may use special Fourier Transform algorithms, designed to handle a data stream with millions of samples. The same applies to finding the $C^{wk}_{\omega_{mj}}$ and $S^{wk}_{\omega_{mj}}$. Such procedures move the information from bulk storage devices, such as disk in and out of the random access central memory, when needed, and as efficiently as possible, bulk storage is necessary whenever all the data cannot be contained in central memory at once (see Brigham, Chapter 12, 1974).

## 3.8 Solving the normal equations and finding the formal variances and covariances of the results.

The goal of the adjustment is to obtain corrections $\Delta\bar{C}^{\alpha}_{nm}$ to the potential coefficients, which amounts to estimating the coefficients themselves when these are totally unknown. The arc parameters are auxiliary variables included for the purpose of separating those effects that are periodic over the whole length of the "tail-biting" data stream, and depend only on the $\Delta\bar{C}^{\alpha}_{nm}$, from those who are aperiodic and may depend on other things. Thanks to these extra variables, the normal matrix G has the arrow structure described in the previous paragraph. But, beyond allowing this structure to appear, the arc parameters are of no real interest in themselves and have no clear physical significance.

After grouping the unknowns and partitioning the problems as explained in paragraph (3.2), the normal equations can be solved by Cholesky factorization of the normal matrix. Calling L to the Cholesky factor, so $G = L\,L^T$, where L is lower triangular, then L can be partitioned into $L_{11}$, $L_{21}$, and $L_{22}$ (corresponding to $G_{11}$, $G_{21}$, and $G_{22}$ as defined by expression (3.2.13)), respectively. Moreover, $L_{11}$ is, in fact, the Cholesky factor of $G_{11}$. Since the Cholesky factor of a block-diagonal matrix is also block-diagonal, and the blocks in the factor are the factors of the original blocks, obtaining $L_{11}$ reduces itself to obtaining the factors of the blocks of $G_{11}$, which breaks the task into a number of much smaller and mutually independent operations. Once $L_{11}$ has been found, obtaining $L_{21}$ and $L_{22}$ is also a straightforward and relatively small task, if one uses expressions (3.8.5a-c) to be introduced later in this paragraph) to this end.
With the partitions of L in hand, the solution of the normals can proceed along the following steps:
First, obtain the vector $\underline{s}_1$, by solving the lower triangular system of equations

$$L_{11}\,\underline{s}_1 = \underline{b}_c \tag{3.8.1a}$$

Next, compute $L_{21}\underline{s}_1$ and find $\underline{\hat{a}}$, the estimate of the arc parameters, by solving the equation

$$F\ \underline{\hat{a}} = \underline{b}_a - L_{21}\ \underline{s}_1 \tag{3.8.1b}$$

where $F = L_{22}\ L_{22}^T$ is a real and symmetric matrix already factorized. Finally, find $\underline{\hat{c}}$, the estimate of the potential coefficients, as the solution to the triangular system of equations

$$L_{11}^T\ \ \underline{\hat{c}} = \underline{s}_1 - L_{21}^T\ \underline{\hat{a}} \tag{3.8.2}$$

For the derivation of these formulas, and of the Cholesky method generally, the reader may consult the book on least squares' theory by P. Meissl (section D, par. 3.2). In Meissl's book, $L_{11}$, $L_{21}$, $L_{22}$, $\underline{b}_c$, $\underline{b}_a$, $\underline{\hat{c}}$ and $\underline{\hat{a}}$ are called $R_{11}^T$, $R_{12}^T$, $R_{22}^T$, $b_1$, $b_2$, $x_1$ and $x_2$, respectively. Obtaining $L_{11}$ is further helped by the fact that, in $G_{11}$, the blocks corresponding to unknowns $\Delta\bar{C}_{nm}^{\alpha}$ with $\alpha = 0$ are identical to those for the $\Delta\bar{C}_{nm}^{\alpha}$ with $\alpha = 1$, and so are the respective factors, so it is sufficient to factorize half of the blocks (say, for $\alpha = 0$) to determine the whole $L_{11}$. As the blocks are dealt with independently, the arithmetic rounding errors in the computation of one block do not propagate to the others. Moreover, the largest block is the one for the zonal coefficients, and its dimension is likely to be about $N_{max}$, or nearly 300. The smallest blocks are of dimension 1, corresponding to the coefficients of order N. Therefore, rounding errors and computing time are not going to be a serious problem.

The greatest effort is in obtaining $L_{11}$. From previous experience (Colombo, 1981a) with factorizing a similar block diagonal matrix, where the blocks were close in nature to those under discussion here, but only half in size, I estimate that calculating $L_{11}$ would require about 6 hours of C.P.U. time (most of it employed in setting up the normals by the method explained in paragraph (3.7)) in a computer similar to the AMDHAL 470v/VL-II used by me in 1981 at the Ohio State University. This assumes working with double precision arithmetic (64-bit words), and the availability of some 4 megabytes of central memory. Accordingly, the solution should be feasible even with existing, sequential machines, although it would remain a quite considerable task. This is a time, however, when quite revolutionary changes in computing machinery are taking place, destined to provide better and much faster devices for scientific calculations than the business-oriented computers common until now. In addition to advances such as the introduction of "pipelining", which speeds up matrix-vector operations generally by orders of magnitude, parallel processing could further accelerate the task, as each block in $L_{11}$, for example, can be computed quite independently from the others. All these improvements will save computing time, but the results may not be better than those produced by a conventional machine, as their accuracy depends only on the lengths of the (memory and arithmetic) registers and not on computer architecture. Whether using an unconventional machine is of any real advantage will depend mainly on economic considerations (relative costs) and on the availability of the machine itself. Obviously there is no gain in using a computer that can do the job twenty times faster, if the charges, per second, are thirty times higher. Numerical accuracy should be no problem with most present-day "mainframe" computers, except when calculating the nominal orbits and forming the residuals, where both arithmetic unit and central memory registers of a length similar to those of ordinary CDC machines are needed. By "arithmetic" I mean, in all cases, double precision arithmetic. Four byte word machines could do the job of computing the reference values only be resorting to extended, or quadruple, precision for the floating-point arithmetic, though this is very time-consuming in today's devices. But these can be used for the adjustment itself without any serious problems, as far as I can see.

In addition to solving for the corrections to the potential coefficients, it is of interest to obtain the formal variances and covariances of the errors in the solution. In principle, they can be found by computing the partition $(G^{-1})_{11}$ of $G^{-1}$, which corresponds to $G_{12}$ in G, according to

$$\begin{aligned}(G^{-1})_{11} &= (G_{11}-G_{12}G_{22}^{-1}G_{21})^{-1}\\ &= E_{\underline{c}} \quad \text{(the error matrix for the } \Delta\hat{\bar{C}}^{\alpha}_{nm}) \qquad (3.8.3)\end{aligned}$$

which is applicable not only to ordinary least squares estimation (see Meissl (1982), section A, paragraph (7.4)) but also to least squares collocation with parameters, as shown by K.P. Schwarz (1976, and 1978). However, while inverting $G_{11}$ alone presents no great problems, because it is block-diagonal in either form of adjustment, the extra term $-G_{12}G_{22}^{-1}G_{21}$ in (3.8.3) causes the inverse to loose all distinct structure other than symmetry: it is one huge full matrix. Nevertheless, if the arc parameters, which are the source of this problem, are much fewer than the $\Delta\hat{\bar{C}}^{\alpha}_{nm}$, as it is likely the case, one would expect $(G^{-1})_{11}$ to be not too different from $G_{11}^{-1}$, which is block-diagonal. So it is likely that the elements of $(G^{-1})_{11}$ which correspond, in position, to those in the blocks of $G_{11}^{-1}$ are going to be considerably larger than those outside these blocks. In other words: it is likely that the correlations in the results should be mostly among $\hat{\bar{C}}_{nm}^{\alpha}$ belonging to the same order and having the same $\alpha$. Therefore, the most interesting part of $(G^{-1})_{11}$ would be the diagonal blocks, and finding these relatively small portions of the whole $G^{-1}$ can be done with a computing effort comparable to that needed to find the solution (3.8.1). What is required is the complete Cholesky factor of the normal matrix G, which has to be computed, anyway, to solve for the $\Delta\hat{\bar{C}}_{nm}^{\alpha}$. The Cholesky factor L has the form

$$\left[\begin{array}{c:c} L_{11} & 0 \\ \hdashline L_{21} & L_{22} \end{array}\right]$$

and satisfies the matrix equation

$$\left[\begin{array}{c:c} L_{11} & 0 \\ \hdashline L_{21} & L_{22} \end{array}\right] \times \left[\begin{array}{c:c} L_{11}^{T} & L_{21}^{T} \\ \hdashline 0 & L_{22} \end{array}\right] = \left[\begin{array}{c:c} G_{11} & G_{12} \\ \hdashline G_{21} & G_{22} \end{array}\right] \qquad (3.8.4)$$

The formulas for calculating the various partitions of L, once $L_{11}$ is known, are

$$L_{11}X = G_{12} \qquad (3.8.5a)$$

$$L_{21} = X^{T} \qquad (3.8.5b)$$

$$L_{22} = (G_{22}-L_{21}L_{21}^{T})^{\frac{1}{2}} \qquad (3.8.5c)$$

where X is obtained by "forward substitution", and $(G_{22}-L_{21}L_{21}^{T})^{\frac{1}{2}}$ indicates the Cholesky factorization of $G_{22}-L_{21}L_{21}^{T}$.

Once the Cholesky factor of G is known by doing the extra operations indicated by expression (3.8.5a-c), one can proceed to compute the desired elements of the inverse of G. One column of $G^{-1}$ is a vector $\underline{g}_j$ satisfying the equation

$$G\,\underline{g}_j = \underline{e} \qquad (3.8.6)$$

(G is symmetrical) where $\underline{e}$ is a vector of the same dimension of G and $\underline{g}_j$, in which all elements are zero except for the jth, which is 1. This equation can be solved by applying the Cholesky method, but only the elements of the solution vector $\underline{g}_j$ corresponding to the diagonal blocks in $G_{11}$ are wanted and, because of the symmetry of $G^{-1}$, only those at and below the main diagonal. To save computing effort, the unknown $\Delta\hat{\bar{C}}^{\alpha}_{nm}$ can be "reordered" by thinking of those belonging to the order m under consideration to be at the end of the sub-vector $\underline{\hat{c}}$, so the corresponding diagonal block in $G_{11}$ is the one at the lower-right corner. Because $L_{11}$ is also block-diagonal, the "forward substitution" phase of the solution can start at a point level with the top row of the (now) last block of $L_{11}$, proceeding down from there until the lowest component of the intermediate solution vector is found. In this way, only a relatively small part of this intermediate vector is computed. The "back substitution" phase starts from the bottom of $\underline{g}_j$ and moves up until the last desired element of $\underline{g}_j$, the one on the main diagonal of $G^{-1}$, is obtained. In this way, if $N_{cm\alpha}$ is the number of $\Delta\hat{\bar{C}}^{\alpha}_{nm}$ with a given m and $\alpha$, and $N_a$ is the total number of arc parameters, only the last $N_{cm\alpha} + N_a$ components of the intermediate solution vector and an equal number of elements of $\underline{g}_j$, at the very most, are computed. The resulting incomplete version of $G^{-1}$is, of course, a "poor man's variance-covariance matrix", but it contains what are likely to be the most significant parts of $G^{-1}_{11}$, such as the formal variances of all the $\Delta\hat{\bar{C}}^{\alpha}_{nm}$ and their correlations within the same order, which are the strongest.
The "reordering" of the unknowns needed to speed-up computations (and to decrease numerical errors) is a *purely notional* one. Reordering does not change the values of the elements of the various blocks of L, G and $G^{-1}$ that get moved around, so there is no need to shuffle them inside the computer, but merely to alter the order in which they are used in the calculation after they have been created in the "natural" ordering of increasing m and, for a given m, of increasing $\alpha$. Obtaining a complete $G^{-1}$ is likely to be unfeasible with existing machines, because of the size and lack of structure of this matrix, and to remain so for the foreseeable future.

It could happen that, during a mission, the altitudes of the spacecraft and their separation are kept more or less constant over long periods of time, but are changed from period to period, so as to get better resolution on particular bands of the spectrum from the partial data streams $\underline{d}_h$ (where h indicates the corresponding sub-interval), and for other reasons (such as reducing aliasing between coefficients), that have been suggested by other workers. Each partial data stream will contribute a subset of observation equations

$$A_h\underline{x}_h + \underline{n}_h = \underline{d}_h \tag{3.8.7}$$

which can be partitioned: $A_h = [A_{c_h} \vdots A_{a_h}]$, $\underline{x}_h = [\frac{\hat{c}_h}{\hat{\underline{a}}_h}]$. Every $A_h$ has the basic structure discussed here, provided that the partial stream is long enough to come sufficiently close to "biting its own tail". The unknown $\Delta\hat{\bar{C}}^{\alpha}_{nm}$ may not be the same in all cases, because $N_{max}$ may change with altitude from one segment to the next, and the arc unknowns will be, of course, all different. To maintain the overall arrow structure of G, all unknowns, whether they are considered or not in the actual observation equations of a given sub-interval, should be entered in a formal way, by assigning to some of them null coefficients in the rows of A corresponding to equations where they do not appear, so the overall order is left unchanged. The adjustment can then be done using the formula

$$\hat{\underline{c}} = (\sum_h G_{11_h})^{-1}\sum_h(A^T_{c_h}W_h\underline{d}_h - G_{12_h}G_{22_h}A^T_{a_h}W_h\underline{d}_h) \tag{3.8.8.}$$

where $\sum_h$ indicates a sum over all values of h, and the individual $G_{11_h} = A^T_{c_h}W_hA_{c_h}$ etc., are set up in the manner explained in paragraph (3.7), filling up with zeros those portions that correspond to unknowns absent from the corresponding subset of observations equations.

## 3.9 <u>The adjustment when $N_{max}$ is sufficiently small.</u>

Assume that the satellites are high enough, so only components of the spherical harmonic expansion of the potential below degree $N_{max}$ can be detected from the relative velocity signal, where $N_{max}$ is sufficiently small for the effect of the "ups and downs" and along-track departures from circularity experienced by the reference orbit to be quite small. Then, the asymmetry of the orbits with respect to the equator can be ignored, in a first approximation, and the results obtained in this way can be refined without much extra effort. There is advantage in this because, for a symmetrical orbit such as a circle, the blocks of G are halved in dimension, though doubled in number, as it will be explained shortly. The asymmetry of the periodical reference orbit of section 2 stems from it being slightly elliptical, with its perigee and apogee fixed in their latitudes in the opposite hemispheres, so the two spacecraft pass lower over one hemisphere than over the other. At the same time, they move slower when they are higher, taking longer to cover one hemisphere than the other. This results in more measurements being made on that on which they are higher, a phenomenon that would tend to counteract the loss of sensitivity due to the greater altitude.

The departures from symmetry modulate the orbit and more than double the number of frequencies present in the signal, compared to the case of a symmetrical and circular orbit discussed in paragraph (2.6). This change is very small for perturbations caused by low degree coefficients, and increases slowly with n[1]. It cannot be ignored in the sort of mission discussed in the main body of this work, because the satellites are then too low, so $N_{max}$ is much too large. But for $N_{max}$ small enough, the situation can be very close to that where the reference orbit is circular, so that the perturbations caused by the $\Delta\bar{C}^{\alpha}_{nm}$ with n-m *even* contain only frequencies $jn_0+m\theta'$ where j is even, and those with n-m *odd*, only frequencies where j is odd. This makes the columns of A for $\Delta\hat{C}^{\alpha}_{nm}$ with n-m even, orthogonal to those for $\Delta\hat{C}^{\alpha}_{nm}$ with n-m odd, so the corresponding elements of G are zero. Ordering the $\Delta\hat{C}^{\alpha}_{nm}$ not only according to n and $\alpha$, but also to the parity of n-m, results in diagonal blocks that are half the size of those when the reference orbit is not circular. In general, with a slightly elliptical orbit, those elements of G that would be zero if the orbit were a circle, would be smaller than the rest inside the larger blocks. So $G_{11}$ can be considered as the sum of $\hat{G}_{11}$, where only the smaller "half" blocks are non-zero, and a "perturbation matrix" $\Delta G_{11}$ of the complementary off-diagonal blocks corresponding to coefficients $\Delta\bar{C}^{\alpha}_{n'm'}$ and $\Delta\hat{C}^{\alpha}_{nm}$ with n-m even and n'-m' odd simultaneously. If the elements of $\Delta G_{11}$ are very small (and its larger eigenvalue is << 1), the following series converges quickly:

(1) See formulas (2.2.16) and (2.2.33-34) in par. (2.2).

$$G_{11}^{-1} = \hat{G}_{11}^{-1}+\Delta G_{11} \sum_{i=0}^{\infty} (-\hat{G}_{11}^{-1}\Delta G_{11})^{i} \hat{G}_{11}^{-1} \qquad (3.9.1)$$

(where $F^i$ = F F...F i times). Truncating at $i = I_{max}$,

$$\underline{\hat{c}} = G_{11}^{-1} \underline{b}_c^{(R)} \approx h_0+\Delta G_{11} \sum_{i=0}^{I_{max}} \underline{h}_i \qquad (3.9.2)$$

where $\underline{h}_i$ is the solution of $-\hat{G}_{11}\underline{h}_i = \Delta G_{11}\underline{h}_{i-1}$, with $\underline{h}_0$, in turn, being the solution of the system $\hat{G}_{11}\underline{h}_0 = \underline{b}_c^{(R)}$. This series expansion reduces the solution of the normals to doing $I_{max}$ solutions involving $\hat{G}_{11}$, with its "half" blocks. Because these blocks are half the size, but twice the number, of those in $G_{11}$, and the solution of a system of equations is proportional to the cube of the number of unknowns, the number of arithmetic operations required is $I_{max}/4$ times the number that is needed to solve the normals in the conventional way. If $I_{max} < 4$, there is a real economy in computing.

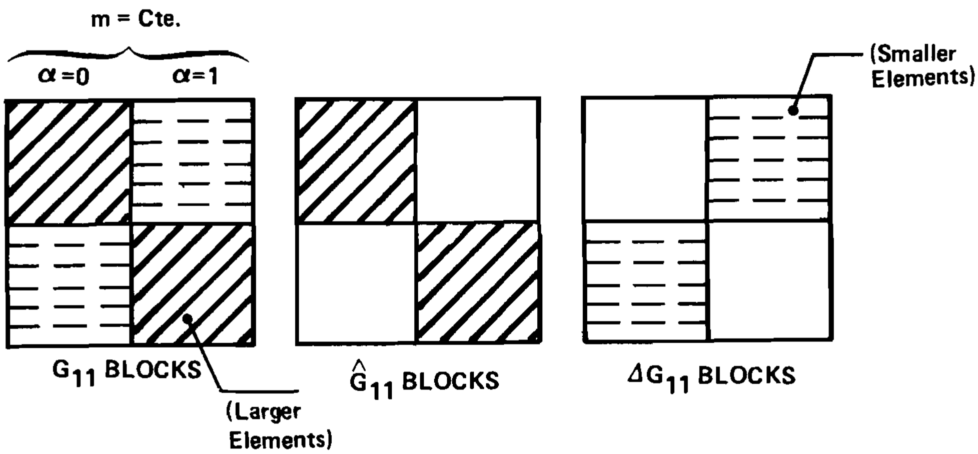


Fig. 3.9.1 The blocks of $G_{11}$, $\hat{G}_{11}$, and $\Delta G_{11}$.

## 3.10 Miscellaneous questions.

Out of many problems associated with the global high resolution mapping of the gravity field by techniques like those described in this section, I shall consider here five questions that have turned up rather often when discussing this topic with colleagues.

(a) Aliasing.

This word means really two rather different things. The first is associated with the impossibility of separating information at two frequencies equidistant from the Nyquist frequency (half the sampling rate) because the sampling process scrambles them. This means that signal coming from a $\Delta\bar{C}^{\alpha}_{nm}$ where $(n+3N)n_0+m\theta' > \frac{1}{2}$ (sampling rate) will have some of its contents mixed-up with those of different order coefficients and, if the scramble is bad enough (too many frequencies above the Nyquist) the coefficient in question will be estimated with great error, and will interfere with the estimation of others. It is also possible that the design matrix may become rank-defficient (and, thus, the normal matrix) if one tries to estimate too many coefficients (see Colombo (1981b), paragraph 1.3)). The thing to do here is to guess as accurately as possible the band of frequencies within which most of the power in the signal is likely to lie, and to select the sampling frequency accoringly. Guessing the bandwith of the signal requires studying its likely power spectrum on the basis of some model of the degree variances $\sigma^2_n$, as proposed, for example, by Wagner and Goad (1982).

The second meaning of "aliasing", at least in satellite geodesy, refers to the bias in the estimates of the $\Delta\bar{C}^{\alpha}_{nm}$ caused by not including enough of them among the unknowns to make a sufficiently realistic model of the signal. This problem arises when there is more information in the signal at high frequencies than it is possible to model, as this would require too much computing effort. Then, the expansion of the disturbing potential is truncated too low, resulting in errors in the results. These errors are likely to affect more the high degree coefficients close to the point of truncation. Let us say that there is significant information in the signal up to degree 400, but it is only practical to model up to degree 350. In reality, one would be happy enough with a global model up to degree 300, because finer detail is more conveniently mapped by local methods, as explained in paragraph (3.12), and a model with more than $10^5$ coefficients (roughly the number up to degree 300) can be awkward to use. One possible solution is to estimate up to degree 350, the maximum practicable, and then to throw away all coefficients above 300. Those are put in the adjustment to absorb the brunt of the aliasing due to the coefficients above 350 that have been ignored, so that the ones up to 300 are not seriously harmed.

(b) Iteration.

The model of the signal given in paragraph (3.4) is not perfect, but only a good approximation, as shown in section 4. Because of its errors, the estimates based on it will be biased. Using the adjusted potential coefficients and all the tracking data, one can obtain better nominal orbits, form new residual measurements, and proceed to estimate the errors in the results of the previous iteration by a new adjustment of the coefficients, repeating the process as many times as feasible, to refine the solution as far as possible. If the initial results were good enough, this procedure should converge to better estimates. Since the normal matrix, as explained in paragraph (3.1), is not iterated with the solution, but the same matrix is used again and again, to save computing at the cost of a minor loss in accuracy, there is a problem if one uses conditioning based on collocation. The conditioning matrix C must have for diagonal terms the inverses of the variances $\sum_{m\alpha} (\Delta\bar{C}^{\alpha}_{nm})^2(2n+1)^{-1}$, but these change from one iteration to the next, as the corrections $\Delta\bar{C}^{\alpha}_{nm}$ become smaller. The variances at the ith iteration can be guessed as equal to the *formal* variances of the errors in the previous iteration. The difference between the old and the new C can be regarded as a perturbation matrix, and the solution dealt with by a refining algorithm like the one proposed in paragraph (3.9), at little extra computing cost. Ordinary least squares does not present, of course, this problem.

(c) Downward continuation and convergence.

The data are gathered by the two satellites as they run their courses nearly two hundred kilometers above the Earth, but it is on the surface of it that the resulting gravity field model has most of its potential applications. To obtain a map of gravity anomalies, geoid undulations, or similar quantities of interest, the potential coefficients obtained from the analysis of the satellites' data can be modified simply by multiplying them by factors such as a, (n+1), etc., and then the values according to these coefficients can be computed by replacing the general coordinates r, $\varphi$, $\lambda$ in the corresponding spherical harmonic expansion with those of the points of interest on the Earth's surface. One problem with this straightforward operation is that the expansion is truncated at $N_{max}$ and, even if one were to assume that it does converge in the limit to the desired quantity, there is no guarantee that it will do so quickly enough for its truncated form to give an accurate result. Spherical harmonic series are a special kind of Fourier series, defined on a sphere, so, like their counterparts on the real line, they can converge slowly and in an oscillatory fashion near those points where the function they approximate is discontinuous. This is known as Gibson's phenomenon, whose existence would reveal itself near strong and abrupt anomalies as groups of ripples surrounding the anomalies in the map, and corresponding to no real feature. This distortion can be reduced by damping or smoothing the high degree terms, i.e. multiplying them by factors less than unity chosen in some appropriate way (it may be enough to use collocation to condition the normals). The ultimate question of whether the series as such converges or not down to the surface is one that has puzzled and worried many geodesists over the years, while it has been generally accepted by them that this is not likely to be a very serious problem in practice. In a recent study by Jekeli (1982), the effect of the main topographic features of the world was examined by computer simulation, to see if they would affect the representation of the geopotential by a series truncated at a high degree, like n = 300. No evidence of real difficulties was found, so one may expect a reasonable behaviour of the truncated series, which means that the map would be reliable over most of the world. The theoretical problem of convergence was settled many years ago by Walsh (1929), who extended a theorem by Runge on the series expansions of analytical functions of complex variables, to the case of spherical harmonic expansions of harmonic functions like the gravitational potential, though this was not known to geodesists until recently. His theorem shows that, if there is a distribution of mass inside the Earth for which the series does not converge at the surface, there must be also another that differs from the first by as little as desired, and whose expansion converges uniformly down to the surface. Moreover, the corresponding $\bar{C}^{\alpha}_{nm}$ in both expansions can differ from each other also by as little as desired, so their differences can be always less than the uncertainty in their estimated values due to observational errors. Conversely, if the series converges, it is always possible to find another expansion whose potential coefficients differ from those of the convergent series by insignificant amounts, corresponding to a mass distribution that differs from the true one by as little as desired, and that does not converge at the surface. This situation has been compared by Moritz (see Chapter 7 of (Moritz, 1980)) to the question of whether a distance measurement corresponds to a "true" distance that is a rational or an irrational number. Not surprisingly, he concludes that both problems are just as physically meaningless. I also have discussed this problem in detail in (Colombo, 1982).

(d) Numerical integration of the nominal orbits.

A gravity anomaly of 1 mgal covering a $1^{o} \times 1^{o}$ area on the ground will be sensed as a change in the relative velocity of the spacecraft of only a few micrometers per second, so, in order to detect features as small as 1 mgal, the measurement noise should not be much more than one micrometer per second. This means that a great deal of trouble and expense are being devoted at present to the design of an unprecendently accurate tracking radar that each spacecraft will carry to sense the movements of the other. As explained in paragraph (3.1), the measurements themselves (after screening for blunders) must be converted into residual observations by substracting from them corresponding values of the relative velocity computed from existing models of the gravity field, the attraction of the Sun and the Moon, etc. This must be done to eliminate effects not included in the theory, and nonlinearities due to inevitably large departures from the periodical reference orbit along which the problem must be linearized to ensure a feasible adjustment. These computed values are obtained using the position and velocity of each satellite along its nominal orbit, and this orbit is calculated by numerical integration. The errors in the numerically integrated velocities will show up in the residual measurements, so one must be careful that the accuracy of the original data is not spoiled by the introduction of a lot of integrator noise when forming the residuals. My experience with the numerical integrator that I have used for getting the results of section 4 suggests to me that, with a computer whose double precision words are more than 100 bits long, like a CDC machine, for example, no serious numerical problems should crop up. Of course, one must also use a very good numerical integrator, but probably nothing beyond the present state of the art.

During some procedures, like the iteration of the solution, for example, it becomes necessary to compute nominal orbits using very high degree spherical harmonic expansions for the gravitational accelerations. This cannot be done by the usual method of adding up the terms of those series at the point of computation, as that would require a tremendously long time. An alternative that has been contemplated for many years, but seldom, if ever, used, because no real need for it has existed so far, is to compute the accelerations on a regular grid and then to interpolate them to the places in the orbits where they are needed. Calculating spherical harmonic expansions on regular grids (constant increments in latitude and in longitude) covering the whole world can be done most efficiently and in a few minutes using today's machines, even when $N_{max}$ is as high as 300. For determining orbits, one needs a three dimensional grid, made up of a succession of contiguous and concentric spherical shells, each divided in a regular array of parallelepipeds the bases of which form a regular two-dimensional grid on the bottom spherical surface of the shell. At the vertices of each cell, the three accelerations $a_r$, $a_\varphi$, $a_\lambda$ are computed using an efficient procedure like those described in (Colombo, 1981b). The interpolation from the vertices to a point inside can be done by some simple and fast scheme, and more points from neighboring cells could be involved. The accuracy of the procedure would depend on the size of the cells, and on the number of vertices used. Linear interpolation of any kind consists in multiplying the values at the nodes by certain weights and then adding them up together. The weights will be different for cells with different mean latitudes and heights, but the same for those in the same shell and latitude, as the grid must have rotational symmetry about the Earth's axis. This means that only the weights of cells in one wedge, cutting across the whole system of shells and running from pole to pole, are needed, because they repeat themselves in the cells outside. The 3-dimensional set of all the values of the three accelerations is bound to be quite huge, as small and, therefore, numerous cells are needed. But it can be stored in mass storage devices, such as magnetic disks, arranged in files of a simple sequencial organization. In each, the shells will be represented by consecutive sub-files, and in those, the rows of cells of equal latitude by smaller consecutive sub-files in which the cells are stored sequentially by longitude. Knowing the approximate initial conditions, the orbit can be integrated with sufficient accuracy to predict through which cells the satellites will pass within a certain period, using only a few terms of the spherical harmonic expansion (central force and oblatness) to compute rough estimates of the orbits. Once the cells needed are identified in this way, the values of the accelerations at their vertices, and the corresponding weights, which may be in a separate file, can be picked up by scanning sequentially through the data set on disk. With a suitable arrangement of the files, they have to be searched only once in this manner.

The crucial point is the interpolation scheme and the size of the cells, as both determine the accuracy of the interpolated values. This accuracy should be high enough to guarantee that the interpolation errors shall not corrupt the residual measurements. The accelerations, in their spherical harmonics form, are functions of r, $\varphi$, and $\lambda$, and can be interpolated as such by a carefully chosen finite differences scheme. During my stay in Delft, E.O. Schrama has tested one such scheme and found that the orbits computed using the interpolated accelerations were virtually as accurate as those calculated without interpolation. He used a field of zonals, complete to degree 300, a ten seconds'integration step, and the same integration subroutine that I have used for the work reported in next section. Mr. Schrama is finishing his surveying degree at the Department of Geodesy, and did his work under the supervision of Professor Rummel. His results give me confidence that simple and very efficient interpolation schemes can be found to compute orbits using very high resolution gravity field models. These schemes can be used, of course, with low degree fields as well, and doing so can result in a great deal of time and money being saved in the routine operation of computing orbits that goes on nowdays in many places round the world.

(e) Two different orbital planes.

The ideas developed here for the case of two satellites following much the same course, so they share a common reference orbit,can be generalized to a situation in which the orbital planes are so different that two separate reference orbits are needed, one for each spacecraft. In this case, the rotational symmetry of the data distribution that underpins the efficient adjustment of the potential coefficients can be obtained by making each reference orbit identical, except for a rotation of the orbital plane, and closed and periodic as described in section 2. The linearized model will contain a larger number of terms, as the out-of-plane perturbations must be included, so the computations must be somewhat more laborious than with coplanar orbits. A possible advantage would be that, as the data is sensitive to East-West forces as well as to North-South ones, the tesserals and sectorials may be estimated better than with the two satellites moving in the same meridional plane (see also the comments at the end of paragraph (1.4)).

3.11 The data as "point" measurements.

The discussion in this report is centered on the estimation of the $\Delta\bar{C}^{\alpha}_{nm}$ from a virtually uninterrupted stream of data, long enough for it to "bite its tail". This is what one hopes to obtain out of a gravity field mapping mission, this is what the mission is supposed to give. But will it? If not, what? A break in the data of a few minutes is not fatal, as one could fill the gap with interpolated values from the measurements taken near both ends of it. Several such gaps would deteriorate the results. One or more larger gaps, where a substantial part of a revolution is missing, could put "paid" to the whole idea. A break due to a malfunction that lasted more than one revolution would do that for sure. At one end of the range of possibilities, not a single observation is missed, the stream is long enough to "bite itself", and the ideas proposed so far can be applied. At the other end, there are so many breaks that only over some regions the coverage is adequate, and one must resort to making local maps of those, giving up on global methods. It is the middle situation that is a problem: a global coverage, but too many breaks. In this case one would like to do local solutions over those areas with the denser coverage, and still get some kind of global model using all the data available, even if some corners must be cut to realize this, so the results are likely to be less than the absolute best. It is a question of doing something at all. In this situation, one could regard the measurements, not as samples in a time series, but as "point" measurement taken with an instrument sensitive to some aspects of the gravity field, which exists in space.

Looking at expression (3.4.16) for the residual signal, this could be written, ignoring the aperiodic terms, as

$$\delta\bar{s} \simeq \sum_{nm\alpha} \Delta\bar{C}^{\alpha}_{nm} \sum_{j} \bar{s}_{nmj} \sin(jn_0t+m\theta'+\bar{\phi}_{m\alpha})$$

$$= \sum \Delta\bar{C}^{\alpha}_{nm} \sum_{j} \bar{s}_{nmj}[\sin(jn_0t+\tilde{\phi}_{m\alpha})\cos(m(\theta't+L_0))+\cos(jn_0t+\tilde{\phi}_{m\alpha})$$

$$\sin(m(\theta't+L_0))]$$

with $\tilde{\phi}_{m\alpha} = \bar{\phi}_{m\alpha}-mL_0$, $L_0$ being the longitude of the ascending node at $\tau_0$, the first time the reference position of the middle point between satellites reaches perigee. But $m(\theta't+L_0) = m\lambda$, so

$$\delta\bar{s} \simeq \sum_{nm\alpha} \Delta\bar{C}^{\alpha}_{nm} \sum_{j} \bar{s}_{nmj}[\sin(jn_0t+\tilde{\phi}_{m\alpha})\cos m\lambda+\cos(jn_0t+\tilde{\phi}_{m\alpha})\sin m\lambda]$$

The values of $\cos(jn_0t+\tilde{\phi}_{m\alpha})$ and $\sin(jn_0t+\tilde{\phi}_{m\alpha})$ repeat themselves for all j only at those times $t_1$, $t_2$, ..., $t_i$ when the corresponding angles $n_0t_i$ are *congruous* with each other, i.e., differ by whole multiples of $2\pi$. Because the reference orbit is periodical, whenever this congruence occurs, both r and F' determine the same point in the orbital plane, for an observer fixed to this plane. Therefore, r and $\varphi$ in Earth-fixed coordinates are also the same. The congruity of $n_0t$ means that the satellite is, at such times, always in an *ascending pass* (the half of a revolution where the satellites crose the equator going North) or in a *descending pass* (the other half). So, if measurements taken along ascending passes are considered separate from the others, the periodical part of the signal in them is a function of position; the same is true, of course, of measurements taken along descending passes.

As the Earth rotates and the orbital plane precesses, the common reference orbit (here I am assuming that the two satellites have been kept close enough to it, in spite of malfunctions) sweeps a surface of revolution round the spin axis, in Earth-fixed space. As the measurements are taken no more than a few hundred meters away from this surface, and the spherical harmonics reflected in them change very slightly over such distances, it is possible to assume that they have been taken actually *on the surface*, and to assign to them the coordinates r, $\varphi$, $\lambda$ of the projection on it of the middle point between satellites. It is possible to subdivide the surface with a grid of lines of constant latitude and longitude, and assign to each block the value of the average of all measurements taken when the satellites were on ascending passes over them. Another set of mean values can be formed using the descending passes. These two sets of grided data can be processed quite efficiently, because they are of the form

$$\underset{(\text{ascending})}{\delta s(i,j)} = \sum_{nm\alpha} \Delta\bar{C}^{\alpha}_{nm} \int_{\varphi_i}^{\varphi_{i+1}} \underset{(\text{ascending})}{(\text{Function of } \varphi)\, d\varphi} \int_{\lambda_j}^{\lambda_{j+1}} \left\{ {\sin \atop \cos} \right\} m\lambda \qquad (3.11.1a)$$

and

$$\underset{(\text{descending})}{\delta s(i,j)} = \sum_{nm\alpha} \Delta\bar{C}^{\alpha}_{nm} \int_{\varphi_i}^{\varphi_{i+1}} \underset{(\text{descending})}{(\text{Function of } \varphi)\, d\varphi} \int_{\lambda_j}^{\lambda_{j+1}} \left\{ {\sin \atop \cos} \right\} m\lambda \qquad (3.11.1b)$$

where $\delta s(i,j)$ indicates an area mean, i,j identify a particular block in the grid, and the grid has rotational symmetry. Algorithms for least squares adjustment and least squares collocation exploiting this symmetry are discussed in detail in (Colombo, 1981b). One problem is that the accuracy of the mean values changes with the number of measurements per block. As indicated in the work just mentioned, if all blocks at the same latitude are assigned a fictitious accuracy equal to the mean accuracy for all such blocks, and an efficient estimation algorithm is set up on this basis, the result, while not absolutely optimum, is the best estimate that such an efficient procedure can provide. This is really a minor problem, because at this stage one is trying to do what one can with whatever one can get. Another problem is the elimination of the non-periodic part of the signal. The secularly growing oscillations could be removed by analyzing overlapping passes, as explained in paragraph (3.4). The free response terms at 0, $n_0$, $2n_0$ and $3n_0$, which are discontinuous from arc to arc, cannot be estimated in this way, or in any other way I can thing of, from overlapping or from intersecting passes. The best thing to do, probably, is to use as much information as possible about the gravity field when computing the nominal orbits, so the initial state errors, and the related oscillations, are kept small. Fortunately, for the Earth there is already a great deal of information, reflected in gravity field models like GEM 10C (Lerch et al., 1981) and similar high resolution maps. These have come about since the late seventies thanks to the availability of a great deal of satellite altimetry data from the GEOS-3 and SEASAT missions, combined with land gravimetry accumulated, through international exchanges, in large data banks. After having obtained a first estimate of the $\Delta\bar{C}^{\alpha}_{nm}$ by processing the satellite-to-satellite tracking data in the way outlined here, it is possible to use this result to recompute the nominal orbits, form new residual measurements, and iterate the solution, hoping to improve in this way on the previous one.

The use of a surface of revolution generated by a periodic orbit has been proposed recently by Rummel and Colombo (1983) for the processing of measurements of a gravity gradiometer carried by a satellite which are "point" measurements, as an alternative technique for mapping the gravity field.

## 3.12 Local solutions.

As an alternative to *global mapping* of the geopotential, which involves estimating the spherical harmonic potential coefficients $\bar{C}^{\alpha}_{nm}$ or equivalent parameters, there is *local mapping* where data collected over a limited region are used to estimate the parameters of a two-dimensional function defined on this region (bicubic splines, 2-D Fourier series, area means, etc.). This function usually represents boundary values on the Earth's surface, such as geoid undulations or gravity anomies, that are susceptible of upward-continuation if they are known over a sufficiently large area. Gravity field maps are commonly called "solutions", presumably because the main job in making them is to solve a rather large system of normal equations in order to fit some function to the data. Be as it may, the usefulness of local solutions is very considerable. If there are not enough data to go around, they are the only possibility and global mapping is out. If the distribution of data is too irregular to use elegant methods based on uninterrupted data streams and "tail biting", they are the only choice, as a global adjustment, though possible in theory, requires too much computing in practice, so it is not feasible. Instead, one makes many local solutions until they

cover the whole world, and then "stitches them up" in some way or another, like a crazy-quilt. But even when a global solution is possible both in theory *and* in practice, local solutions still have a very important, if *complementary*, role to play. There are regions in our planet (and, presumably, in others which some day may be studied by these methods) where the gravitational field presents strong and sharp anomalies; such regions stand out in a map of the ocean surface obtained from satellite altimetry, and are intimately related to cracks and folds in the crust. Where these cracks and folds occur, a number of geophysical processes are at work shaping the crust and upper mantle, so the study of these regions is of great interest. Typically, along a major fault or mountain system there are strong and narrow anomalous features in the field, perhaps a hundred kilometers across and a thousand or more in length. The details of these anomalies, even if they were detectable with a satellite pair, would be near the limit of resolution of a spherical harmonic model with a maximum degree of 300 or thereabouts, at least in one of their dimensions. To be able to see them clearly one might need a maximum degree of 400 or higher, with close to 200000 coefficients. On the other hand, most of the world is actually very "flat" and uneventful in terms of gravity anomalies. So many of the coefficients would be dedicated to modelling what happens over a small portion of our planet, and on the rest of it they would be just fitting the noise in the data, assuming that one uses ordinary least squares adjustment. When using collocation, or a similar kind of conditioning, the map will probably not reflect the finest detail, regardless of how many coefficients are estimated, because such methods tend to give a smoothed picture of the field.
A local solution, on the other hand, can provide as detailed a picture of an anomalous region as the data would allow, by using all the parameters to describe just that region and nothing else. But it would be needlessly time-consuming to map the whole world in little bits, most of which would cover rather featureless areas. So global and local solutions complement each other: the ones providing the main trends and broader outlines, the others focusing on interesting details while, perhaps, combining satellite-to-satellite tracking data with other measurements and with knowledge of geological structures, something much easier to do in local maps.

One problem with local solutions is that they tend to be numerically unstable. To some extent, obtaining a global solution first, and using it as a reference field to create residual data for the local ones afterwards, could reduce the numerical difficulties by removing the trends and leaving only fast-changing anomalies to be mapped. Trends de-estabilize, as anybody who has tried fitting low-degree polynomials to closely packed data knows. Removing trends is, of course, only part of the answer to instability, and the important question of how to obtain good local solutions remains an open one at this time.

Summary.

The data to be analyzed consist of the measured values of the line-of-sight velocity minus the values calculated from orbits integrated numerically on the basis of existing force models; these differences are the residual measurements. The integrated orbits, fitted to all tracking data available, are known as nominal orbits. The residual measurements are formed to eliminate disregarded effects, such as the attraction of the Sun and Moon, tides, polar motion, relativistic effects, etc., as well as nonlinearities which could be otherwise too large to ignore, because of the rather large departures of the actual courses from the reference orbit along which the problem is linearized. The formation of these residuals, and the use of the spacecraft thrusters to manoeuvre them back to the reference orbit, cancelling out undesired long-period perturbations, tend to ensure the validity of the use of a linear model of the actual signal, including only the effects of gravitational anomalies and initial state errors. The observation equation based on this model has, the same as the perturbations discussed in section 2, the form of a Fourier series, slightly modulated by the lack of circularity of the reference orbit. By choosing this orbit such that its period is a whole multiple of the duration of the data stream, which is a whole number $N_d$ of days long and also an exact number $N_r$ of revolutions, $N_d$ and $N_r$ being relative primes, the normal equations of the adjustment have an *arrow* structure. This structure appears when the $\Delta\hat{\bar{C}}^{\alpha}_{nm}$ are grouped together by order, and within each order by $\alpha$, and the other unknowns, the arc parameters, are grouped separately, by arc. The "shaft" and "tip" of the "arrow" are strings of diagonal blocks, the "fins" on the bottom and on the right edge of the normal matrix are full rectangles, and the rest is all zeroes. The arc parameters replace the original initial state errors, which are not fully estimable from satellite-to-satellite tracking data, and absorb all non-periodical effects associated with discontinuities between the arcs, and with zonal resonances peculiar to the choice of a periodical reference orbit. By adding a diagonal matrix of degree variances to the normal matrix, least squares adjustment becomes least squares collocation with parameters, at least if one ignores the slight correlations between these parameters and the $\Delta\bar{C}^{\alpha}_{nm}$. Collocation tends to produce a more smooth map of the field, reflected in smaller, attenuated values for the higher degree coefficients, compared to least squares adjustment. It also helps to improve the numerical stability of the solution, which, for some separations of the spacecraft, can be affected adversely by the appearance of attenuation bands in the spectrum of the signal stemming from the differential nature of the measurements. The number of coefficients to be solved for can be very large, but there are practical limits to its size. For many applications, going beyond $N_{max}$ = 300 may not be altogether necessary, and carrying so many coefficients around would cause unnecessary problems to the users. As most high frequency anomalies of interest are concentrated in a few regions, such as the vecinities of trenches, ridges and cordilleras, any information about those features can be extracted more easily by local solutions, which can achieve high resolution with a moderate number of parameters. The idea of making local maps, or solutions, complements, rather than competes, with that of making global ones. The arrow structure of the normals helps enormously to alleviate the computing burden, but it depends on the availability of a virtually uninterrupted stream of data during the mission. If things go badly enough, local solutions, which pose no such requirements, are the only way of obtaining results. If there are major breaks in the data stream, but still there is enough data coverage to allow a global solution, it may be better not to treat the data as a time-series, but to regard the residual measurements as point measurements. This ignores their aperiodic part, which might be made quite small if as much information as it is available nowadays on the anomalous field (after two successful satellite altimeter missions) were used to calculate the nominal orbits.

## 4. NUMERICAL TESTS OF THE MODEL.

### Preliminary comments and overview.

This section presents the results of computer studies carried out to verify how accurately the linearized model of sections 1 and 3 represents the actual relative line-of-sight velocity (or relative range-rate) between the two satellites. This has been done by computing simulated range-rate from numerically integrated orbits in a high degree ($n = 300$) *zonal* field. The common reference orbit has been chosen *exactly polar*, to simplify some of the calculations, while the use of a zonal field means that a "tail-biting" data stream can consist of just one revolution of the spacecraft, as all variations in the field take place with latitude only. These choices bring important simplifications to the problem from the point of view of the total computing time needed. More comprehensive tests, involving tesserals to high degree and order, would have required a much longer arc, whose calculation was well beyond the means available for this project. The limited tests show, nevertheless, some important characteristics of both signal and model not encountered in the work of previous authors who have used analytical perturbations to tackle the satellite-to-satellite tracking problem, and have been forced to use very low degree spherical harmonic expansions (up to degree and order 8) to be able to include tesserals and sectorials without prohibitively lengthy calculations. One of these characteristics is the increasing effect with harmonic degree of the departures from circularity of the reference orbit on the perturbations. This orbit has been obtained by the procedure described in paragraph (2.1), and it closes (i.e., returns to the same initial state after one revolution) almost perfectly (a few microns difference in position, and a few microns per second in velocity). The departure from perfect circularity is of about ±10 km, corresponding to a mean ellipse with an eccentricity of about $1.5 \times 10^{-3}$. The departure of the mean height from the desired 160 km, is a few meters. The initial state of the nominal orbits included various "errors", or departures from the "true orbits", of the order of 10 m and $10^{-3}$ m $s^{-1}$, while the "true" orbits departed from the reference orbit by about 20 m at the starting point, and nearly 600 m in radial distance at the point of greatest separation, thus testing the effectiveness of linearizing the problem along the inflexible, closed reference orbit of section 2. The integrator presented some minor problems, suggesting the need to use an improved version in more realistic simulations, as well as for the analysis of real data. These problems seemed tobe caused by the limited arithmetic of the computer (14 significant figures in double precision) and the large range in the size of the perturbations caused by the harmonics included in the simulated gravitational field. This difficulty could be overcome by using machines with 20 or more significant figures in double precision, already widely available.

The study concentrated in the comparison between the *lumped* Fourier coefficients of the simulated signal, and their values according to the model. To obtain these coefficients in a manner that made the comparison possible, the signal had to be corrected to eliminate the nonperiodical part (secularly increasing oscillations due to zonal resonance, etc.), which in an adjustment would be absorbed by the arc parameters introduced in the previous section. After this was done, the agreement was better than 1% at most frequencies.

The section begins by describing the various orbits: reference, nominal and true, their discrepancies and errors (included to see how they affect the results), their respective gravitational fields, the method of integration and associated problems, and a detailed discussion of the periodical reference orbit. The second paragraph introduces the concept of lumped Fourier coefficients, the formulas for calculating their theoretical values according to the model, and the error measures used to quantify the agreement between these theoretical values and the simulated ones. Paragraph (4.3) refers to the effect of neglecting some first order terms when deriving the linearized observation equations in section 3, and also to the influence of the slight eccentricity of the orbit on high degree perturbations. Finally, the section closes with a discussion of the results of the comparisons between the simulated line-of-sight perturbations and their theoretical counterparts, for a distance between both satellites of either 100 km or 300 km and with various orbital errors.

## 4.1 Characteristics of the simulations.

### (a) "True" and "nominal" orbits, and "orbital errors".

The relative line-of-sight velocity (or relative *range-rate*) between two satellites was simulated by integrating two separate polar orbits, starting from initial conditions identical to the positions and the velocities at two points on the reference orbit (also polar) symmetrical respect to perigee ($F' = 0$), or else differing from such position and velocity by specified quantities. The velocity on each "true" orbit was projected along the line-of-sight directions and the difference of the projections was the simulated "true" instantaneous relative velocity. The residuals $\delta s$ were created by substracting from the "true" range-rate the range-rate corresponding to the nominal orbits. These were integrated using a field truncated at degree $n = 30$. These orbits had the same initial conditions as the "true" ones, sometimes modified by "orbit estimation errors" added to them. These "errors", for each nominal orbit, were about 10 m in radial component, a change in along-track velocity sufficient to cancel out the resulting relative drift between "true" and "nominal" orbits (see expression (2.4.5)), and $\pm 50$ m in the across-track direction. The condition of zero relative drift was chosen to simulate, roughly, the nature of the errors following the adjustment of the initial conditions of a reasonably long arc (several days of tracking from ground stations as well as between satellites), where any substantial rate of drift would eventually result in errors between predicted and observed quantities sufficiently large to be detected and corrected by the adjustment procedure. The size of the radial errors was choosen considering that, with sufficient tracking and the use of as much information on the gravitational field as is contained in present day Earth models complete to degree and order 180 (such as GEM 10C, (Lerch et al., 1981)), a 10 m uncertainty for a low satellite is probably reasonable. Across-track errors are harder to detect, if for no better reason that the satellite-to-satellite tracking data should convey insignificant information about the out-of-plane motions, in the case of a common orbital plane studied here. The discrepancy between the start of each "true" orbit and the reference orbit was chosen identical for both, on the theory that controling manoeuvres to keep the spacecraft near their reference course would be largely directed by the information in the relative line-of-sight velocity measurements, as these would be the bulk of the data available. As a result, relative discrepancies can be corrected better than absolute ones, and the satellites are likely to be put (just after a corrective manoeuvre) almost on the same orbit, parallel to the reference one but separated from it by a common offset. This offset must vary as time goes by, but, as both spacecraft are close to each other, its change is likely to be determined by much the same causes acting simultaneously on both, so their distances from the reference orbit are always nearly equal. The initial offsets were of some 20 m, which must result in a peak separation of about 60 m (see expression (2.4.2)), a separation from the desired reference trajectory easy to detect from the analysis of tracking data. All orbits were chosen as exactly polar (except for the across-track errors in the nominal ones) to simplify the computations, particularly those of the Fourier coefficients of the forcing terms of the variational equations (see paragraph (2.3)). The "true" field, and those of the reference and nominal orbits, were all purely zonal, and the coefficients for those zonals common to all three were identical. The choice of a zonal field was dictated by the limited availability of computing resources, as the only way to test the theory up to a sufficiently high degree (n = 300) without the very large effort in developing and running the programs which would be needed if tesserals and sectorials were included. This happened because I had chosen to look at the Fourier coefficients of the simulated signal, which can be computed in a way that makes them comparable to their analytical counterparts only from arcs lasting an exact number of revolutions, or "biting their own tails", so the data stream is periodic. Otherwise, a significant excess (or defect) in completing the last turn could result in numerical artifacts that falsify the results. To obtain a "tail biting" orbit with high degree tesserals, one has to integrate a very long arc, which was both beyond my resources and (probably) of the accuracy of the double precision arithmetic of my machine. On the other hand, one revolution is enough for the orbit to close upon itself in a zonal field, because such field has rotational symmetry. As the orbit is polar, it must encounter all the "roughness" of such a field along its way, as all anomalies depend on latitude only. The mean height of the satellites was 160 km above the mean Earth's radius, and the separation was, alternatively, 300 km and 100 km. This height and separation correspond to current assumptions made about the likely mission parameters, based on the results of years of error analyses (see references in the Introduction).

(b) The field.

The zonal field was defined by the choice of potential coefficients up to degree 300. The value of GM = 0.39860047 × $10^{15}$ and of the mean terrestrial radius a = 6378139 m were adopted. The dimensionless potential coefficients up to degree 9 were identical in value to the first nine zonal coefficients of GEM 9 (Lerch et al., 1977) (see Common statement "CC" in the programs of Appendix III). For degrees from 10 to 100, the coefficients were made equal to the square roots of the corresponding degree variances $\sigma_n^2$ *for the full harmonics*, according to the approximate spectral law (Rapp, 1979)

$$\sigma_n = \{(\frac{R^2}{GM})^2 \frac{1}{n-1} [\frac{A_1}{n+B_1} s_1^{n+2} + \frac{A_2}{(n-2)(n+B_2)} s_2^{n+2}]\}^{\frac{1}{2}} \times 10^{-5} \quad (4.1.1)$$

where $A_1$ = 3.405 $mgal^2$, $A_2$ = 140.03 $mgal^2$, $B_1$ = 1, $B_2$ = 2, $s_1$ = 0.998006, $s_2$ = 0.914232.

Above degree 100, the coefficients were all zero, except for the last two zonals (degrees 299 and 300), which were both equal to $10^{-5}$. As all the zonal harmonics are positive over the North pole, a choice of alternating signs for the coefficients from n = 10 up to n = 100, with the first two positive, the following two negative, etc. (always an even and an odd zonal in each pair) was adopted to avoid a "mountain" in the anomalous field over that pole. The alternating signs eliminated this "mountain", resulting in more or less evenly distributed irregularities from pole to pole and, therefore, in a more "natural" kind of anomalies. The choice of coefficients was a compromise between two contradictory goals: first, to use values that were strongly determined by present knowledge of the terrestrial field, to make the selection as impersonal as possible; second, to minimize the effect of the finite floating point arithmetic of the computer. At 160 km, the perturbations in gravitational acceleration due to an individual harmonic of degree 300 is *more than eleven orders of magnitude smaller* than that due to the zero harmonic, or central force term. The floating point double precision arithmetic at my disposal was able to carry up to 14 significant figures, so a perturbation of a single coefficient at the high end of the spectrum would have been smaller than the likely accumulation of rounding errors when the accelerations are computed by adding up the terms of their spherical harmonic expansions. This meant choosing "larger than life" coefficients, capable of creating much larger perturbations. Up to degree 100 it was enough to merely concentrate all the power of the harmonic of a given degree in the respective zonal, but above $n = 100$ several tests indicated that even this size of zonal was not big enough. As the use of such large zonals created correspondingly large resonant perturbations (along track drift and secularly increasing once per revolution oscillations), the departure of the "true" orbits from the reference and nominal ones amounted to hundreds of meters. So coefficients between degree 101 and 298 were set to zero, while coefficients of degree 299 and 300 were made both equal to $10^{-5}$, which is much larger than the total spectral power at those degrees would imply. This choice was needed because, otherwise, due to the attenuation bands mentioned in paragraph (3.5), the signal at some frequencies may fall below the orbital integration errors, resulting in misleadingly large discrepancies between theoretical and simulated perturbations. This did happen as soon as the size of the coefficients was reduced by just one order of magnitude. All the problems mentioned above would have been obviated by using extended precision arithmetic, or 28 significant figures, but this would have costed about four times more computing time, because all calculations of this kind are done by software rather than hardware. There are, however, machines currently available that can carry about 27 significant figures in double precision (implemented by hardware).

(c) The numerical integrator.

All orbital integrations were done using the subroutine COWELL provided by Chris Reigber, the current Director of the Deutsches Geodätisches Forschungs Institut (DGFI) in Munich. The algorithm of this subroutine is an eight order predictor-corrector based on a variant of the Cowell method developed by Kulikov (see Levallois, Book IV, pp. 213-225, 1970). This integrator did a most remarkable job, considering that it was never meant to calculate orbits in fields of degree larger than 20 or 30. One small problem was the presence of a tiny spike-shaped error at the start of each arc, resulting in a nearly constant error in the amplitude of all the cosine Fourier coefficients of the perturbations. This spike was probably caused by the different way in which the first point after the initial conditions is calculated, compared to the rest of the orbit. The algorithm is selfstarting, so the first computed point is found by a series of iterative refinements of a first estimate. To reduce this effect, I tried iterating once the corrector step in every subsequent calculation, but this modification of the original program brought only a slight improvement at the cost of doubling the computing time, so I did not use it. By choosing the field in the manner outlined above to get fairly strong perturbations at most frequencies compared to the integrator errors, and by correcting the simulated cosine lumped Fourier coefficients in the simple manner explained in paragraph (4.2), this problem was largely superated. For a calculation with realistic coefficients, as would be needed to form residuals from actual data in the event of iterating the solution, for example (see paragraph (3.10b)), a better integrator must be chosen. This can be the same type of integrator, but of higher order, or a different one altogether. The question requires careful study to achieve a good compromise between accuracy and computing efficiency, but present-day theory and computers should be sufficient to fulfill this purpose.

The integration step chosen for all orbits in the final calculations was about 2.8 seconds. Halving this interval resulted in negligible changes in the results. The same step was used for the reference, nominal, and "true" orbits, and it divided the reference orbital period an exact number of times. This number was a power of two (2048) adopted in order to reduce the computer time needed to get the various Fourier coefficients with a Fast Fourier Transform algorithm.

(d) The periodical reference orbit.

The initial conditions were obtained by means of the "closing" and "circularizing" procedures described in paragraph (2.1), to ensure that, when integrated in a field consisting of the central force term and the first nine zonals as defined by the coefficients mentioned in part (b) of this paragraph, the orbit would return at the end of one revolution to virtually the same place with the same velocity, and it would be also

reasonably close to a circle. Starting with Cook's formulas for the classical "frozen" orbit, the initial conditions were selected for a polar orbit of about 160 km height using expressions (2.1.5a-e). After finding the precise instant when the orbit reached its apogee ($F' = \pi$), the radial velocity was calculated for that instant. The integration was then repeated after changing the initial radial distance by 1 km. The new radial velocity at apogee, substracted from the original one and divided by 1 km was used as the approximate value of $\frac{d\dot{r}_\pi}{dr_0}$ needed for determining the corrections to the initial state. The procedure was iterated three times, after which two iterations of the "circularizing" procedure were executed, followed by a final application (one iteration) of the "closing" procedures. Each iteration of the "closing" procedure needed two separate runs, the derivative of $\dot{r}_\pi$ being obtained afterwards with a pocket calculator. Each iteration of the "circularizing" algorithm required a separate run. Clearly, elegant and easy-to-use software was not the overriding goal in writing the programs for calculating the reference orbit or, indeed, any of the other programs. With limited time, I settled for making sure that the programs were working by carrying out numerous checks, and kept attempts at optimizing the code to a bare minimum, except for the programs directly concerned with the orbital integrations, where I put greater effort in eliminating redundant calculations and made use of the efficient subroutine LEGEND (Colombo, 1981a) to compute the normalized Legendre functions needed both for the numerical integration of the orbits and for calculating the forcing terms of the variationals.
Economizing time and effort took precedence over cutting computing costs, where savings were mostly achieved through the choice of a polar orbit and a purely zonal field.

The orbits were computed in an inertial system of Cartesian coordinates (x,y,z), where the x axis was normal to the orbit plane, the y axis formed the intersection of this plane and the equator, and the z axis was aligned with the Earth's axis, positive towards the North. The initial conditions obtained for the closed orbit were

$$x(0) = 0.\text{m} \quad y(0) = 0.\text{m} \quad z(0) = 6526447.57571 \text{ m}$$

$$\dot{x}(0) = 0.\text{ms}^{-1} \quad \dot{y}(0) = -7812.98318978 \text{ ms}^{-1} \quad \dot{z}(0) = 0.\text{ms}^{-1}$$

The maximum departures from a circular orbit were ±10 km, and the mean height was 160002.33 m, just 2.33 m off the desired value. At the end of the orbital period, which was equal to 5263.369068 seconds, the position and velocity components were

$$x(T_0) = 0.\text{m} \qquad y(T_0) = -0.169 \times 10^{-5} \text{ m} \qquad z(T_0) = 6526447.57570 \text{ m}$$

$$\dot{x}(T_0) = 0.\text{ms}^{-1} \quad \dot{y}(T_0) = -7812.98318978 \text{ ms}^{-1} \quad \dot{z}(T_0) = -0.13 \times 10^{-7} \text{ ms}^{-1}$$

so the misclosures in position was of the order of 10 microns, and that in velocity, of the order of 1 microns per second.

To test the procedure further, an orbit with an inclination of $70^{\circ}$ (rather low for a geodetic satellite) was also studied. The initial conditions turned out to be (for a mean height of 160 km)

$$x(0) = 2240660.96245 \text{ m} \quad y(0) = 0.\text{m} \qquad z(0) = 6156165.39899 \text{ m}$$

$$\dot{x}(0) = 0.\text{ms}^{-1} \qquad \dot{y}(0) = -7795.38760561 \text{ ms}^{-1} \quad \dot{z}(0) = 0.\text{ms}^{-1}$$

the orbital period was 5282.38457 seconds, and the misclosures in position and velocity after one revolution, of about 10 microns and 0.1 microns per second, respectively. No "circularizing" of this orbit was attempted. All programs used are listed in Appendix III.

According to expression (3.4.8), it is necessary to know $\overline{\cos \eta_{10}} \simeq \cos n_0 c$ and $\overline{\sin \eta_{10}} \simeq \sin n_0 c$, the mean values of the cosine and the sine of the angle (see paragraph (1.4)) and the Fourier coefficient $b_1$ corresponding to the fundamental $n_0$ of $\sin \eta_{10}(t)$. These values are, for 100 km separation:

$$\overline{\cos \eta_{10}} = -0.9997361435 \qquad \overline{\sin \eta_{10}} = -0.2294022357 \times 10^{-1}$$

$$b_1 = -0.1538200005 \times 10^{-2}$$

Also

$$b_2 = 0.6416433923 \times 10^{-3}$$

For 300 km separation:

$$\overline{\cos \eta_{10}} = -0.9999700635 \qquad \overline{\sin \eta_{10}} = 0.7647337494 \times 10^{-2}$$

$$b_1 = -0.1538919912 \times 10^{-2}$$

Also

$$b_2 = 0.6421539551 \times 10^{-3}$$

## 4.2 Using lumped coefficients to study the accuracy of the linearized model.

The signal in the simulations was the instantaneous value of the relative velocity. In order to test the theory, the Fourier coefficients of the simulated signal, corrected for the effects of orbital errors and secular terms, were compared to their corresponding values according to the formulas of section 3. The "true" coefficients were obtained by analyzing one revolution worth of the simulated range-rate, computed at regular intervals. As the only purpose of the calculations was to test the linearized model for the observations, noise and other instrumental errors were not included, so one could say that the "true" values were "perfect data". Moreover, as the quality of the approximation must be the same for the perturbations on the instantaneous velocity as for the averaged velocity (the Fourier coefficients of both differ only by a known smoothing factor), the tests were done with instantaneous velocity. The theory of section 3 refers to the Fourier coefficients $s_{nm\alpha j}$ of the perturbations caused by each harmonic separately, but to test 300 harmonics in this way, one at the time, would have required more computer time than was available, and produced masses of numbers which could only be interpreted by summarizing them in some way (i.e., throwing most of them away). So, instead of individual perturbations, I have choosen to look at the whole effect on the signal of all the harmonic terms of the disturbing potential, as represented by the *lumped Fourier coefficients*

$$\hat{s}_{jm} = \sum_{\alpha=0}^{1} \sum_{n=m}^{N_{max}} \Delta\bar{C}^{\alpha}_{nm} \, s_{nmj} \qquad (4.2.1)$$

According to this definition and to expression (3.4.8),

$$\hat{s}_{jm} = \sum_{\alpha n} \Delta\bar{C}^{\alpha}_{nm\alpha}\{2\omega_{mj}[\sin n_0 c\ b_{nmj} \cos \omega_{mj} c - \cos n_0 c\ g_{nmj} \sin \omega_{mj} c]$$

$$+ b_1[\omega_{m(j-1)}\ b_{nm(j-1)} \sin \omega_{m(j-1)} c - \omega_{m(j-1)}\ b_{nm(j+1)}$$

$$\cdot \sin \omega_{m(j+1)} c]\} \tag{4.2.2}$$

where the $b_{nmj}$ and $g_{nmj}$ are given by (2.4.8-9). For the actual calculations see explanation in paragraph (AIII.4), Appendix III. By starting the orbits precisely at the instant when the mean point along the line-of-sight chord between the satellites was at perigee (F' = 0), and assuming that the node of the (nonprecessing) polar reference orbit is $L = L_0 = 0$, the phases of the perturbations in expression (3.4.8) are, for $\alpha = m = 0$ (zonals) and $t_0 = \tau_0 = 0$,

$$\phi_{m\alpha} = 0$$

according to (2.2.47) and (2.2.50), so the periodic part of the signal due to the disturbing zonals is an odd function of time, in accordance with the rotational symmetry of the chosen field and the selection of the crossing of perigee as the starting time. This symmetry extends to the secularly increasing oscillations cause by zonal resonance, that must have the form

$$At \cos n_0 t$$

which is an odd function of time too. *Even* perturbations can only be due, if the model is correct, to nominal orbit errors, as indicated by expressions (2.4.6-7) for the variations in radial and along-track velocity. Such even secular oscillations have the form

$$Bt \sin n_0 t$$

In the course of an adjustment along the lines of section 3, the nonperiodical part of the signal (increasing oscillations, oscillations due to initial state errors that change from arc to arc) are more or less absorbed by the *arc parameters* of expression (3.4.18). To eliminate this part from the simulated signal in a manner roughly equivalent to an adjustment, the

Fourier coefficients of this signal were subject to some corrections. In the first place, due to numerical integration errors, the cosine Fourier coefficients in the absence of orbit errors took on a very small and nearly constant non-zero value (because of the symmetry mentioned above, they should have been all zero). The effect of integrator errors on the sine coefficients was less clear. To compensate for this, the value of one of the high frequency non-zero cosine coefficients was substracted from the rest, leaving negligible residual values. The sine coefficients were not treated in this way.

Since the simulated arc lasts exactly one revolution of the reference orbit, the arc parameters for the zero, first, second and third harmonic in a real adjustment would absorb the zero and first harmonics completely, and also the *differences* between the analytical and the observed second and third harmonics, at least with noiseless data. Therefore, the differences between the analytical and the observed coefficients must be zero: for one-revolution arcs the theory is apparently "error-free" up to the third harmonic. The real corrections must take place, in this special case, from the fourth harmonic up. These corrections are needed to remove the effects of the secularly increasing oscillations. If the theory were perfect, the differences between the (noiseless) simulated and the analytical lumped coefficients would be identical to the Fourier coefficients of the increasing oscillations *analyzed over one orbital period*,

$$F(At \cos n_0 t)_j = h_s j(j^2-1)^{-1} \tag{4.2.3}$$

for the sine terms, and

$$F(Bt \sin n_0 t)_j = h_c (j^2-1)^{-1} \tag{4.2.4}$$

for the cosine terms, where $F(s)_j$ indicates the operation of taking the Fourier coefficient for the jth harmonic component of s. The coefficients $h_c$ and $h_s$ are proportional to the respective arc parameters, and were "estimated" by averaging $h_s$ and $h_c$ computed from individual differences between observed and analytical coefficients, according to (4.2.3-4) above, over several frequencies. This completes the "rough adjustment" of the arc parameters. Substracting the differences from the "observed" coefficients gives the *corrected lumped coefficients* for the periodical part of the signal, which can be compared to their theoretical counterparts, and should be identical if the theory were exact. Therefore, the percentage error per coefficient

$$\varepsilon_j\% = \frac{\hat{s}_{j:(\text{corrected})} - \hat{s}_{j:(\text{analytical})}}{\hat{s}_{j:(\text{corrected})}} \times 100 \qquad (4.2.5)$$

is a measure of the accuracy of the theory at a given frequency. As there are more than 300 frequencies to be considered, it is easier to give a general picture of the quality of the model by providing the total percentage r.m.s. error within a narrow band of a few individual frequencies:

$$\varepsilon_{(k\ \text{band})}\% = \left\{ \frac{\sum_{j=k}^{k+\Delta j} [(\text{corr.}(\text{cos.})\text{coeffs.})_j^2 + (\hat{s}_{j(\text{corr.})} - \hat{s}_{j(\text{anal.})})^2]}{\sum_{j=k}^{k+\Delta j} [(\text{uncorr.}(\text{cos.})\text{coeffs.})_j^2 + \hat{s}_{j(\text{uncorr.})}^2]} \right\}^{\frac{1}{2}} \times 100 \qquad (4.2.6)$$

as well as the total percentage rms error over all the frequencies of interest[1]. In the present case, the highest frequency with significant power was 304 cycles per revolution, and the lowest, for the reasons given before, was four cycles per revolution (lower frequencies are, of course, interesting, but they are "error-free" in this particular case). This total rms error is, then

$$\varepsilon_T\% = \left\{ \frac{\sum_{j=4}^{304} [(\text{corr.}(\text{cos.})\text{coeffs.})_j^2 + (\hat{s}_{j(\text{corr.})} - \hat{s}_{j(\text{anal.})})^2]}{\sum_{j=4}^{304} [(\text{uncorr.}(\text{cos.})\text{coeffs.})_j^2 + \hat{s}^2_{j(\text{uncorr.})}]} \right\}^{\frac{1}{2}} \times 100 \qquad (4.2.7)$$

Summing up: after making a small correction to the observed lumped cosine coefficients to reduce the numerical integration errors, the effects of the increasing oscillations due to zonal resonance were estimated according to expressions (4.2.3-4) and then substracted from the observed coefficients corresponding to frequencies above the third harmonic. Up to the third, all aperiodical effects would be absorbed exactly by the arc parameters, amounting, for noise-free data, to a perfect agreement with the analytical coefficients, so no correction was needed. This can only happen in a one-revolution arc, or one that lasts an exact number

[1] Where "(cos.)coeffs." and "$\hat{s}_j$" stand for "cosine" and "sine" coefficients, respectively.

of revolutions, and should be reasonably close to the thruth for those which do not, but are so long that the fraction of cycle left over at the end is a small part of the overall period (provided the noise level is low). The corrected lumped coefficients were then compared to their analytical counterparts by computing percentage errors for the Fourier coefficients of the individual frequencies in the band from 4 to 304 cycles per revolution; the percentage error in rms over narrow bands (5 cycles wide) across the same overall range; and the total percentage rms error (or "error-to-signal" ratio) in that range. As (except for the low frequency effects of nominal orbit errors) only the sine terms convey any interesting information, the relative errors per individual frequency, listed in Appendix IV, were computed only for the sine coefficients, as the cosine terms are negligible after correction.

## 4.3 The main effect of neglected first order terms, and the influence of the eccentricity on perturbations of high degree.

The linearized model is based on expression (1.4.12), which was obtained from the complete differential of the signal by ignoring those terms in (1.4.11) that represent the effect of perturbations on the direction of the line of sight. These terms contain $\beta_i$ and $\gamma_i$ but not their derivatives, and their only appreciable influence on the results is an increasing oscillation of the form At cos t $n_0$t. I found this oscillation after introducing small modifications to the values of $\bar{C}^0_{20}$ and $\bar{C}^0_{30}$ in order to cancel out the one-cycle-per-revolution components in the forcing terms $D_{\bar{C}_{nm}}^{\alpha}\, a_r$ and $D_{\bar{C}_{nm}}^{\alpha}\, a_u$ that cause zonal resonances.

The modifications eliminated completely these resonances from $\beta_i$, $\gamma_i$ and their derivatives, but the increasing oscillations reappeared, to a small extent, in the line-of-sight velocity. They were the only significant departure from the predictions of the simplified model. My explanation of this is as follows: the ignored terms contain the product of $\gamma_1$ and $\gamma_2$ times the difference in reference velocity normal to the line-of-sight. $\gamma_1$, $\gamma_2$ include a secular along-track drift Gt, while the normal relative velocity, if the satellites are close to each other, is much the same as the difference in their radial velocities. The radial velocity, when the time origin corresponds to F' = 0, is an odd function of time, so its increment over the constant interval separating the passage of both spacecraft through the same point is an even function. For an ellipse of small eccentricity, it takes the approximate form of K cos $n_0t$ (where K is proportional to $n_0$ and the intersatellite distance). As $\gamma_1$ and $\gamma_2$ are multiplied by K cos $n_0t$, the product because of the secular drift, must include a term At cos $n_0t$, where A = KG. When correcting the coefficients in the manner described in the previous paragraph, or through the inclusion of the appropriate arc parameter in the adjustment, this extra secular oscillation gets lumped together with the one coming from the first order terms retained in the model, and both are taken care of together.

As explained in paragraph (2.6), if the reference orbit were perfectly circular, the perturbations caused by a single spherical harmonic of the disturbing potential would be zero at those frequencies $j\ n_0 + m\theta'$ where j is opposite in parity to the degree n. So, for an *odd* zonal (m = 0), all *even* harmonics of $n_0$ must be zero. If one includes the effect of the eccentricity to the first power only, as in previous attempts to build a simple linearized model based on analytical perturbations using the "literal" approach and classical theory, the result is identical to that of assuming that the reference orbit is circular, because this result is valid only for orbits of vanishing eccentricity (see, for example, Wagner and Goad (1982)[1], expression (33)). The influence of the non-circularity of the orbit increases with the degree n, because of the non-linear relationship between the values of the coordinates and of the spherical harmonics. At the high end of the spectrum, the simulations showed that the harmonics of opposite parity from n were not just different from zero, but also comparable in amplitude to those of the same parity. An example, for the odd zonal n = 299, is given in Table (4.1), where the amplitudes of a few consecutive frequencies are listed to show what is a prevailing pattern across the spectrum. The fact that the highest significant frequency for n = 300 is around n = 304 suggests that, if one uses the "literal" formulation and classical theory, at least all powers of the eccentricity up to the fourth should be included when the degree is close to 300. On the other hand, for low degrees ($n \leq 20$), the first power might be sufficient.

(1) Or equation (31) in C.A. Wagner's later paper in J.G.R. (Red), December 1983, Vol. 88, No. B12, pp. 10309-10321.

| cycles per revolution (j) | $jn_0 \times b_{nmj}$ $\times 10^{-4}$ | $jn_0 \times g_{nmj}$ $\times 10^{-5}$ |
|---|---|---|
| 20 | -.377 | .125 |
| 21 | -.838 | .208 |
| 22 | -.342 | $.592 \times 10^{-1}$ |
| 23 | -.765 | $.749 \times 10^{-1}$ |
| 24 | -.314 | $.940 \times 10^{-2}$ |
| 25 | -.704 | $-.274 \times 10^{-1}$ |
| 26 | -.290 | $-.294 \times 10^{-1}$ |
| 27 | -.652 | -.107 |
| 28 | -.269 | $-.601 \times 10^{-1}$ |
| 29 | -.607 | -.172 |
| 30 | -.251 | $-.850 \times 10^{-1}$ |

Table (4.1): Fourier coefficients of the time derivatives of the radial and along track variations created by the 299th zonal (periodical part only). Here $\bar{C}^{0}_{299\ 0} = 10^{-5}$.

## 4.4 Numerical results of the accuracy tests.

When comparing the lumped coefficients of the simulated signal to those calculated according to the model, one must keep in mind that the former are affected by numerical integrator errors and by nonlinear effects caused by the separation between the nominal and the "true" orbits. This separation comes close to 200 meters in radial distance at the end of the first quarter of the orbit period, and to nearly 600 meters at the end of the third, and consists of large, increasing once-per-revolution oscillations due to zonal resonance. The separation of the reference orbit from the nominal orbit follows the same pattern, but it is much smaller (10 m and 30 m, respectively). Therefore, the "true" orbits depart by more than half a kilometer from the one along which the problem is linearized, giving a reasonable test for the applicability of this linearization to a real situation, where the satellites are likely to move away hundreds of meters from their "reference positions". Because of the numerical errors of the integrator, the discrepancies between theory and "reality" are likely to appear worse than they really are, so the results listed in the tables at the end of this section are probably on the conservative side. As pointed out in paragraph (4.1(b)), the use of "larger than life" zonal coefficients to compensate for the limited number of significant figures in the number system of the computer led to these very exagerated oscillations which, in a real situation, are likely to grow much more gradually, over many revolutions, until eventually a correcting manoeuvre to return the spacecraft to the neighbourhood of the reference orbit becomes necessary. The need for such manoeuvres could be reduced by including many more zonals in the field of the reference orbit than the first nine used here. These zonals could be obtained from existing information on the field, and/or from a preliminary solution up to a lower degree than 300, say n = 180, that could be done after only one month of the satellites being in orbit. Such a preliminary solution would be also useful for predicting the future trajectories of the spacecraft in order to correct them more precisely when they stray too far from their desired course.

As shown in the tables, the discrepancies between computed and simulated lumped coefficients does not exceed 1% at most frequencies in the band from 4 to 304 cycles per revolution, and in most cases are of a few parts per thousand or even less, in agreement with the accuracy of the approximations made when deriving Hill's equations, which were of the order of the flattening (1/300) and of the orbital eccentricity (the eccentricity of the reference orbit is close to $1.5 \times 10^{-3}$). As already explained in paragraph (4.3) in the special case considered here, the arc parameters are supposed to eliminate all discrepancy between "true" and theoretical coefficients up to 3 cycles per revolution, so only the "uncorrected" and the "analytical" coefficients are listed for those frequencies, but not the percentage errors, which would be all zero after correction (for noisless data). The main discrepancies occur in the attenuation bands that appear when the separation is 300 km (for 100 km, the first band is well above the highest significant frequency in the spectrum). Inside the bands, the error can be seen to grow gradually until it exceeds 1%. In fact, looking at the detailed table for 300 km, to be found in Appendix IV, one can see that the error approaches 100% at two points, but this happens only where the signal is extremely weak, and the effect on the overal rms error is quite insignificant. Probably, a considerable part of this larger discrepancies may be caused by the integration errors and limited arithmetic of the computer, as suggested by the fact that, reducing the size of the last two zonals (n = 299 and n = 300) makes this errors much larger in percentage, compared to the simulated values. As the theory should work better for smaller perturbations, and not worse (it is a first order theory) this contradictory behaviour can be explained, at least partially, by numerical errors and not by defficiencies in the theory itself.

Tables (4.2) to (4.6) are reproductions of part of the computer printout of the program described in (AIII.4). They show, first, the zonal field and the orbits (maximum degree of the "true" and the "nominal" fields, here called *reference*, a misnomer), height, separation, reference orbital period, number of integration steps in one period, the common displacement of the orbits from the common reference (listed as "DZCE"), and the errors in the initial states of the nominal orbits: DX11, DX12 for the across-track displacements of the first and second satellite, DR1 and DR2 for the radial errors, and DUP1 and DUP2 for the along-track velocity errors. This information is followed by the listing of the lumped coefficients from 0 to 3 cycles per revolution. Under the heading "corrected (cos / sin)" appear the coefficients of the simulated data after correcting them for the integrator error in the cosine terms, and for the secularly increasing oscillations. The discrepancies with the "analytical" coefficients in the column on the right, for 2 and 3 cycles per revolution, are due primarily to the modulation of the once-per-revolution term in the free response, and would be cancelled out by the corresponding arc parameters. Once the free oscillations are known from the Fourier analysis of the simulated data, it is possible to use the theory to estimate how much of them is "spilling over" the neighbouring frequencies through modulation. This gives estimates of the discrepancies that are in rough agreement. That, in the case of Table (4.6), this agreement was way off, suggests a relatively strong influence of the across-track orbit errors, which are here the only differences among the two nominal and the two "true" orbits (the radial and along-track errors and DZCE are all 0). As the first order theory of section 1 excludes the possibility of such an influence by perturbations normal to the orbital plane, these must be due to second and higher order effects. In any event, they are confined to the lowest frequencies, and the arc parameters for those frequencies should absorbe them. The rms of the simulated signal, both in their uncorrected and corrected forms, the "analytical" value of the latter, according to the theory, and the overall percentage errors are listed next, totalized over bands five cycles per revolution wide, up to the 304th harmonic. Finally, the total rms of the error from 4 to 304 cycles is compared to (a) the total rms of the signal (resonant oscillations and all), and (b) the rms of the corrected, periodical part of the signal. The results show clearly the presence of attenuation bands, with 300 km separation, where the signal is overpowered by integrator errors and nonlinear effects. The total errors (from 4 cycles to 304) are always below 1%. All computations were carried out with the AMDAHL 470 V7B of the Technical University of Delft.

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 300000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = .0 DX11 = .0 DX12 = .0 DR1 = .0 DUP1 = .0
DR2 = .0 DUP2 = .0
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

FROM 0 CY/REV. TO 3 CY/REV :

| CY/REV | CORRECTED | (COS, SIN) | ANALYTICAL. |
|---|---|---|---|
| 0 | .188049D-07 | | |
| 1 | -.894372D-02 | .654939D-04 | |
| 2 | .231722D-06 | .181704D-04 | .416936D-04 |
| 3 | -.423704D-06 | .364512D-04 | .260653D-04 |

SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM 4 TO 304 CY/REV.

| BAND | RMS UNCORRECTED | RMS CORRECTED | RMS ANALYTICAL | % (SIN+COS) ERROR |
|---|---|---|---|---|
| 4 - 14 | .560978D-04 | .731815D-04 | .733002D-04 | .232767 |
| 15 - 24 | .129322D-03 | .135994D-03 | .135982D-03 | .621289D-01 |
| 25 - 34 | .693438D-03 | .694904D-03 | .694447D-03 | .730045D-01 |
| 35 - 44 | .776036D-03 | .776627D-03 | .776330D-03 | .485214D-01 |
| 45 - 54 | .551326D-03 | .550612D-03 | .550468D-03 | .438234D-01 |
| 55 - 64 | .365829D-03 | .366063D-03 | .366014D-03 | .439519D-01 |
| 65 - 74 | .257575D-03 | .256917D-03 | .256925D-03 | .511845D-01 |
| 75 - 84 | .155790D-03 | .155749D-03 | .155765D-03 | .741701D-01 |
| 85 - 94 | .107370D-03 | .106692D-03 | .106753D-03 | .112255 |
| 95 - 104 | .494289D-04 | .491960D-04 | .492333D-04 | .223844 |
| 105 - 114 | .269134D-04 | .256878D-04 | .258140D-04 | .502619 |
| 115 - 124 | .159993D-04 | .148945D-04 | .150316D-04 | .953066 |
| 125 - 134 | .488465D-05 | .426503D-05 | .433980D-05 | 3.46983 |
| 135 - 144 | .107387D-04 | .116389D-04 | .114990D-04 | 1.27916 |
| 145 - 154 | .236110D-04 | .244978D-04 | .243567D-04 | .592562 |
| 155 - 164 | .361672D-04 | .370059D-04 | .368750D-04 | .371651 |
| 165 - 174 | .475307D-04 | .483224D-04 | .482082D-04 | .262753 |
| 175 - 184 | .570403D-04 | .577890D-04 | .576970D-04 | .198984 |
| 185 - 194 | .641205D-04 | .648303D-04 | .647658D-04 | .160210 |
| 195 - 204 | .682829D-04 | .689574D-04 | .689245D-04 | .140205 |
| 205 - 214 | .691338D-04 | .697763D-04 | .697784D-04 | .138422 |
| 215 - 224 | .663773D-04 | .669904D-04 | .670298D-04 | .156611 |
| 225 - 234 | .598046D-04 | .603908D-04 | .604682D-04 | .199219 |
| 235 - 244 | .492678D-04 | .498286D-04 | .499433D-04 | .281338 |
| 245 - 254 | .346432D-04 | .351780D-04 | .353267D-04 | .461530 |
| 255 - 264 | .161104D-04 | .165904D-04 | .167567D-04 | 1.11945 |
| 265 - 274 | .133609D-04 | .129764D-04 | .128117D-04 | 1.62608 |
| 275 - 284 | .466453D-04 | .461613D-04 | .459598D-04 | .527555 |
| 285 - 294 | .107577D-03 | .107126D-03 | .106897D-03 | .290752 |
| 295 - 304 | .272660D-03 | .272291D-03 | .272283D-03 | .227380 |

TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL. (BAND FROM 4 CY/REV TO 304 CY/REV)
(A) .9046809412D-01 (B) .9031086767D-01

Table (4.2)

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 300000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = -20.000000 DX11 = .0 DX12 = .0 DR1 = 10.0000 DUP1 = .238751D-01
DR2 = 9.00000 DUP2 = .214876D-01
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

FROM 0 CY/REV. TO 3 CY/REV : CY/REV CORRECTED (COS, SIN) ANALYTICAL.

| CY/REV | CORRECTED (COS, SIN) | | ANALYTICAL |
|---|---|---|---|
| 0 | -.293455D-05 | | |
| 1 | -.947773D-02 | -.232205D-02 | |
| 2 | -.214287D-05 | .188321D-04 | .416936D-04 |
| 3 | .279433D-05 | .348839D-04 | .260653D-04 |

SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM 4 TO 304 CY/REV.

| BAND | RMS UNCORRECTED | RMS CORRECTED | RMS ANALYTICAL | % (SIN+COS) ERROR |
|---|---|---|---|---|
| 4 - 14 | .565147D-04 | .730669D-04 | .733002D-04 | .620702 |
| 15 - 24 | .129558D-03 | .136016D-03 | .135982D-03 | .109063 |
| 25 - 34 | .693522D-03 | .694940D-03 | .694447D-03 | .156426 |
| 35 - 44 | .776220D-03 | .776791D-03 | .776330D-03 | .185400 |
| 45 - 54 | .551520D-03 | .550828D-03 | .550468D-03 | .226827 |
| 55 - 64 | .365977D-03 | .366203D-03 | .366014D-03 | .267146 |
| 65 - 74 | .257698D-03 | .257060D-03 | .256925D-03 | .305660 |
| 75 - 84 | .155874D-03 | .155835D-03 | .155765D-03 | .359715 |
| 85 - 94 | .107408D-03 | .106753D-03 | .106753D-03 | .394879 |
| 95 - 104 | .494582D-04 | .492344D-04 | .492333D-04 | .526396 |
| 105 - 114 | .268124D-04 | .256263D-04 | .258140D-04 | .849218 |
| 115 - 124 | .158485D-04 | .147798D-04 | .150316D-04 | 1.82896 |
| 125 - 134 | .476741D-05 | .419546D-05 | .433980D-05 | 7.42800 |
| 135 - 144 | .109502D-04 | .118235D-04 | .114990D-04 | 2.92911 |
| 145 - 154 | .238459D-04 | .247045D-04 | .243567D-04 | 1.51918 |
| 155 - 164 | .364012D-04 | .372131D-04 | .368750D-04 | 1.07377 |
| 165 - 174 | .477447D-04 | .485110D-04 | .482082D-04 | .867127 |
| 175 - 184 | .572145D-04 | .579392D-04 | .576970D-04 | .760560 |
| 185 - 194 | .642353D-04 | .649223D-04 | .647658D-04 | .712140 |
| 195 - 204 | .683192D-04 | .689721D-04 | .689245D-04 | .708571 |
| 205 - 214 | .690747D-04 | .696965D-04 | .697784D-04 | .750016 |
| 215 - 224 | .662087D-04 | .668021D-04 | .670298D-04 | .848496 |
| 225 - 234 | .595165D-04 | .600839D-04 | .604682D-04 | 1.03633 |
| 235 - 244 | .488556D-04 | .493984D-04 | .499433D-04 | 1.40007 |
| 245 - 254 | .341109D-04 | .346283D-04 | .353267D-04 | 2.23946 |
| 255 - 264 | .155152D-04 | .159761D-04 | .167567D-04 | 5.50006 |
| 265 - 274 | .139979D-04 | .136154D-04 | .128117D-04 | 7.44136 |
| 275 - 284 | .475011D-04 | .470515D-04 | .459598D-04 | 2.58269 |
| 285 - 294 | .108522D-03 | .108086D-03 | .106897D-03 | 1.52363 |
| 295 - 304 | .272531D-03 | .272175D-03 | .272283D-03 | 1.30012 |

TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL. (BAND FROM 4 CY/REV TO 304 CY/REV)
(A) .4064009309 (B) .4057170473

Table (4.3)

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 100000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = .0 DX11 = .0 DX12 = .0 DR1 = .0 DUP1 = .0
DR2 = .0 DUP2 = .0
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

FROM 0 CY/REV. TO 3 CY/REV : CY/REV CORRECTED (COS, SIN) ANALYTICAL.

| CY/REV | CORRECTED | (COS, SIN) | ANALYTICAL |
|---|---|---|---|
| 0 | -.264407D-08 | | |
| 1 | -.204751D-01 | -.203807D-04 | |
| 2 | -.414886D-06 | -.388638D-04 | .138998D-04 |
| 3 | -.863398D-07 | .326021D-04 | .868815D-05 |

SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM 4 TO 304 CY/REV.

| BAND | RMS UNCORRECTED | RMS CORRECTED | RMS ANALYTICAL | % (SIN+COS) ERROR |
|---|---|---|---|---|
| 4 - 14 | .301910D-04 | .245705D-04 | .245635D-04 | .876589 |
| 15 - 24 | .489005D-04 | .468581D-04 | .468811D-04 | .366296 |
| 25 - 34 | .254100D-03 | .253660D-03 | .253495D-03 | .785435D-01 |
| 35 - 44 | .296939D-03 | .296759D-03 | .296650D-03 | .456506D-01 |
| 45 - 54 | .228369D-03 | .228586D-03 | .228504D-03 | .474843D-01 |
| 55 - 64 | .168432D-03 | .168363D-03 | .168329D-03 | .431452D-01 |
| 65 - 74 | .135110D-03 | .135313D-03 | .135271D-03 | .536785D-01 |
| 75 - 84 | .965006D-04 | .965192D-04 | .965019D-04 | .603655D-01 |
| 85 - 94 | .819482D-04 | .821604D-04 | .821278D-04 | .743961D-01 |
| 95 - 104 | .488341D-04 | .489319D-04 | .489156D-04 | .112007 |
| 105 - 114 | .409586D-04 | .413335D-04 | .412762D-04 | .149882 |
| 115 - 124 | .444918D-04 | .448354D-04 | .447805D-04 | .136936 |
| 125 - 134 | .479012D-04 | .482183D-04 | .481644D-04 | .130029 |
| 135 - 144 | .511890D-04 | .514834D-04 | .514303D-04 | .125472 |
| 145 - 154 | .543597D-04 | .546344D-04 | .545821D-04 | .122440 |
| 155 - 164 | .574204D-04 | .576779D-04 | .576267D-04 | .120474 |
| 165 - 174 | .603825D-04 | .606248D-04 | .605750D-04 | .119342 |
| 175 - 184 | .632627D-04 | .634914D-04 | .634433D-04 | .118872 |
| 185 - 194 | .660855D-04 | .663022D-04 | .662562D-04 | .118959 |
| 195 - 204 | .688868D-04 | .690926D-04 | .690493D-04 | .119532 |
| 205 - 214 | .717193D-04 | .719153D-04 | .718751D-04 | .120539 |
| 215 - 224 | .746616D-04 | .748487D-04 | .748123D-04 | .121973 |
| 225 - 234 | .778347D-04 | .780136D-04 | .779818D-04 | .123842 |
| 235 - 244 | .814326D-04 | .816040D-04 | .815777D-04 | .126189 |
| 245 - 254 | .857865D-04 | .859510D-04 | .859314D-04 | .129093 |
| 255 - 264 | .915134D-04 | .916716D-04 | .916603D-04 | .132684 |
| 265 - 274 | .999254D-04 | .100078D-03 | .100077D-03 | .137168 |
| 275 - 284 | .114498D-03 | .114645D-03 | .114661D-03 | .142891 |
| 285 - 294 | .150041D-03 | .150182D-03 | .150232D-03 | .151064 |
| 295 - 304 | .276308D-03 | .276422D-03 | .276675D-03 | .201159 |

TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL. (BAND FROM 4 CY/REV TO 304 CY/REV)
(A) .1212166476 (B) .1212191840

Table (4.4)

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 100000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = -20.000000 DX11 = .0 DX12 = .0 DR1 = 10.0000 DUP1 = .238751D-01
DR2 = 9.00000 DUP2 = .214876D-01
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

FROM 0 CY/REV. TO 3 CY/REV :

| CY/REV | CORRECTED | (COS, SIN) | ANALYTICAL. |
|---|---|---|---|
| 0 | -.278070D-05 | | |
| 1 | -.206580D-01 | -.240821D-02 | |
| 2 | -.271378D-05 | -.384219D-04 | .138998D-04 |
| 3 | .313797D-05 | .321411D-04 | .868815D-05 |

SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM 4 TO 304 CY/REV.

| BAND | RMS UNCORRECTED | RMS CORRECTED | RMS ANALYTICAL | % (SIN+COS) ERROR |
|---|---|---|---|---|
| 4 - 14 | .304822D-04 | .245814D-04 | .245635D-04 | 1.62123 |
| 15 - 24 | .489992D-04 | .468569D-04 | .468811D-04 | .337910 |
| 25 - 34 | .254127D-03 | .253666D-03 | .253495D-03 | .154767 |
| 35 - 44 | .297000D-03 | .296811D-03 | .296650D-03 | .176796 |
| 45 - 54 | .228464D-03 | .228693D-03 | .228504D-03 | .228545 |
| 55 - 64 | .168487D-03 | .168415D-03 | .168329D-03 | .268311 |
| 65 - 74 | .135215D-03 | .135428D-03 | .135271D-03 | .335451 |
| 75 - 84 | .965550D-04 | .965754D-04 | .965019D-04 | .395486 |
| 85 - 94 | .820707D-04 | .822940D-04 | .821278D-04 | .481955 |
| 95 - 104 | .489323D-04 | .490367D-04 | .489156D-04 | .727434 |
| 105 - 114 | .412049D-04 | .415982D-04 | .412762D-04 | .866568 |
| 115 - 124 | .447391D-04 | .450994D-04 | .447805D-04 | .823002 |
| 125 - 134 | .481556D-04 | .484882D-04 | .481644D-04 | .807199 |
| 135 - 144 | .514503D-04 | .517590D-04 | .514303D-04 | .799938 |
| 145 - 154 | .546264D-04 | .549145D-04 | .545821D-04 | .797411 |
| 155 - 164 | .576907D-04 | .579607D-04 | .576267D-04 | .798089 |
| 165 - 174 | .606542D-04 | .609083D-04 | .605750D-04 | .801252 |
| 175 - 184 | .635336D-04 | .637735D-04 | .634433D-04 | .806488 |
| 185 - 194 | .663533D-04 | .665805D-04 | .662562D-04 | .813574 |
| 195 - 204 | .691492D-04 | .693651D-04 | .690493D-04 | .822388 |
| 205 - 214 | .719740D-04 | .721795D-04 | .718751D-04 | .832876 |
| 215 - 224 | .749063D-04 | .751025D-04 | .748123D-04 | .845055 |
| 225 - 234 | .780672D-04 | .782548D-04 | .779818D-04 | .858977 |
| 235 - 244 | .816508D-04 | .818306D-04 | .815777D-04 | .874738 |
| 245 - 254 | .859886D-04 | .861611D-04 | .859314D-04 | .892473 |
| 255 - 264 | .916978D-04 | .918637D-04 | .916603D-04 | .912352 |
| 265 - 274 | .100091D-03 | .100251D-03 | .100077D-03 | .934575 |
| 275 - 284 | .114646D-03 | .114800D-03 | .114661D-03 | .959382 |
| 285 - 294 | .150176D-03 | .150324D-03 | .150232D-03 | .987271 |
| 295 - 304 | .275530D-03 | .275649D-03 | .276675D-03 | 1.33300 |

TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL. (BAND FROM 4 CY/REV TO 304 CY/REV)
(A) .7272736313 (B) .7272888757

Table (4.5)

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 300000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = .0 DX11 = 50.0000 DX12 = -50.0000 DR1 = .0 DUP1 = .0
DR2 = .0 DUP2 = .0
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

FROM 0 CY/REV. TO 3 CY/REV :

| CY/REV | CORRECTED | (COS, SIN) | ANALYTICAL. |
|---|---|---|---|
| 0 | .187621D-07 | | |
| 1 | -.894354D-02 | .654926D-04 | |
| 2 | .413765D-06 | -.120338D-05 | .416936D-04 |
| 3 | -.423282D-06 | .364033D-04 | .260653D-04 |

SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM 4 TO 304 CY/REV.

| BAND | RMS UNCORRECTED | RMS CORRECTED | RMS ANALYTICAL | % (SIN+COS) ERROR |
|---|---|---|---|---|
| 4 - 14 | .560503D-04 | .731790D-04 | .733002D-04 | .237548 |
| 15 - 24 | .129304D-03 | .135994D-03 | .135982D-03 | .621680D-01 |
| 25 - 34 | .693434D-03 | .694904D-03 | .694447D-03 | .730119D-01 |
| 35 - 44 | .776034D-03 | .776627D-03 | .776330D-03 | .485174D-01 |
| 45 - 54 | .551328D-03 | .550612D-03 | .550468D-03 | .438043D-01 |
| 55 - 64 | .365828D-03 | .366063D-03 | .366014D-03 | .439689D-01 |
| 65 - 74 | .257577D-03 | .256916D-03 | .256925D-03 | .512361D-01 |
| 75 - 84 | .155790D-03 | .155749D-03 | .155765D-03 | .743389D-01 |
| 85 - 94 | .107372D-03 | .106692D-03 | .106753D-03 | .112558 |
| 95 - 104 | .494295D-04 | .491959D-04 | .492333D-04 | .224554 |
| 105 - 114 | .269166D-04 | .256875D-04 | .258140D-04 | .503960 |
| 115 - 124 | .160021D-04 | .148942D-04 | .150316D-04 | .955154 |
| 125 - 134 | .488644D-05 | .426489D-05 | .433980D-05 | 3.47622 |
| 135 - 144 | .107363D-04 | .116391D-04 | .114990D-04 | 1.28106 |
| 145 - 154 | .236086D-04 | .244980D-04 | .243567D-04 | .593289 |
| 155 - 164 | .361649D-04 | .370061D-04 | .368750D-04 | .372017 |
| 165 - 174 | .475285D-04 | .483225D-04 | .482082D-04 | .262951 |
| 175 - 184 | .570382D-04 | .577891D-04 | .576970D-04 | .199092 |
| 185 - 194 | .641186D-04 | .648304D-04 | .647658D-04 | .160263 |
| 195 - 204 | .682810D-04 | .689574D-04 | .689245D-04 | .140223 |
| 205 - 214 | .691320D-04 | .697763D-04 | .697784D-04 | .138415 |
| 215 - 224 | .663756D-04 | .669905D-04 | .670298D-04 | .156578 |
| 225 - 234 | .598030D-04 | .603908D-04 | .604682D-04 | .199148 |
| 235 - 244 | .492662D-04 | .498287D-04 | .499433D-04 | .281204 |
| 245 - 254 | .346417D-04 | .351781D-04 | .353267D-04 | .461265 |
| 255 - 264 | .161092D-04 | .165905D-04 | .167567D-04 | 1.11873 |
| 265 - 274 | .133619D-04 | .129763D-04 | .128117D-04 | 1.62501 |
| 275 - 284 | .466465D-04 | .461811D-04 | .459598D-04 | .527230 |
| 285 - 294 | .107578D-03 | .107126D-03 | .106897D-03 | .290625 |
| 295 - 304 | .272661D-03 | .272291D-03 | .272283D-03 | .227377 |

TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL. (BAND FROM 4 CY/REV TO 304 CY/REV)
(A) .9051794792D-01 (B) .9036022736D-01

Table (4.6)

Summary.

The polar orbits were integrated in a purely zonal reference field and over one revolution only, to stay within the available computing resources. The "true" field extended to degree 300, and the field of the nominal orbits to degree 30, while the periodic reference orbit corresponded to the first nine zonals, taken from GEM 9 (together with GM and the mean Earth radius). The height was about 160 km, and two separations were considered, 100 km and 300 km. Because of the limited number of significant figures (14) in the arithmetic of the computer, zonal coefficients much larger than those in the terrestrial field were chosen above degree 9. The numerical integrator behaved reasonably well, but a better one is needed for more realistic simulations, or for a real application. Computers that can work with 20 significant figures in double precision already are widely available, and their use might eliminate all the problems associated with arithmetic rounding errors. The "estimation errors" in the nominal orbits were about 10 m, with along-track velocity errors cancelling out the drift respect to the "true" orbits. The periodical reference orbit was computed following the "closing and circularizing" procedures described in paragraph (2.1). The result had a misclosure, after one revolution, of about 10 microns and 1 micron per second. The mean height was only meters off the desired 160 km, and the swing above and below this height was of some ±10 km. The reference orbital period could be determined with sufficient accuracy to rid the results of numerical artifacts due to imperfect knowledge of the fundamental frequency of the signal. Simulated and analytical lumped Fourier coefficients show an agreement better than 1% at most frequencies up to 304 cycles per revolution, where most of the signal power is confined.

## CONCLUSIONS.

It may be possible to process all the data collected by a satellite pair to obtain a very high resolution map of the global gravity field with the method described in this report if:

(a) The satellites are kept in a tight formation near their common reference orbit by controlling them with their drag-compensating rockets;
(b) the common orbit virtually closes upon itself at the end of a period of time long enough to resolve all the spherical harmonic potential coefficients associated with detectable perturbations;
(c) the stream of data during that period is virtually uninterrupted.

If the last condition cannot be fulfilled, but there is enough data coverage, a global solution may be attempted by the procedure sketched in paragraph (3.11), treating the periodical part of the signal as a function of position, and reducing the aperiodical part by estimating the nominal orbits as accurately as possible, at least regarding the relative motion of the satellites. This may be helped by the great deal of information on the gravity field already available, after the satellite altimeter experiments of the Seventies, by the use of the relative line-of-sight velocity data in the adjustment of the orbit, and by iteration of the whole procedure. If there are not enough data even for this, then a patchwork of local solutions may be the only way that is left. Even if a global solution is possible, local mapping remains an important complementary tool for resolving the finest detail in some of the most anomalous (and scientifically interesting) areas.

The results of the tests described in section 4 support the assumption that the model for the observation equations of the adjustment developed here represents the signal to better than 1% at most frequencies with significant spectral lines. Further tests, involving sectorial and tesseral harmonics, should be carried out to confirm this.

The choice of common orbit with rotational symmetry, and the introduction of arc parameters to take care of the aperiodical perturbations, produce a very sparse normal matrix (be it of ordinary adjustment, of least squares collocation, or of any such method where conditioning affects only the diagonal elements). With a suitable arrangement of the unknowns, it is possible to obtain an arrow pattern that makes feasible finding a solution complete to degree and order 300 with most "main-frame" computers now available, because of the reasonable demands for storage, computing time, and numerical accuracy. This means that the choice of a "super computer", with parallel processing, huge central memory and the latest type of hardware, is optional, though probably advantageous. Regarding numerical accuracy, most operations, with the exception of the integration of the nominal orbits, can be done with 64 bits (8 bytes) floating point arithmetic.

The sparseness of the normal equations does not depend on the actual linearized model, because this property can be obtained with very different models, as previous work has shown. There is the question of whether it is necessary to try and improve the description of the signal any further. My own opinion is that a model good to one percent is good enough for any likely application. Whatever the final choice, the main thing is to make sure that it is truly that good, and this may require more realistic (and ingenuous) probing that what has been done so far.

The approach to the numerical calculation of the Fourier coefficients of the analytical perturbations, avoiding getting entangled in long "literal" formulas, may be of some general interest in satellite geodesy and celestial mechanics. The analytical theory for near-polar, near-circular orbits based on Hill's equations may be useful in studying the orbits of satellites whose purpose is to survey the world as completely as possible, so they are given high inclinations.

Although there are many questions of detail still to be studied, I think, on ending this research, that the data analysis is no longer a major problem. The main obstacles that might be found on the way towards getting accurate and detailed maps of gravity from space are likely to be technological, economical and motivational. Of the three, the third may prove the most important, because, regardless of the comings and goings of governments and their policies, if it is recognized by the scientific community that enough of its members want to have such maps for sufficiently good reasons, this recognition will keep the subject firmly on the agenda.

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## APPENDIX I.

### Solving the variationals.

The variational equations obtained in section 1 were

$$\ddot{\beta}_k = D_{p_k} a_r + 3n_0^2\beta_k - 2n_0\dot{\gamma}_k \qquad (1.3.8)$$

$$\ddot{\gamma}_k = D_{p_k} a_u + 2n_0\dot{\beta}_k \qquad (1.3.9)$$

This two equations are linear and time-invariant (constant coefficients) so they can be solved by the method of the Laplace Transform (LT).
This method converts the problem of solving differential equations into that of solving algebraic equations, which is usually much easier. Before explaining how this is done with (1.3.8-9), some basic points about this method that are needed here shall be mentioned.
The LT of a function of time f(t) defined in the interval $0 < t < \infty$ is

$$F(s) = \int_0^\infty e^{-st} f(t)dt \qquad (AI.1)$$

or, symbolically,

$$F(s) = \mathcal{L}\{f(t)\}$$

where s is an independent complex variable. F(s) is the LT of f(t), and f(t) is the *inverse LT* of F(s); f(t) and F(s) constitute a *transform pair*. Here is a list of transform pairs that are necessary to understand the main reasoning in this Appendix (a and $a_j$ are constants):

| f(t) | F(s) | |
|---|---|---|
| $1$ | $\frac{1}{s}$ | (AI.2a) |
| $t$ | $\frac{1}{s^2}$ | (AI.2b) |
| $af(t)$ | $aF(s)$ | (AI.2c) |
| $\sum_j a_j f_j(t)$ | $\sum_j a_j F_j(s)$ | (AI.2d) |
| $\frac{d^n f(t)}{dt^n}$ | $s^n F(s) - s^{n-1} f(0) - \sum_{k=2}^{n} s^{n-k} \frac{d^{k-1}}{dt^{k-1}} f(0)$ | (AI.2e) |

| f(t) | F(s) | |
|---|---|---|
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ | (AI.2f) |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ | (AI.2g) |
| $e^{at}$ | $\frac{1}{(s-a)}$ | (AI.2h) |
| $\int_0^t g(t)dt$ | $\frac{L\{g(t)\}}{s}$ | (AI.2i) |

If N(s) and D(s) are two polynomials in s of degrees n and d, respectively, and n < d, and if D(s) has both multiple and simple roots, so it can be written, in general, as

$$D(s) = a(s-s_1)^{n_1}(s-s_2)^{n_2}\ldots(s-s_i)^{n_i}\ldots(s-s_j)\ldots(s^2+z_k)\ldots \quad \text{(AI.3)}$$

(where the integer $n_i > 1$ indicates the degree of multiplicity of the ith root, while $s_j$ stands for a single root, and $z_k$ (real) for a pair of complex conjugate roots $iz_k^{\frac{1}{2}}$ and $-iz_k^{\frac{1}{2}}$ ($i = [-1]^{\frac{1}{2}}$)); then the rational function N(s)/D(s) can be written as a sum of partial fractions as follows:

$$\frac{N(s)}{D(s)} = \sum_{p=0}^{n_1-1} \frac{A_{p1}}{(s-s_1)^{n_1-p}} + \sum_{q=0}^{n_2-1} \frac{A_{q2}}{(s-s_2)^{n_2-q}} + \ldots + \sum_{u=0}^{n_i-1} \frac{A_{ui}}{(s-s_i)^{n_i-u}} + \ldots$$

$$\ldots + \frac{A_j}{(s-s_j)} + \ldots + \frac{A_k}{(s-(iz_k^{\frac{1}{2}}))} + \frac{\bar{A}_k}{(s-(-iz_k^{\frac{1}{2}}))} \quad \text{(AI.4)}$$

where $\bar{A}_k$ is the conjugate of $A_k$.
Moreover, in the case of simple roots such as $s_j$, $iz_k^{\frac{1}{2}}$ and $-iz_k^{\frac{1}{2}}$, (regardless of whether they are conjugate of not)

$$A_j = \frac{N(s_j)}{\frac{d}{ds}D(s_j)}, \quad A_k = \frac{N(iz_k^{\frac{1}{2}})}{\frac{d}{ds}D(iz_k^{\frac{1}{2}})}, \text{ etc.} \quad \text{(AI.5)}$$

After these preliminaries one can proceed to solve the variationals. The first step is to take the LTs of the right and left hand sides of each equation using the table of transforms given above. Once this is done, the result is a pair of linear algebraic equations where the LTs of $\beta_k$ and $\gamma_k$ are the unknowns. The starting time from which the equations are integrated is $t_0$, so in order to be able to work in the interval from 0 to $\infty$ one must make the change in variable

$$t' = t - t_0 \tag{AI.6}$$

This change in variable does not modify the shape of the differential equations. Calling

$$B_k(s) = \mathcal{L}\{\beta_k(t')\}$$

$$G_k(s) = \mathcal{L}\{\gamma_k(t')\}$$

$$A_{rk}(s) = \mathcal{L}\{D_{p_k} a_r(t')\}$$

$$A_{uk}(s) = \mathcal{L}\{D_{p_k} a_u(t')\} \quad ,$$

the algebraic transforms of the equations, with $B_k(s)$ and $G_k(s)$ as unknowns, are

$$s^2 B_k(s) - s\beta_k(0) - \dot{\beta}_k(0) = A_{rk}(s) + 3n_0^2 B_k(s) - 2n_0(sG_k(s) - \gamma_k(0)) \tag{AI.7}$$

$$s^2 G_k(s) - s\gamma_k(0) - \dot{\gamma}_k(0) = A_{uk}(s) + 2n_0(sB_k(s) - \beta_k(0)) \tag{AI.8}$$

where use has been made of (AI.2c-e), and $\beta_k(0)$, $\gamma_k(0)$, $\dot{\beta}_k(0)$, $\dot{\gamma}_k(0)$ are the initial conditions for $t' = 0$ (or $t = t_0$).
Their solution is

$$B_k(s) = \frac{s(A_{rk}(s) + s\beta_k(0) + \dot{\beta}_k(0) + 2n_0\gamma_k(0)) - 2n_0(A_{uk}(s) + s\gamma_k(0) + \dot{\gamma}_k(0) - 2n_0\beta_k(0))}{s(s^2 + n_0^2)} \tag{AI.9}$$

$$G_k(s) = \frac{(s^2 - 3n_0^2)(A_{uk}(s) + s\gamma_k(0) + \dot{\gamma}_k(0) - 2n_0\beta_k(0)) + 2n_0 s(A_{rk}(s) + s\beta_k(0) + \dot{\beta}_k(0) + 2n_0\beta_k(0))}{s^2(s^2 + n_0^2)} \tag{AI.10}$$

Two cases must be considered:

(a) $p_k$ *is a component of the initial state of the orbit.* As explained in paragraph 2.4, in this case the forcing terms $D_{p_k} a_r$, $D_{p_k} a_u$ are both identically 0 for all t, so their transforms $A_{rk}(s)$, $A_{uk}(s)$ are zero as well, according to the definition (AI.1). The initial conditions $\beta_k(0)$, $\dot{\beta}_k(0)$, $\gamma_k(0)$, $\dot{\gamma}_k(0)$ are all zero, except for the one related to the component of the initial state $p_k$. Clearly $B_k(s)$ and $G_k(s)$ have the form N(s)/D(s) with the degree of the denominator higher than that of the numerator. The roots in the numerator of $B_k(s)$ are: $s_1 = 0$, $s_2 = \bar{s}_3 = in_0$, where the overbar denotes the conjugate of a complex number. The roots of the denominator of $G_k(s)$ are the same as for $B_k(s)$, except that the 0 root $s_1$ is now a double one. Expanding these functions in partial fractions according to (AI.4):

$$B_k(s) = \frac{B_1}{s} + \frac{B_2}{(s-in_0)} + \frac{\bar{B}_2}{(s+in_0)} \tag{AI.11}$$

$$G_k(s) = \frac{G_{01}}{s^2} + \frac{G_{11}}{s} + \frac{G_2}{(s-in_0)} + \frac{\bar{G}_2}{(s+in_0)} \tag{AI.12}$$

where the $B_1$, $B_2$, $G_{01}$, etc., are constants that depend on $\beta_k(0)$, $\dot{\beta}_k(0)$, $\gamma_k(0)$, $\dot{\gamma}_k(0)$ and $t_0$. Taking the inverse LTs of both members with the help of (AI.2a-2d) and (AI.2h):

$$\beta_k(t') = B_1 + B_2 e^{in_0 t'} + \bar{B}_2 e^{-in_0 t'} \tag{AI.13}$$

$$\gamma_k(t') = G_{01}t' + G_{11} + G_2 e^{in_0 t'} + \bar{G}_2 e^{-in_0 t'} \tag{AI.14}$$

Making use of the basic relationships

$$e^{in_0 t'} = \cos n_0 t' + i \sin n_0 t' \tag{AI.15}$$

$$e^{-in_0 t'} = \cos n_0 t' - i \sin n_0 t' \tag{AI.16}$$

after some manipulations and a change in variable from t' back to $t = t'+t_0$:

$$\beta_k(t) = B_{0k} + B_{1k} \cos n_0 t + B_{2k} \sin n_0 t \tag{AI.17}$$

$$\gamma_k(t) = G_{0k} + G_{1k} \cos n_0 t + G_{2k} \sin n_0 t + G_{3k} t \tag{AI.18}$$

where the new coefficients $B_{0k}$, etc. are functions of those in (AI.13-14). Given that $\beta_k(t)$ and $\gamma_k(t)$ are both real functions, all the coefficients $B_{0k}$, $G_{0k}$, etc. must be real too. Now, after the change in variable from t' back to t, they become not only functions of the initial conditions, but of the starting time $t_0$ as well. As the forcing terms are both 0 in this case, expressions (AI.17-18) give also the general form of the homogeneous response of the system (1.3.8-9).

(b) $p_k$ *is a potential coefficient* $\bar{C}^{\alpha}_{nm}$. In this case, as shown in paragraph 2.4, all the initial conditions are zero, and the forcing terms are given by expressions (2.3.48-49). Therefore, using $B_{nm\alpha}(s)$, $G_{nm\alpha}(s)$ and $A_{rnm\alpha}(s)$, $A_{unm\alpha}(s)$ to denote the LTS of $\beta_{nm\alpha}(t)$, $\gamma_{nm\alpha}(t)$ and $D_{\bar{C}^{\alpha}_{nm}}\, a_r$, $D_{\bar{C}^{\alpha}_{nm}}\, a_u$ (following the notation of paragraph (2.4)):

$$B_{nm\alpha}(s) = \frac{sA_{rnm\alpha}(s) - 2n_0 A_{unm\alpha}(s)}{s(s^2+n_0^2)} \tag{AI.19}$$

$$G_{nm\alpha}(s) = \frac{(s^2-3n_0^2)A_{unm\alpha}(s) + 2n_0 s A_{rnm\alpha}(s)}{s^2(s^2+n_0^2)} \tag{AI.20}$$

according to (AI.9-10). Changing, for the same reason as before, from t to $t' = t-t_0$, expressions (2.3.48-49) become:

$$D_{\bar{C}^{\alpha}_{nm}}\; a_r(t) = \sum_{j=-(n+3N)}^{n+3N} a_{rnmj} \cos(\omega_{jm}t' + \phi_{jm\alpha}) \tag{AI.21}$$

$$D_{\bar{C}^{\alpha}_{nm}}\; a_u(t) = \sum_{j=-(n+3N)}^{n+3N} a_{unmj} \sin(\omega_{jm}t' + \phi_{jm\alpha}) \tag{AI.22}$$

where

$$\omega_{jm} = jn_0 + m\theta' \tag{AI.23}$$

$$\phi_{jm\alpha} = \omega_{jm}t_0 + \hat{\phi}_{m\alpha} \tag{AI.24}$$

and $\hat{\phi}_{m\alpha}$ is defined in paragraph 2.2, expression (2.2.50).
Taking the LTs of the forcing terms with the help of (AI.2d) and (AI.2f-g), since $\cos(\omega_{jm}t'+\phi_{jm\alpha}) = \cos\phi_{jm\alpha}\cos\omega_{jm}t' - \sin\phi_{jm\alpha}\sin\omega_{jm}t'$ and $\sin(\omega_{jm}t'+\phi_{jm\alpha}) = \cos\phi_{jm\alpha}\sin\omega_{jm}t' + \sin\phi_{jm\alpha}\cos\omega_{jm}t'$, then

$$A_{rnm\alpha}(s) = \sum_{j=-(n+3N)}^{n+3N} a_{rnmj} \frac{(s\cos\phi_{jm} - \omega_{jm}\sin\phi_{jm\alpha})}{(s^2+\omega_{jm}^2)}$$

$$A_{unm\alpha}(s) = \sum_{j=-(n+3N)}^{n+3N} a_{unmj} \frac{(s\sin\phi_{jm} + \omega_{jm}\cos\phi_{jm\alpha})}{(s^2+\omega_{jm}^2)}$$

Replacing these last two expressions in (AI.19-20)

$$B_{nm\alpha}(s) = \sum_{j=-(n+3N)}^{n+3N} \frac{[a_{rnmj}s(s\cos\phi_{jm\alpha}-\omega_{jm}\sin\phi_{jm\alpha}) - 2n_0 a_{unmj}(s\sin\phi_{jm\alpha}+\omega_{jm}\cos\phi_{jm\alpha})]}{s(s^2+n_0^2)(s^2+\omega_{jm}^2)} \quad (AI.25)$$

$$G_{nm\alpha}(s) = \sum_{j=-(n+3N)}^{n+3N} \frac{[a_{unmj}(s^2-3n_0^2)(s\sin\phi_{jm\alpha}+\omega_{jm}\cos\phi_{jm\alpha}) + 2n_0 s a_{rnmj}(s\cos\phi_{jm\alpha}-\omega_{jm}\sin\phi_{jm\alpha})]}{s^2(s^2+n_0^2)(s^2+\omega_{jm}^2)} \quad (AI.26)$$

Once more $B_{nm\alpha}(s)$ and $G_{nm\alpha}(s)$ are ratios of polynomials with the denominators of greater degree than the numerators. In the case of $B_{nm\alpha}(s)$ the roots of D(s) are

$$s_1 = 0,\ s_2 = \bar{s}_3 = in_0,\ s_{jm1} = \bar{s}_{jm2} = i\omega_{jm}$$

where $-(n+3N) \leq j \leq n+3N$ and m is fixed.

The denominator of $G_k(s)$ has the same roots, except that the zero $s_1$ is double instead of simple. Expanding in partial fractions

$$B_{nm\alpha}(s) = \frac{B_1}{s} + \frac{B_2}{(s-in_0)} + \frac{\bar{B}_2}{(s+in_0)} + \sum_{j=-(n+3N)}^{n+3N} \frac{B^n_{jm\alpha}}{(s-i\omega_{jm})} + \frac{\bar{B}^n_{jm\alpha}}{(s+i\omega_{jm})} \quad (AI.27)$$

$$G_{nm\alpha}(s) = \frac{G_{01}}{s^2} + \frac{G_{11}}{s} + \frac{G_2}{(s-in_0)} + \frac{\bar{G}_2}{(s+in_0)} + \sum_{j=-(n+3N)}^{n+3N} \frac{G^n_{jm\alpha}}{(s-i\omega_{jm})} + \frac{\bar{G}^n_{jm\alpha}}{(s+i\omega_{jm})} \quad (AI.28)$$

where $B_1$, $B_2$, $\bar{B}_2$ and $G_{01}$, $G_{11}$, $G_2$, $\bar{G}_2$ represent contributions from all the terms of the summations in (AI.25-26).

According to (AI.5) the coefficients of the fractions corresponding to the complex pairs of roots are

$$B^n_{jm\alpha} = \frac{-a_{rnmj}\omega^2_{jm}(\cos\phi_{jm\alpha}+i\sin\phi_{jm\alpha})-2n_0 a_{unmj}\omega_{jm}(\cos\phi_{jm\alpha}+i\sin\phi_{jm\alpha})}{2\omega^2_{jm}(\omega^2_{jm}-n_0^2)} \quad (AI.29)$$

$$G^n_{jm\alpha} = \frac{-2n_0 a_{rnmj}\omega^2_{jm}(\cos\phi_{jm\alpha}+i\sin\phi_{jm\alpha})-a_{unmj}(\omega^2_{jm}+3n_0^2)\omega_{jm}(\cos\phi_{jm\alpha}+i\sin\phi_{jm\alpha})}{2\omega^3_{jm}(\omega^2_{jm}-n_0^2)i} \quad (AI.30)$$

and their complex conjugates.

The inverse LT of (AI.27-28) is

$$\beta_{nm\alpha}(t') = B_1+B_2e^{in_0t'}+\bar{B}_2e^{-in_0t'}+\sum_{j=-(n+3N)}^{n+3N} B^n_{jm\alpha}\, e^{i\omega_{jm}t'}+\bar{B}_{jm}e^{-i\omega_{jm}t'} \qquad \text{(AI.31a)}$$

$$\gamma_{nm\alpha}(t') = G_{01}t+G_{11}+G_2e^{in_0t'}+\bar{G}_2e^{-in_0t'}+\sum_{j=-(n+3N)}^{n+3N} G^n_{jm\alpha}\, e^{i\omega_{jm}t'}+\bar{G}_{jm}e^{-i\omega_{jm}} \qquad \text{(AI.31b)}$$

But

$$\cos\phi_{jm\alpha} + i\sin\phi_{jm\alpha} = e^{i\phi_{jm\alpha}}$$

$$\cos\phi_{jm\alpha} - i\sin\phi_{jm\alpha} = e^{-i\phi_{jm\alpha}}$$

and making use of (AI.29-30)

$$\beta_{nm\alpha}(t')= B_1+ B_2e^{in_0t'}+\bar{B}_2e^{-in_0t'}+\sum_{j=-(n+3N)}^{n+3N} b_{nmj}\left(\frac{e^{i(\omega_{jm}t'+\phi_{jm\alpha})}+e^{-i(\omega_{jm}t'+\phi_{jm\alpha})}}{2}\right) \qquad \text{(AI.32a)}$$

$$\gamma_{nm\alpha}(t')= G_{01}t'+G_{11}+G_2e^{in_0t'}+\bar{G}_2e^{-in_0t'}+ + \sum_{j=-(n+3N)}^{n+3N} g_{nmj}\left(\frac{e^{i(\omega_{jm}t'+\phi_{jm\alpha})}-e^{-i(\omega_{jm}t'+\phi_{jm\alpha})}}{2i}\right) \qquad \text{(AI.32b)}$$

where

$$b_{nmj} = -\frac{(\omega_{jm}\alpha_{rnmj}+2n_0\alpha_{unmj})}{\omega_{jm}(\omega^2_{jm}-n^2_0)} \qquad \text{(AI.33)}$$

$$g_{nmj} = -\frac{(2\omega_{jm}n_0\alpha_{rnmj}+(3n_0^2+\omega^2_{jm})\alpha_{unmj})}{\omega^2_{jm}(\omega^2_{jm}-n^2_0)} \qquad \text{(AI.34)}$$

Moreover

$$\cos(\omega_{jm}t'+\phi_{jm\alpha}) = \frac{e^{i(\omega_{jm}t'+\phi_{jm\alpha})}+e^{-i(\omega_{jm}t'+\phi_{jm\alpha})}}{2} \qquad \text{(AI.35)}$$

and

$$\sin(\omega_{jm}t'+\omega_{jm\alpha}) = \frac{e^{i(\omega_{jm}t'+\phi_{jm\alpha})}-e^{-i(\omega_{jm}t'+\phi_{jm\alpha})}}{2i} \qquad \text{(AI.36)}$$

The summations in (AI.31-32) are the particular integral part of the solution. Using the notation of paragraph (2.4) and (AI.35-36) above:

$$\tilde{\beta}_{nm\alpha}(t) = \sum_{j=-(n+3N)}^{n+3N} b_{nmj} \cos(\omega_{jm}(t'+t_0)+\hat{\phi}_{m\alpha}) \qquad \text{(AI.37)}$$

$$\tilde{\gamma}_{nm\alpha}(t) = \sum_{j=-(n+3N)}^{n+3N} g_{nmj} \sin(\omega_{jm}(t'+t_0)+\hat{\phi}_{m\alpha}) \qquad \text{(AI.38)}$$

because $\phi_{jm\alpha} = \omega_{jm}t_0 + \hat{\phi}_{m\alpha}$. Changing back from t' to $t = t' + t_0$ after replacing (AI.37-38) in (AI.31-32) and of several manipulations of the homogeneous part:

$$\beta_{nm\alpha}(t) = B_{0nm\alpha} + B_{1nm\alpha} \cos n_0 t + B_{2nm\alpha} \sin n_0 t + \sum_{j=-(n+3N)}^{n+3N} b_{nmj} \cos(\omega_{jm}t+\hat{\phi}_{m\alpha}) \qquad \text{(AI.39)}$$

$$\gamma_{nm\alpha}(t) = G_{0nm\alpha} + G_{1nm\alpha} \cos n_0 t + G_{2nm\alpha} \sin n_0 t + G_{3nm\alpha} t + \sum_{j=-(n+3N)}^{n+3N} g_{nmj} \sin(\omega_{jm}t+\hat{\phi}_{m\alpha}) \qquad \text{(AI.40)}$$

This is the complete solution of the variationals for $p_k = \bar{C}^{\alpha}_{nm}$. The coefficients $B_{0nm\alpha}$, etc., and $G_{0nm\alpha}$, etc. are all real, because $\beta_{nm\alpha}(t)$ and $\gamma_{nm\alpha}(t)$ are both real. All these coefficients, with the exception of $b_{nmj}$ and $g_{nmj}$ (as shown in expressions (AI.33-34) are functions of the starting time $t_0$, in order that the initial conditions may be fulfilled. This highlights the independence of $\tilde{\beta}_{nm\alpha}$ and $\tilde{\gamma}_{nm\alpha}$ from $t_0$, which is essential to the adjustment of the potential coefficients in the manner discussed in section 3. Expressions (AI.17-18), (AI.37-38), and (AI.39-40) correspond to (2.4.1a-h) and (2.4.6-11) in paragraph (2.4).

To verify the correctness of (AI.33-34) and (AI.37-38) as expressions of the particular integral terms and their coefficients, the reader can put these terms and their derivatives, together with $D_{\bar{C}^{\alpha}_{nm\alpha}} a_r$ and $D_{\bar{C}^{\alpha}_{nm\alpha}} a_u$ according to (2.3.48-49) in the variational equations and see that these are actually fulfilled (though the initial conditions are not met; that is the function of the homogeneous terms).

The forcing functions include terms of frequencies 0 and $n_0$ for which the analysis just completed breaks down, as the amplitudes $b_{nmj}$ and $g_{nmj}$ become indefinite. This is known as *resonance* and it appears, among other reasons, because the equations of motion from which the variationals have been derived correspond to a drag-free satellite and contain no dissipative terms. Even in the presence of drag the solutions can have very large terms near or at the two critical fequencies. The fact that the general expressions previously obtained become indefinite merely indicates that the terms at 0 and $n_0$ require a separate treatment. In the case of a perfectly circular orbit, paragraph (2.5) shows that the only frequencies present are of the form $(n-2p)n_0 + m\theta'$, with $0 \leq p \leq n$, so only *even zonals* can produce zero frequency terms, and only *odd zonals* can originate terms of frequency $n_0$. As the orbit is not circular, even and odd zonals will contribute to both frequencies.

(A) Zero frequency term:

Each zonal contributes a constant forcing term $a_{rn00}\cos 0t = a_{rn00}$ (there is no $a_{un00}\sin 0t$ term, $\sin 0t = 0$ for all t). Since $a_{rn00}$ is constant, the choice of $t_0$ does not affect the conclusions, so $t_0 = 0$ can be adopted for convenience. Then expressions (AI.19-20) become

$$B^{j=0}_{n00}(s) = s\left(\frac{a_{rn00}}{s}\right)(s(s^2+n_0^2))^{-1} = \frac{a_{rn00}}{s(s^2+n_0)^2} \tag{AI.41}$$

and

$$G^{j=0}_{n00} = \frac{2n_0 a_{rn00}}{3^2(s^2+n_0^2)} \tag{AI.42}$$

where the superscript j = 0 indicates that these relationships correspond to the zero frequency parts of $B_{n00}(s)$ and $G_{n00}(s)$ only.

Expanding in partial fractions and reasoning as before leads to expressions of the form

$$\beta^{j=0}_{n00} = A + B\cos n_0 t \tag{AI.43}$$

and

$$\gamma^{j=0}_{n00} = C + Dt + E\sin n_0 t \tag{AI.44}$$

If $t_0 \neq 0$, t can be replaced with $t - t_0$ directly.

(B) Zonals, frequency $n_0$:

To begin with, the following additional formulas are necessary:

$$\frac{1}{(s^2+n_0^2)^2} = L\left\{\frac{\sin n_0t - n_0t \cos n_0t}{2n_0^3}\right\} \tag{AI.45}$$

$$\frac{s}{(s^2+n_0^2)^2} = L\left\{\frac{t \sin n_0t}{2n_0}\right\} \tag{AI.46}$$

$$\frac{s^2}{(s^2+n_0^2)^2} = L\left\{\frac{\sin n_0t + n_0t \cos n_0t}{2n_0}\right\} \tag{AI.47}$$

$$\frac{s^3}{(s^2+n_0^2)^2} = L\{\cos n_0t - \tfrac{1}{2}n_0t \sin n_0t\} \tag{AI.48}$$

$$n_0^2 \int_0^t t\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}n_0t\ dt = n_0t\begin{Bmatrix}-\cos n_0t\\ \sin n_0t\end{Bmatrix} + \begin{Bmatrix}\sin n_0t\\ \cos n_0t\end{Bmatrix} - \begin{Bmatrix}0\\ 1\end{Bmatrix} \tag{AI.49}$$

Replacing $\omega_{jm}$ with $n_0$ in (AI.25-26):

$$B^{j=1}_{n00}(s) = \frac{s\ a_{rn00}(s \cos n_0t_0 - n_0 \sin n_0t_0) - 2n_0 a_{un00}(s \sin n_0t + n_0 \cos n_0t_0)}{s(s^2+n_0^2)^2} \tag{AI.50}$$

$$G^{j=1}_{n00}(s) = \frac{a_{un00}(s^2-3n_0^2)(s \sin n_0t_0 + n_0 \cos n_0t_0) + 2n_0 s\ a_{rn00}(s \cos n_0t_0 - n_0 \sin n_0t_0)}{s^2(s^2+n_0^2)^2} \tag{AI.51}$$

so $B^{j=1}_{n00}(s)$ and $G^{j=1}_{n00}(s)$ (j=1 means the effect of the fundamental) are of the form

$$B^{j=1}_{n00}(s) = \frac{as^2+bs+c}{s(s^2+n_0^2)^2} = \frac{as}{(s^2+n_0^2)^2} + \frac{b}{(s^2+n_0^2)^2} + \frac{c}{s(s^2+n_0^2)^2}$$

and

$$G^{j=1}_{n00}(s) = \frac{ds^3+es^2+fs+g}{s^2(s^2+n_0^2)^2} = \frac{ds}{(s^2+n_0^2)^2} + \frac{e}{(s^2+n_0^2)^2} + \frac{f}{s(s^2+n_0^2)^2} + \frac{g}{s^2(s^2+n_0^2)^2}$$

From the last two expressions, and (AI.45-49) follows that the inverse Laplace transforms $\beta^{j=1}_{n00}$ and $\gamma^{j=1}_{n00}$ are functions of $t' = t - \tau_0$ (where $\tau_0$ is the instant when the satellite first reaches perigee, i.e. $F' = 0$) of the general form

$$\beta_{n00}^{j=1}(t) = A + B \sin n_0 t' + Bt' \sin n_0 t' \qquad \text{(AI.52)}$$

and

$$\gamma_{n00}^{j=1}(t) = F + Gt' + H \cos n_0 t' + It \cos n_0 t' \qquad \text{(AI.53)}$$

(for the last result, (AI.2i) has been used together with (AI.49) twice). A good reference for the theory and the use of Laplace transforms is the book by Spiegel (1965).

## APPENDIX II.

### Complementary Orbit Theory.

This Appendix completes the treatment of the analytical, first order perturbation theory for near-circular, near-polar orbits introduced in sections 1 and 2. The first part of the Appendix deals with perturbations normal to the plane of the reference orbit, and the second part considers the extension of the theory to include non-periodical reference orbits.

### (I) Perturbations normal to the reference orbital plane.

The perturbations perpendicular to the orbital plane are governed by the variational equation (1.3.7):

$$\ddot{\alpha}_k = D_{p_k} a_z - n_0^2 \alpha_k$$

where

$$a_z = D_z V; \quad \alpha_k = D_{p_k} z$$

and the parameter $p_k$ can be either a component of the initial conditions $z(t_0)$ or $\dot{z}(t_0)$; or it can be a potential coefficient $\bar{C}^{\alpha}_{nm}$. In the first case, the forcing term is zero, and $\alpha_k$, $\dot{\alpha}_k$ describe the free response of a harmonic oscillator of natural frequency $n_0$. In the second case, the forcing term can be developed in a Fourier series, as were those of the in-plane perturbations discussed in section 2.

### AII.1 Fourier development of the forcing terms.

To obtain the Fourier representation of $D_{\bar{C}^{\alpha}_{nm}} a_z$ as a function of time, consider first the spherical harmonic expansion of the potential

$$V(r,\varphi,\lambda) = \sum_{nm\alpha} \bar{C}^{\alpha}_{nm} \bar{X}_{nm\alpha}(r,\varphi,\lambda)$$

where, according to (1.2.3-4)

$$\bar{X}_{nm\alpha}(r,\varphi,\lambda) = \frac{GM}{r} \left(\frac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi)\cos(m\lambda - \alpha\frac{\pi}{2}) \qquad \text{(AII.1.1)}$$

Changing from spherical coordinates to orbital variables (see Fig. (2.2.1) in paragraph (2.2))

$$\bar{X}_{nm\alpha} = GM\, a^n r^{-(n+1)} \sum_{p=0}^{n} F_{nmp}(i) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} ((n-2p)F+mL+\phi_\alpha) \tag{AII.1.2}$$

which follows from (2.2.3) and the similarity between this and (2.2.8). Here, once more,

$$\phi_\alpha = \begin{cases} 0 & \text{if } \alpha = 0 \\ -\frac{\pi}{2} & \text{if } \alpha = 1 \end{cases}$$

Now

$$D_{\bar{C}^\alpha_{nm}}\, a_z = D_{\bar{C}^\alpha_{nm}}\, D_z V = D_{\bar{C}^\alpha_{nm}}\, D_z\, \bar{C}^\alpha_{nm}\, \bar{X}_{nm\alpha} = D_z \bar{X}_{nm\alpha}$$

and

$$dz = r \sin F\, di$$

so

$$\begin{aligned} D_{\bar{C}^\alpha_{nm}}\, a_z &= \frac{1}{r \sin F} D_i\, \bar{X}_{nm\alpha} \\ &= \frac{GM\, a^n r(F)^{-(n+2)}}{\sin F} \sum_{p=0}^{n} D_i F_{nmp}(i) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} ((n-2p)F+mL+\phi_\alpha) \end{aligned} \tag{AII.1.3}$$

where r is a function of F alone, because the orbit is periodical. The forcing term is continuous everywhere and, therefore, square integrable in the interval $0 \leq F \leq 2\pi$. For this reason, it can be expanded in a convergent Fourier series. This is also true of $D_i\bar{X}_{nmp}$, but the factor

$$f_n(F) = \frac{GM\, a^n r(F)^{-(n+2)}}{\sin F}$$

is singular at $F = k\pi$, where k = 0, 1, 2, ..., so $f_n(F)$ has no trigonometric expansion. The product of $f_n(F)$ and $D_i\bar{X}_{nm\alpha}$ is not singular there, because $D_i\bar{X}_{nm\alpha} \to 0$ as $F \to k\pi$ in such way that the limit of the product is finite.

The singularities in $f_n(F)$ present a difficulty if one tries to follow an approach similar to that of paragraph (2.2). To overcome it, consider the "notch function"

$$N(F) = \begin{cases} 1 & \text{if } |F-k\pi| > \varepsilon \quad (k = 0, 1, 2, \ldots) \\ 0 & \text{otherwise} \end{cases} \tag{AII.1.4}$$

(it resembles a horizontal bar with vertical slits or *notches* cut across at regular intervals). The product

$$f_n'(F) = N(F)f_n(F) \tag{AII.1.5}$$

is everywhere bounded and square integrable, if $\lim\limits_{F\to k\pi} N(F)f_n(F)$ is taken as the value of $f_n'(k\pi)$. Therefore, $f_n'$ can be expanded in Fourier series. Moreover, it is an even function of $F' = F-\frac{1}{2}\pi$, and it contains only odd frequency components because it has the property of half-wave antisymmetry:

$$f_n'(F') = -f_n'(F'+\pi)$$

Consequently

$$f_n'(F') = \sum_{q(\text{odd})}^{\infty} f_{nq}' \cos qF'$$

This expansion can be truncated at a sufficiently high value K of q, so

$$f_n'(F') \simeq f_n''(F') = \sum_{q=1\ (\text{odd})}^{K} f_{nq}' \cos qF' \tag{AII.1.6}$$

(which closely resembles (2.2.16a). Introduce the square integrable function

$$D_{\bar{C}_{nm}^{\alpha}}\, a_z'' = f_n''(F') \sum_{p=0}^{n} D_i F_{nmp}(i) \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} ((n-2p)F'+mL+(n-2p)\tfrac{\pi}{2} + \phi_\alpha) \tag{AII.1.7}$$

which can also be expanded in Fourier series for any $\varepsilon > 0$ and all K. From the definition of N(F) it is clear that

$$D_{\bar{C}_{nm}^{\alpha}}\, a_z = \lim_{K\to\infty} D_{\bar{C}_{nm}^{\alpha}}\, a_z''$$

for $|F-k\pi| > \varepsilon$, so $\lim\limits_{\varepsilon\to 0} \lim\limits_{K\to\infty} D_{\bar{C}_{nm}^{\alpha}}\, a_z'' = D_{\bar{C}_{nm}^{\alpha}}\, a_z$ for all F (and all F').

As *both* $D_{\bar{C}_{nm}}^{\alpha}\ a_z''$ *and* $D_{\bar{C}_{nm}}^{\alpha}\ a_z$ *are square integrable*, both the Fourier expansion of $D_{\bar{C}_{nm}}^{\alpha}\ a_z''$ *and its individual Fourier coefficients* become those of $D_{\bar{C}_{nm}}^{\alpha}\ a_z$ in the limit. So, to find the Fourier expansion of $D_{\bar{C}_{nm}}^{\alpha}\ a_z$ it is sufficient to find that of $D_{\bar{C}_{nm}}^{\alpha}\ a_z''$ and then to take limits. Repeating the reasoning of paragraph (2.2) that lead from (2.2.12) and (2.2.16) to (2.2.40), starting from (AII.1.6) and (AII.1.7) one arrives to

$$D_{\bar{C}_{nm}}^{\alpha}\ a_z'' = \sum_{j=-(n+K+2N)}^{n+K+2N} a_{znmj}'' \cos((jn_0+m\theta')t+\phi_{m\alpha}) \qquad \text{(AII.1.8)}$$

where t = 0 when first F' = 0. If H is a sufficiently high positive integer, so the series for $D_{\bar{C}_{nm}}^{\alpha}\ a_z$ can be truncated at n+H just as (2.2.48) was at n+3N, and if $a_{znmj} = \lim_{\varepsilon\to 0} a_{znmj}''$, then

$$\boxed{D_{\bar{C}_{nm}}^{\alpha}\ a_z = \sum_{j=-(n+H)}^{n+H} a_{znmj} \cos((jn_0+m\theta')t+\hat{\phi}_{m\alpha})} \qquad \text{(AII.1.9)}$$

where, for general time and longitude origins $\tau_0$ and $L_0$, $\hat{\phi}_{m\alpha}$ is as defined by (2.3.50). The summation limit, and therefore H, will depend on the degree n and on the eccentricity of the orbit, increasing with both, just as in the case of $D_{\bar{C}_{nm}}^{\alpha}\ a_r$ and $D_{\bar{C}_{nm}}^{\alpha}\ a_u$. In fact, H.≃ 3N is probably more than enough for the applications that are likely to occur. The values of the coefficients $a_{znmj}$ can be obtained in the same way that was shown in paragraph (2.3) for the $a_{rnmj}$ and $a_{unmj}$ of the in-plane forcing terms. First, $D_{\bar{C}_{nm}}^{\alpha}\ a_z$ can be computed at equal time intervals along the numerically integrated reference orbit with the formula

$$D_{\bar{C}_{nm}}^{\alpha}\ a_z(r,\varphi,\lambda) = \frac{1}{r}[D_{\bar{C}_{nm}}^{\alpha}\ D_\varphi V \sin\mu - D_{\bar{C}_{nm}}^{\alpha}\ D_\lambda V \cos\varphi^{-1} \cos\mu] \quad \text{(AII.1.10)}$$

where $\mu$ is the angle between the along-track unit vector $\underline{u}_0^{0'}$ and the local north-pointing unit vector $\underline{s}_0^0$, given by (2.3.6). Finally, the $a_{znmj}$ can be calculated by analyzing the sequence of values of $D_{\bar{C}_{nm}}^{\alpha}\ a_z$ with a Fast Fourier Transform procedure.

AII.2 The analytical solution.

There are two cases: when the parameter to be estimated, $p_k$, is a component of the initial state, and when it is a coefficient $\bar{C}^{\alpha}_{nm}$.

(1) $p_k$ *is a component of the initial state.*
Reasoning as in paragraph (2.4), and using a similar notation, calling $\alpha_{z(t_0)} = D_{z(t_0)}z(t)$, $\alpha_{\dot{z}(t_0)} = D_{\dot{z}(t_0)}z(t)$, and $t' = t-t_0$, where $t_0$ is the beginning of the orbital arc in question,

$$\alpha_{z(t_0)} = \cos n_0 t' \qquad \text{(AII.2.1a)}$$

$$\alpha_{\dot{z}}(t_0) = n_0^{-1} \sin n_0 t' \qquad \text{(AII.2.1b)}$$

Combining the perturbations caused by $\Delta z(t_0)$ and $\Delta\dot{z}(t_0)$:

$$\Delta z(t) = \Delta z(t_0)\cos n_0 t' + \Delta\dot{z}(t_0) n_0^{-1} \sin n_0 t' \qquad \text{(AII.2.2a)}$$

$$\Delta\dot{z}(t) = n_0^{-1}(-\Delta z(t_0)\sin n_0 t' + \Delta\dot{z}(t_0) n_0^{-1} \cos n_0 t') \qquad \text{(AII.2.2b)}$$

(2) $p_k$ *is a potential coefficient.*
The particular integral for a forcing term of the form $h(t) = A\cos(\omega t+\phi)$ is $\tilde{\alpha} = A(n_0^2-\omega^2)^{-1}\cos(\omega t+\phi)$, as it can be verified by replacing $\alpha_k$ with $\tilde{\alpha}$, and $D_{\bar{C}^{\alpha}_{nm}}\, a_z$ with $h(t)$ in the variational equation. So, when the forcing function is the Fourier series in the right side of (AII.1.9), the complete solution is

$$\alpha_{nm\alpha}(t) = A_{1nm\alpha}\cos n_0 t + A_{2nm\alpha}\sin n_0 t + \sum_{j=-(n+H)}^{n+H} a_{nmj}\cos((jn_0+m\theta')t+\hat{\phi}_{m\alpha}) \qquad \text{(AII.2.3)}$$

where

$$a_{nmj} = \frac{a_{znmj}}{(n_0^2-(jn_0+m\theta')^2)} \qquad \text{(AII.2.4)}$$

According to this result, resonance occurs when $(jn_0+m\theta') = n_0$, the natural frequency of the harmonic oscillator. For a planet like the Earth, spinning too fast for $\theta'$ to be zero, and where $m\theta'$ is not a multiple of $n_0$ (at least for n and m not larger than $N_{max}$, the highest degree considered) the only possibility is when $j = \pm 1$ and $m = 0$. Therefore, resonance is, once more, a purely zonal effect. Taking the Laplace transforms of both sides of the variational (1.3.7) for $\alpha = 0$, $m = 0$ and $j = 1$, keeping in mind that always $\alpha_{n00}(t_0) = \dot{\alpha}_{n00}(t_0) = 0$,

$$s^2 A^{j=1}_{n00}(s) = \mathcal{L}\{a_{zn01} \cos n_0\hat{t}\} - n_0^2 A^{j=1}_{n00}(s) \tag{AII.2.5}$$

where $A^{j=1}_{n00}(s) = \mathcal{L}\{\alpha^{j=1}_{n00}(t)\}$, $\alpha^{j=1}_{n00}$ is the component of frequency $n_0$ in $\alpha_{n00}$ and $\hat{t} = t-\tau_0$, $\tau_0$ being the time of the first passage through perigee ($F' = 0$). Since $\mathcal{L}\{\cos n_0\hat{t}\} = s(s^2+n_0^2)^{-1}$, it follows that

$$A^{j=1}_{n00}(s) = a_{zn01} s(s^2+n_0^2)^{-2} \tag{AII.2.6}$$

Applying the inverse Laplace transform to both sides

$$\alpha^{j=1}_{n00}(t) = \frac{a_{zn01}}{2n_0} \hat{t} \sin n_0\hat{t} \tag{AII.2.7}$$

This expression is always zero at perigee and apogee, where $F' = n_0\hat{t} = k\pi$, $k = 0, 1, 2, \ldots$, and greatest at the nodes, where $n_0\hat{t} = k\,\frac{1}{2}\pi$, $k = 1, 3, 5, \ldots$ This produces a secular increase of the distance of the nodes of the true orbit from those of the reference orbit, and since the inclination of the major axis of the mean ellipse does not change, this is equivalent to a rotation of the orbital plane about the Earth's main axis of inertia (which in this work is regarded as identical to the spins axis, and is called merely "the Earth's axis"). For a polar orbit, $i = \pm\frac{1}{2}\pi$, so the $D_i F_{nmp}(i)$ in (AII.1.3) become all zero, and so does $D_{\bar{C}^{\alpha}_{nm}} a_z$, its Fourier coefficients, including $a_{zn01}$, and the precession rate $a_{zn01}/2n_0$. This agrees with the absence of forces excerted by the zonals perpendicular to a meridian plane, so such a plane does not precess. At any other inclination, this precession is influenced mostly by the even zonals, particularly in near-circular orbits. The reason for this is that for a circular orbit $F' = n_0 t$ at all time, assuming $\tau_0 = 0$ for convenience. Replacing $F'$, together with $f''(t)$ according to (AII.1.6) in (AII.1.7) results in the expression, for m = 0 (after taking limits, etc.),

$$D_{\bar{C}_{nm}}^{\alpha}\ a_z(t) = \sum_{j=-(n+H)}^{n+H} \alpha_{zn0j} \cos jn_0t \qquad \text{(AII.2.8)}$$

where $j = (n-2p\pm q)$, $0 \leq p \leq n$ and $q = 1, 3, 5, 7, \ldots$ Here even zonals produce odd harmonics of $n_0$, and odd harmonics even ones. The resonant frequency $n_0$ is an odd harmonic, so only even zonals can cause resonance, and precession. For non-circular orbits this is no longer quite true, but the effect of the even zonals prevails as long as the orbits are not too eccentric.

(II) Non-periodical reference orbit.

In the orbital theory developed so far, a constant assumption has been that the reference orbit is closed and periodical in its own plane. This property is essential to the efficient estimation of the potential coefficients, as explained in section 3, because of the symmetries that then appear in the formulation of the satellite-to-satellite tracking problem. But a periodical orbit was not needed to derive the variationals in paragraph (1.3), so these can be the basis for a theory that can deal with more general problems, as long as the orbits are near-circular and have high inclinations, as both properties are required for the validity of Hill's equations.

## AII.3 Generalization of the theory.

If the true orbit cannot be fitted by a periodical reference orbit closely enough to use linear perturbation theory, a non-periodical reference must be employed. Such a reference cannot, in general, be obtained by numerical integration because this would have to be done over long periods of time, which creates a number of practical problems, and also because its shape could be too complicated for satisfactory mathematical treatment. So, instead of an integrated orbit, one may use a line in space that follows the true orbit close enough for long enough, but need not be an orbit at all. A possibility ready at hand is the use of the secularly precessing Keplerian ellipse, corresponding to the combined effects of the central force term and the secular perturbations caused by the powerful second zonal, that is employed in classical treatments such as the "variation of constants" approach. This choice permits a relatively simple mathematical development of the forcing terms of the variationals at the price of some loss in accuracy. The ellipse in question has fixed eccentricity $e$ and semimajor axis $a$, and lies in a plane of constant inclination $i$, precessing at a constant rate according to expression (1.3.10). The "unperturbed position" of the satellite along this "orbit" passes through its perigee at equal intervals, and F' (the angle between that position and the semimajor axis measured from perigee), or "true anomaly", varies periodically with a fundamental angular frequency $n_0 = 2\pi\ T_0^{-1}$, where the orbital period $T_0$ is the interval between two consecutive crossings of the perigee. The major axis, for its part, turns slowly in the plane of the ellipse about a focus fixed at the geocenter, so its angular distance to the ascending node, or argument of perigee $\omega$, varies at a steady rate and, above the "critical inclination", in the same direction as the motion of the satellite. If $\Delta\omega = \omega - \frac{\pi}{2}$, and L is the longitude of the node,

$$\Delta\omega = \dot{\omega}(t-\tau_0) \qquad \text{(AII.3.1a)}$$

$$L = \theta'(t-\tau_0) \qquad \text{(AII.3.1b)}$$

$$F' = n_0 t + (\text{periodic terms}) \qquad \text{(AII.3.1c)}$$

where $\tau_0$ is the time when perigee is first reached, and the constant angular rates $n_0$ and $\dot{\omega}$ are (a is the Earth's equatorial radius, $n_0$ is usually known as $\dot{M}$)

$$n_0 = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}}\left[1 - \frac{\bar{C}_{20}^{0}\sqrt{45}\ a^2}{4(1-e^2)^{2/3}a^2}\,(3\cos^2 i-1)\right] \qquad \text{(AII.3.2a)}$$

$$\dot{\omega} = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}}\frac{\bar{C}_{20}^{0}\sqrt{45}\ a^2}{4(1-e^2)^2 a^2}\,(1-5\cos^2 i) \qquad \text{(AII.3.2b)}$$

(see, for example, Kaula, expressions (3.74)). Moreover, because of the symmetry of the ellipse about its semimajor axis, and the antisymmetry of the reference velocity of the satellite respect to the same axis,

$$r^{-(n+2)} = \sum_{q=0}^{N} h_{nq} \cos qF' \qquad \text{(AII.3.3)}$$

and

$$F' = n_0\hat{t} + \sum_{i=1}^{N} f_i \sin i n_0\hat{t} \qquad \text{(AII.3.4)}$$

(where $\hat{t} = t-\tau_0$ and $h_{n0} \simeq a^{-(n+2)}$ for small $e$) which look the same as (2.2.16a) and (2.2.32), with N being in this case a convenient upper limit for truncation, increasing as n and the eccentricity do. The precessing ellipse can be used to derive Hill's variationals in much the same way as this was done in section 1, provided, once more, that the orbit is near-circular (small $e$) and near-polar ($i$ close to $\pm\frac{1}{2}\pi$). The rotating frame (z,r,u) must turn at the constant rate $n_0$, and the $\vec{r}$, $\vec{u}$ axes must remain in the plane of the ellipse, so they precess along with it, as before. The solution of the variationals for perturbations of the initial state are exactly as before, because Hill's equations do not depend, according to their approximate derivation, on the exact shape of the near-circular orbit. To find the analytical solutions corresponding to the potential coefficients $\bar{C}^{\alpha}_{nm}$, it is necessary to start by developing the forcing terms $D_{\bar{C}^{\alpha}_{nm}} a_z$, $D_{\bar{C}^{\alpha}_{nm}} a_r$, $D_{\bar{C}^{\alpha}_{nm}} a_u$ in Fourier series. To show how this can be done, consider $D_{\bar{C}^{\alpha}_{nm}} a_r$. Expression (2.2.12), from which the corresponding derivation for the periodical orbit started, is quite general and can be used here as well. If $\omega \neq \frac{1}{2}\pi$ (equality is allowed only in the periodical case) then $\Delta\omega \neq 0$, and $F = F'+\Delta\omega+\frac{\pi}{2}$. Replacing this in (2.2.12),

$$D_{\bar{C}^{\alpha}_{nm}} a_r' = r^{-(n+2)} \sum_p \tilde{\alpha}_{rnmp} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\begin{matrix}n-m \text{ even}\\ n-m \text{ odd}\end{matrix} \left((n-2p)(F'+\Delta\omega+\frac{\pi}{2})+mL+\phi_\alpha\right) \tag{AII.3.5}$$

where $\phi_\alpha = \begin{cases} 0 & \text{if } \alpha=0 \\ -\frac{\pi}{2} & \text{if } \alpha=1 \end{cases}$. The *actual values* of the $\tilde{\alpha}_{rnmp}$ are, of course, different because the reference is no longer the same, but the form of the expression remains. Reasoning as in paragraph (2.2), from (AII.3.3) and (AII.3.5) follows, just as (2.2.27) from (2.2.12) and (2.2.16a)

$$D_{\bar{C}^{\alpha}_{nm}} a_r' = \sum_{k=-(n+N)}^{n+N} \sum_{p=0}^{n} \bar{\alpha}_{rnmkp} \cos(kF'+(n-2p)\Delta\omega+mL+\phi_{m\alpha}) \tag{AII.3.6}$$

where $\phi_{m\alpha} = -\alpha\,\frac{\pi}{2} - (1-(-1)^m\,\frac{\pi}{4})$, as before, and the summation with respect to p has been retained to take care of the extra term in the arguments, corresponding to $\Delta\omega$. Changing from F', L, and $\Delta\omega$ to their time-dependent forms according to (AII.3.1a-c), repeating the steps leading from (2.2.27) to (2.2.48-50), and carrying out the additional derivation for the out-of plane perturbations along the lines of paragraph (AII.1), one arrives at

$$D_{\bar{C}_{nm}^{\alpha}}\, a_r(t) = \sum_{j=-(n+K)}^{n+K} \sum_{p=0}^{n} a_{rnmjp} \cos((jn_0+(n-2p)\dot{\omega}+m\theta')t+\hat{\phi}_{m\alpha}^{np}) \qquad \text{(AII.3.7)}$$

$$D_{\bar{C}_{nm}^{\alpha}}\, a_u(t) = \sum_{jp} a_{unmjp} \sin((jn_0+(n-2p)\dot{\omega}+m\theta')t+\hat{\phi}_{m\alpha}^{np}) \qquad \text{(AII.3.8)}$$

$$D_{\bar{C}_{nm}^{\alpha}}\, a_z(t) = \sum_{jp} a_{znmjp} \cos((jn_0+(n-2p)\dot{\omega}+m\theta')t+\hat{\phi}_{m\alpha}^{np}) \qquad \text{(AII.3.9)}$$

where K increases with $e$ and with n, $\sum_{jp} \equiv \sum_{j=-(n+k)}^{n+k} \sum_{p=0}^{n}$, and

$$\phi_{m\alpha}^{np} = \phi_{m\alpha}-(jn_0+(n-2p)\dot{\omega}+m\theta')\tau_0 \qquad \text{(AII.3.10)}$$

The coefficients $a_{rnmjp}$, $a_{unmjp}$ and $a_{znmjp}$ are independent of $\theta'$, $\dot{\omega}$ and $\tau_0$, and $L_0$, so they can be calculated as explained in paragraph (2.3). Using these expansions of the forcing terms, the corresponding analytical solutions of the variationals have the same form as before (expressions (2.4.6-9) and (AII.2.3-4)), the only differences being the need for an extra summation with respect to p in each formula, the replacement of the frequencies $(jn_0+m\theta')$ by $(jn_0+(n-2p)\dot{\omega}+m\theta')$, and the extra phase-shifts $(n-2p)\dot{\omega}\tau_0$. Because of the new terms $(n-2p)\dot{\omega}$, which can be zero, regardless of $\dot{\omega}$, only if $p = \frac{1}{2}n$, even zonals can still produce resonant effects at the 0 and $n_0$ angular frequencies, but not the odd zonals. For the near-polar orbits to which this theory applies, however, $\dot{\omega}$ is always quite small, so the odd zonals can still cause strong near-resonances.

The coefficients $a_{rnmjp}$, $a_{unmjp}$, $a_{znmjp}$, when derived in the way outlined here, appear as related to the inclination functions $F_{nmp}(i)$, to its derivatives, and to the Fourier coefficients $h_{nq}$ and $f_i$, of $r^{-(n+2)}$ and $F'(t)$, respectively. Another "literal" formulation could be obtained, using the classical analytical perturbations of keplerian elements, in terms of inclination functions, eccentricity functions $G_{npq}(e)$ (see Kaula), and their derivatives. Putting those perturbations together to describe the variations in (z, r, u) leads, eventually, to the same type(1) of

(1)The "classical" method is based on a first order approximation to the *nonlinear perturbation equations* of Lagrange; the one introduced here is built around Hill's approximation to the *linearized equations of motion*. Both are different ways of linearizing the problem which should give, with the same elliptical primary, or "reference line", expressions of the same form for $\alpha$, $\beta$ and $\gamma$, but slightly different numerical results (as long as the eccentricity is small and the inclination high).

formulas as the present theory, but with the Fourier coefficients given as *explicit* functions of the eccentricity, while in the present one they are not. The use of any "literal" approach is that, though not as suitable for numerical computations as a method like the one proposed in paragraph (2.3), it does allow further insight into what happens when the orbit parameters are modified.

## APPENDIX III.

### Description and listing of computer programs.

This Appendix explains and lists the main programs used for obtaining the numerical results given in section 4 and in appendix IV. Because of lack of time, the programs were commented very suscintly or not at all, so the following paragraphs, each dedicated to an individual program, contain a brief "running comment" on the function of the various segments of the code, mostly in the actual order in which they appear in the listing. The line numbers referred to are those in the column to the right of the FORTRAN statements. The actual listing appears at the end of each paragraph.
Broadly, the programs (and associated subroutines) fall in two classes: those used to compute the closed reference orbit by the method described in paragraph (2.1), and those used to first simulate and then analyze the line-of-sight velocity between two satellites and to compare the results of the numerical simulation (in the form of "lumped coefficients") with those of the analytical theory of sections 2 and 3 and appendices I and II.

### AIII.1 Program for closing the reference orbit.

This program implements the procedure described in paragraph (2.1), equations (2.1.8-12). It begins by using the values of the first 9 zonals, of GM, and of the mean Earth's radius "AE" to find the initial conditions (or state) for a satellite whose mean height is "HS". This is done by equating the initial (osculating) Keplerian elements to the mean elements of a "frozen" orbit using Cook's theory (equations (2.1.3-4)) and then changing them into Cartesian, equatorial, inertial coordinates, resulting in simpler equations of motion that are also easier to integrate numerically. The values of the zonals are defined in the DATA statement, lines 15 and 16 in the listing.
The chosen integration step, in seconds, is "H", and "EPS", "TELEM", and "INIT" are variables used by the orbit integration subroutines and have to be initialized before these are called. These subroutines are described in paragraph (AIII.5). The same applies to "IR" and "DSR2I". "NMAX" is the highest degree of the spherical harmonic expansion of the field considered, in this case NMAX = 9. "DTOUT" is the interval, in seconds, between two consecutive states of the satellite listed in the

printout. "A" is the mean geocentric distance of the spacecraft, and "DINCL" its initial argument of perigees in degrees, here chosen as $90^{o}$ to meet the condition for a "frozen" orbit. Statements 45 through 47 initialize certain arrays used by the subroutine "LEGEND", which computes the normalized Legendre polynomials needed to calculate the force when integrating the orbit. Statements 51 to 68 compute the initial conditions, first as Keplerian elements (according to Cook's theory) and then convert these to Cartesian form.

The procedure for closing the orbit is iterative, but, because of insufficient time to develop it properly, this program can only carry out one full iteration every two runs, and, the changes in initial conditions obtained from one run must be entered "by hand" in the following one, which is done here in statements 70-72. In the first run, both 70 and 71 are "commented" (adding a "C" in column 1) to render them unoperative, while "DRO" (the correction to the radius) is set to 0. In the next run, DRO is set to 1 km, and the difference $\Delta\dot{r}_0$ between the values of the radial velocity at the point where $F' = \frac{1}{2}\pi$, obtained from each run, is used to evaluate $\frac{d\dot{r}_\pi}{d\dot{y}_0} \simeq \frac{\Delta\dot{r}_\pi}{\Delta\dot{y}_0} = \frac{\Delta\dot{r}_\pi}{\Delta r_0 n_0}$, which then is used according to (2.1.10b) to find the "actual" correction DYDO to the velocity, and the corresponding correction DRO to the initial radial distance (equation (2.1.10a)).

The value of the angular frequency $n_0$ of the orbit (or "WO") is initially set to the approximate value $n_0 = \frac{GM}{r_0^3}$, and later changed by the program to an updated value corresponding to the estimated half-period of the actual orbit obtained in line 140. At the start of a new iteration, in lines 70 and 71 the values of geocentric distance and initial velocity are set equal to those obtained from the previous iteration, DRO is set once more to 0, and the whole procedure is then repeated. This awkward method could be avoided by adding a few additional lines of coding in order to "close the loop" and let the whole iterative procedure repeat itself automatically until it meets the convergence criterion. I would have done this if I had had a few extra weeks to polish the software (written somewhat in haste to meet a deadline) but this was not the case.

Statements 79 to 92 initialize the orbit computation loop that follows, adding the latest corrections to the initial position and velocity and estimating the orbital period "TORB" according to the approximate angular frequency W0. The half period "TT" and the three-quarter period "TIT" are obtained from this, and also the radial distance and velocity at the starting time are computed and then printed out, together with the Cartesian initial conditions. The main loop, where the orbit is calculated and the middle point and the final point estimated, to determine the value of $\dot{r}_\pi, T_0/2$ and also the final misclosure (to see if the procedure has converged to a virtually closed orbit) starts at line 93 and runs right to the end, in line 180. The integration over a time increment H is done by subroutine "COWELL". The updated state appears in arrays "X" and "XP" (for Cartesian components of position and of velocity, respectively); X(1), X(2), X(3) correspond to "x", "y", "z", and similarly for XP. Radial distance and total velocity are computed (lines 86-87) to be printed out, together with the full state and time, at points along the orbit "DTOUT" seconds apart (see lines 38-39). Setting INIT = 1 causes the integrator to skip the self-starting procedure after the orbit computation has got under way. Lines 103 to 105 test whether the orbit has reached either the critical neighbourhood of the mid-point, or it is getting close to the end. The flag "MIAU" is set to 0 at the beginning, so the procedure concentrates on finding the mid-point. Once this is done, MIAU = 1 causes the program to search for the end-point, and to estimate the misclosure by comparing it to the starting point. Regardless of whether the search is for the mid- or for the end-point, the method is the same: sense the first time when the satellite has gone beyond the point in question, reverse its motion by changing the sign of the integration step, and divide this step by 10; then proceed to integrate in reverse until the point has been just crossed once more, and repeat the procedure going to and fro with finer and finer sampling in time, until the estimated crossing time changes by less than the desired accuracy from one turn to the next. The actual criterion for deciding on which side of the critical point the satellite happens to be, a geometrical reasoning that is valid for high values of the inclination, is explained in the comments of lines 115-124.

The results, printed according to lines 151-152 and 178-179, consist of "RDOTM", or $\dot{r}_\pi$, "TM", or the time $T_0/2$ where $F' = \frac{\pi}{2}$, "RO", "VO", "RF", "VF", or the initial ("O") and final ("F") values of the geocentric distance and the total velocity; "VZF" is $\dot{z}$, which should be 0 for an ideally closed, symmetrical orbit. Therefore, a comparison of the initial geocentric distance, velocity, and $\dot{z}$ with their final values measures the misclosure. "RMX", "RMN", "RAV", "DIFF" are the maximum, minimum and average radial distance (obtained on the first half of the orbit, on the assumption that it is always nearly symmetrical), and the difference between the greatest and the smallest height, or vertical "swing" of the trajectory. These last values indicate the departure of the orbit from circularity, as well as from the desired mean height. The intermediate values of TM, the half-period, and of $\Delta\dot{r}_\pi$, needed to see how much they change with every integration reversal, are printed according to lines 143-144. In the listing given here, only two reversals per critical point are allowed (lines 160 and 164). Array "R" serves to store consecutive values of the geocentric distance during the first half of the orbit, to obtain RMX, RMN, etc. "CC" stores the given values of the zonals, later to be transferred (via a copy to array "CN" in COMMON "COEFFS") to the integrating subroutines. Arrays "RLEGO", "RLEG1", "RLNN" (line 14) are required by subroutine LEGEND (lines 46-47). Statement 74 ensures that the mean value of the orbital radius does not change from iteration to iteration.
The program calls subroutines LEGEND and COWELL (see their description in paragraph AIII.5).

```
//GDFGOSCA  JOB  (3114,81),OSCAR,TIME=(0,10)
/*JOBPARM LINES=2,INFORM,Q=F
//    EXEC FTG1CG
//FORT.SYSIN DD *
C
C             PROGRAM FOR CLOSING ORBIT
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(27,9),XP(27,9),CC(310)
      COMMON/GEOCON/GM,AE
      COMMON/INTEG/H,EPS
      COMMON/IO/IO1,IN1,IN2,IN3,IN4,IN5,IN6,IN7
      COMMON/COEFFS/ CN(310),DSR2I,IR,NMAX
      DIMENSION RLEGO(35),RLEG1(35),RLNN(35),R(1000)
      DATA CC/1.D0,0.D0,-484.166D-6,.958475D-6,.541539D-6,.0684389D-6,
     *-.151207D-6,.0933127D-6,.0509491D-6,.027331D-6,300*0.D0/
C
C             KIM = 1  WHEN CLOSING THE ORBIT (MID-POINT CHECK);
C             KIM = 0  WHEN CHECKING CLOSURE   (END-POINT CHECK).
C
      IO1 = 6
      NCOUNT = 0
      PI = 4.D0*DATAN(1.D0)
      DO 1     N = 1,310
    1 CN(N) = CC(N)
      H = 12.5D0
      EPS = 1.D-12
      TELEM = 0.D0
      INIT = 0
      GM = .39860047D15
      AE = 6378139.D0
      HS = 160016.D0
      T = TELEM
      KIM = 1
      IR = 0
      NMAX = 9
      DSR2I = 1.D0/DSQRT(2.D0)
      DTOUT = 1000.D0
      INT = DTOUT/H+1.D-5
      A = AE+HS
      DINCL = 90.D0
      DINC = DINCL*PI/180.D0
      DCI = DCOS(DINC)
      DSI = DSIN(DINC)
      M = 1
      CALL LEGEND(M,DCI,DSI,RLEG1,NMAX,IR,RLNN)
      CALL LEGEND(M,0.D0,1.D0,RLEGO,NMAX,IR,RLNN)
C
C             CHOOSE INITIAL CONDITIONS FOR "FROZEN ORBIT".
C
      W0 = DSQRT(GM/A**3)
      DK = 3.D0*DSQRT(5.D0)*W0*CN(3)*(AE/A)**2*(1.D0-5.D0/4.D0*DSI**2)
      DC = 0.D0
      DO 5       N = 3,NMAX,2
      N1 = N+1
      DC = DC+CN(N1)*(AE/A)**N*(N-1)/DSQRT(2.D0*N+1.D0)*RLEGO(N+1)
     * *RLEG1(N+1)
    5 CONTINUE
      DC = DC*0.5D0*W0
      E = DC/DK
```

```
      RP = A*(1.D0-E)
      X(1,5) = DSQRT(RP**2-(RP*DSI)**2)
      X(2,5) = 0.D0
      X(3,5) = RP*DSI
      XP(1,5) = 0.D0
      XP(2,5) = -DSQRT(GM/A*(1.D0+E)/(1.D0-E))
      XP(3,5) = 0.D0
C
C           CHOOSE PERTURBATIONS TO INITIAL CONDITIONS.
C
      X(3,5) = 6526460.3518460D0
      XP(2,5) = -7812.9679321953D0
      DYD0 = -0.015257583UD0
      DR0 = DYD0/W0
C ***********************************************
      DZ = 0.1D0*A
C
C           ORBIT COMPUTATION
C
      X(1,5) = X(1,5)+DR0*DCI
      X(3,5) = X(3,5)+DR0*DSI
      XP(2,5) = XP(2,5)+DYD0
      DISCOL = 0.D0
      MIAU = 0
      TORB = 2.D0*PI/W0
      R0 = DSQRT(X(1,5)**2+X(2,5)**2+X(3,5)**2)
      V0 = DSQRT(XP(1,5)**2+XP(2,5)**2+XP(3,5)**2)
      TT = 0.25*TORB
      TIT = 3.D0*TT
      WRITE(IO1,11) (X(I,5),I=1,3),(XP(I,5),I=1,3),R0,V0
   11 FORMAT(//' INITIAL CONDITIONS'//,(1X,3G20.14))
      NINT = (TORB+500.D0)/H+1.D-5
 1000 DO 100      NI = 1,NINT
      CALL COWELL(X,XP,TELEM,T,NCA,INIT)
      INIT = 1
      IP = T/H+1.D-5
      R0 = DSQRT(X(1,5)**2+X(2,5)**2+X(3,5)**2)
      V = DSQRT(XP(1,5)**2+XP(2,5)**2+XP(3,5)**2)
      IF(NCOUNT.EQ.0) R(NI) = R0
      IF((IP/INT)*INT.EQ.IP) WRITE(IO1,12) (X(I,5),I=1,3),(XP(I,5)
     1 ,I=1,3),R0,V,T
   12 FORMAT(//(1X,3G20.14))
      IF(T.GT.TT.AND.DABS(A)-DABS(X(3,5)).LT.DZ) GO TO 20
      GO TO 100
   20 IF(T.LT.TIT.AND.MIAU.EQ.1) GO TO 100
C
C           FIND THE  F' = PI   AND  F' = 2*PI  CROSSINGS.
C
      PNI1 = 1.D0
      C1 = -X(1,5)
      C2 = -XP(1,5)
      DETE = (X(2,5)*XP(3,5)-X(3,5)*XP(2,5))
      PNI2 = (C1*XP(3,5)-X(3,5)*C2)/DETE
      DISC = (-X(1,5)*PNI2+X(2,5)*PNI1)/DSQRT(PNI2**2+PNI1**2)
C
C           THE MERIDIAN PERPENDICULAR TO THE ORBITAL PLANE
C           MUST CONTAIN THE NORMAL TO THE LATTER. THE
C           PROJECTION OF THIS NORMAL ON THE  (X,Y) PLANE
C           IS THE TRACE OF THAT MERIDIAN ON (X,Y). FOR
C           LARGE INCLINATIONS, THE SINE OF  F'-PI  IS
```

```
C           PROPORTIONAL TO THE PROJECTION OF THE POSITION
C           VECTOR OF THE SATELLITE ONTO THE NORMAL TO THE
C           TRACE OF THE MERIDIAN ON (X,Y),(-PNI2,PNI1) .
C
C
C           INTERPOLATE  RADIAL VELOCITY  RDOT, R, V .
C
      RDOT = (X(1,5)*XP(1,5)+X(2,5)*XP(2,5)+X(3,5)*XP(3,5))/RO
      IF(DISC*DISCOL.LT.0.D0) GO TO 30
      DISCOL = DISC
      RDOL = RDOT
      ROL = RO
      VOL = V
      VZOL = XP(3,5)
      GO TO 100
   30 DTI = H*DISCOL/(DISCOL-DISC)
      RDOTF = RDOL+(RDOT-RDOL)/H*DTI
      IF(MIAU.EQ.1) GO TO 50
      IF(KIM.EQ.0) MIAU = 1
      TM = T-H+DTI
      WO = PI/TM
      RDOTM = RDOTF
      WRITE(IO1,222) RDOTM,TM
  222 FORMAT(//' RDOTM,TM =',2G20.14)
      IF(KIM.EQ.1) GO TO 110
      GO TO 60
   50 RF =ROL+(RO-ROL)/H*DTI
      VF = VOL+(V-VOL)/H*DTI
      VZF = VZOL+(XP(3,5)-VZOL)/H*DTI
      TF = T-H+DTI
      WRITE(IO1,51) RDOTM,TM,RO,VO,TF,RF,VF,VZF
   51 FORMAT(//' RDOTM = ',G20.14//' TM,RO,VO,TF,RF,VF,VZF = '/,
     * (1X,4G20.14))
      GO TO 110
   60 RDOL = RDOT
      ROL = RO
      VOL = V
      DISCOL = DISC
  100 CONTINUE
  110 IF(NCOUNT.GT.2) STOP
      H = -H/10.D0
      INIT = 0
      NCOUNT = NCOUNT+1
      RDOL =RDOT
      ROL = RO
      VOL = V
      DISCOL = DISC
      IF(NCOUNT.GT.1) GO TO 1000
      NK = T/(H*10.D0)+1.D-5
      RMX = -1.D0
      RMN = 1.D40
      DO 120 N = 1,NK
      IF(R(N).GT.RMX) RMX = R(N)
      IF(R(N).LT.RMN) RMN = R(N)
  120 RAV = RAV+R(N)
      RAV = RAV/NK
      DIFF = RMX-RMN
      WRITE(IO1,115) RMX,RMN,RAV,DIFF
  115 FORMAT(//' RMX,RMN,RAV,DIFF ='/,1X,4G20.14)
      GO TO 1000
      END
```

## AIII.2 Program for "circularizing" the reference orbit.

This program takes the initial conditions of the closed orbit, obtained with the routine described in the previous paragraph, to begin a search for another orbit which is still closed but somewhat more circular than the initial one, and has also the prescribed mean height, a condition the original closed orbit generally does not fulfill. To this effect it uses the same orbit integration subroutines and the same field of 9 zonals as the previous program. The successive values of the geocentric distance along the first half of the orbit, computed every H seconds, is stored in the array "RS". The corrections to the initial state $\Delta\dot{y}_0$ and $\Delta r_0$ are called DYDO and DRO, respectively, and are computed in the rotating frame of coordinates that turns with the satellite, instead of in the quasi-inertial frame to which equations (2.1.11-14) correspond. This is done to simplify the numerical integration of the orbit, which is carried out in the loop formed by statements 73 to 83. Here also the state (position and velocity) is printed out every 50 integration steps (see statements 46-47 and 79-82). The initial conditions are printed just before the loop starts (lines 71-72). Statements 84 to 98 form a loop where (a) the normal equations for the adjustment of $r_0$, $\dot{y}_0$ to minimize the quadratic functional $\phi$, which is the measure of the overall departure from circularity, are formed; (b) the maximum and the minimum geocentric distances along the orbit are found. The adjustment proper takes place in statements 99-102. Then the maximum and minimum geocentric distances, the corrections to the initial radius and velocity, the maximum swing in height, the height bias, and the rms of the departure from circularity along the orbit are printed out before going back to start a new iteration. Altogether three iterations are allowed in the program as coded (statements 19, 103 and 110). Statements 22-24 in the initial segment define the values of $D_{\dot{y}_0}\dot{r}_\pi$, $D_{r_0}\dot{r}_\pi$ and of their ratio, which is needed to ensure that the "closedness" constraint (2.1.12) is satisfied. This program invokes subroutine "COWELL" and its associates to perform the numerical integration.

As the "closedness" constraint is only a linear approximation to a non-linear condition, the resulting "circularized" orbit may have a larger misclosure than the original one, so the "closing" procedure may have to be applied once more, the output of this program becoming the input to the one in the previous paragraph.

```
//GDFGOSCA   JOB   (3114,81),OSCAR,TIME=(0,10)
/*JOBPARM LINES=2,INFORM,Q=F
//    EXEC FTG1CG
//FORT.SYSIN DD *
C
C             PROGRAM FOR CIRCULARIZING THE REF. ORBIT
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(27,9),RS(1000),XP(27,9),CC(310)
      COMMON/GEOCON/GM,AE
      COMMON/INTEG/H,EPS
      COMMON/IO/IO1,IN1,IN2,IN3,IN4,IN5,IN6,IN7
      COMMON/COEFFS/ CN(310),DSR2I,IR,NMAX
      DATA CC/1.D0,0.D0,-484.166D-6,.958475D-6,.541539D-6,.0684389D-6,
     *-.151207D-6,.0933127D-6,.0509491D-6,.027331D-6,300*0.D0/
      IO1 = 6
      DO 1     N = 1,310
    1 CN(N)  = CC(N)
      NCOUNT = 0
      DYD0 = 0.D0
      DR0 = 0.D0
      DYORM = 0.5152D-3/0.1062D0
      DRORM = -0.29128D-5
      DKIN = -DYORM/DRORM
      PI = 4.D0*DATAN(1.D0)
      TM = 2638.7404786D0
      W0 = PI/TM
      DK = -DKIN/(1.D0+W0*DKIN)
      DINCL = 90.D0
      DINC = DINCL*PI/180.D0
      DCI = DCOS(DINC)
      DSI = DSIN(DINC)
      NT = 200.D0
      H = 2.D0*TM/NT
      EPS = 1.D-12
      NPO = 0
      TELEM = 0.D0
      INIT = 0
      GM = .39860047D15
      AE = 6378139.D0
      HS = 160016.D0
      T = TELEM
      IR = 0
      NMAX = 9
      DSR2I = 1.D0/DSQRT(2.D0)
      DTOUT = H*50
      INT = DTOUT/H+1.D-5
      X(1,5)  = (AE+HS)*DCI
      X(2,5)  = 0.D0
      X(3,5)  = (AE+HS)*DSI
      XP(1,5) = 0.D0
      XP(2,5) = -7805.9927335843D0
      XP(3,5) = 0.D0
      X10 = X(1,5)
      X20 = X(2,5)
      X30 = X(3,5)
      XP10 = XP(1,5)
      XP20 = XP(2,5)
      XP30 = XP(3,5)
 2000 X10 = X10+DR0*DCI
```

```
      X30 = X30+DRO*DSI
      XP20 = XP20-DYDO-WO*DRO
      X(1,5) = X10
      X(2,5) = X20
      X(3,5) = X30
      XP(1,5) = XP10
      XP(2,5) = XP20
      XP(3,5) = XP30
      RAV = AE+HS
      RO = DSQRT(X(1,5)**2+X(2,5)**2+X(3,5)**2)
      WRITE(IO1,40) (X(I,5),I=1,3),(XP(I,5),I=1,3)
   40 FORMAT(//' INITIAL CONDITIONS'//,(1X,3(G20.14,2X)))
      DO 1000    NI = 1,NT
      CALL COWELL(X,XP,TELEM,T,NCA,INIT)
      INIT = 1
      NPO = NPO+1
      RS(NPO) = DSQRT(X(1,5)**2+X(2,5)**2+X(3,5)**2)
      TM = T/60.D0
      IP = T/H+1.D-5
      IF((IP/INT)*INT.EQ.IP) WRITE(IO1,50) TM,(X(I,5),I=1,3),
     # (XP(I,5),I=1,3)
   50 FORMAT(///1X,G12.5/,3(1X,G20.14))
 1000 CONTINUE
      NPO = 0
      T = 0
      INIT = 0
      SQ = 0.D0
      RMX = RO
      RMN = RO
      CDR1 = 0.D0
      AVDR = 0.D0
      DO 1500    NI = 1,NT
      CDR1 = (RS(NI)-RAV)*DCOS(WO*NI*H)+CDR1
      AVDR = (RS(NI)-RAV)+AVDR
      SQ = SQ+(RS(NI)-RAV)**2
      IF(RS(NI).GT.RMX) RMX = RS(NI)
      IF(RS(NI).LT.RMN) RMN = RS(NI)
 1500 CONTINUE
      CC1 = 2.D0/WO+3.D0*DK
      CC2 = 2.D0/WO+4.D0*DK
      DYDO = -1.D0/(NT*(0.5D0*CC1**2+CC2**2))*(-CC1*CDR1+CC2*AVDR)
      DRO = DK*DYDO
      NCOUNT = NCOUNT+1
      RDIFF = RMX-RMN
      AVRD =AVDR/NT
      RMS = DSQRT(SQ/NT)
      WRITE(IO1,1510) RMX,RMN,DRO,DYDO,   RDIFF,AVRD,RMS
 1510 FORMAT(//' RMX,RMN,DRO,DYDO,RDIFF,AVRD,RMS'/,(1X,4G20.10))
      IF(NCOUNT.LT.2) GO TO 2000
      STOP
      END
//GO.SYSIN DD *
//
```

AIII.3 Program to simulate the line of sight velocity residuals with respect to "computed" or "nominal" values.

This program computes first, and then stores on disk for further analysis by the procedure described in paragraph AIII.4, the difference between "true" and "nominal" line of sight values. The "true" values correspond to two satellite orbits integrated numerically from initial states that, in general, do not lie exactly on the common closed reference orbit, but are each near to a "reference" point on it, both "reference" points being symmetrical with respect to the perigee (F' = 0). The relative velocities are found by differencing the projections of the velocities of the two satellites along the common line-of-sight direction, calculation that is carried out by subroutine SIGNAL. The "nominal" values correspond to a pair of "nominal" orbits starting from the same initial states as the "true" ones, plus some small perturbations or "initial state estimation errors" chosen to make the relative drift along-track between "true" and "nominal" orbits zero (as it happens, approximately, when real orbit estimates are used to calculate residuals). The "nominal" along-track velocities are computed also by subroutine SIGNAL.

The integration step is H seconds long, and is chosen so that its ratio to the period of the reference orbit "TORB" (obtained as part of the determination of this orbit by the program of paragraph AIII.1) is a power of 2. This choice is dictated by the need to analyze the residuals by means of a Fast Fourier Transform algorithm during the execution of the program of paragraph AIII.4. The maximum degree in the expansion of the "true" field is "NMAXIM", while that for the field of the "nominal" orbit is "NREF". As before, the field for the reference orbit has a maximum degree of 9, and all three fields consist exclusively of zonals (although this is not a limitation imposed by the present program, but by the subroutines that compute the gravitational acceleration during the integration of the orbits). The desired mean height of the satellites is "HSS", the number of integration steps for the whole orbit is "NINT" (in the runs which produced the numerical results of section 4 and appendix IV, NMAXIM = 300 and NINT = 2048, so "H" was less than 3 seconds). "DZCE" is a common shift in the initial state of the satellites, parallel to the z or $x_3$ axis, from their position on the reference orbit. "DX11" and "DX12" represent displacements in the x (or across-track) direction for each satellite independently (a "1" at the end of a variable's name usually indicate the leading satellite, and a "2" the trailing one). The reference orbit is supposed to be a *polar* one. "DR1" and "DR2" designate differences in position in the radial direction between the "true" and the "nominal" orbits at their starting points. "SEP" is the approximate initial distance between the satellites, in meters. "WO" is the mean angular velocity $n_0$ ($2\pi$/orbit period) of the reference orbit. The two "reference" satellites pass the same point on this orbit TSEP seconds apart, which is the time uniformly rotating satellite with angular velocity WO would take to describe an arc of circle of length "SEP". Initialization ends at line 51, where subroutine PCOEFF is called to compute the values of the zonal coefficients up to degree "NMAXIM" (that of the "true" field). The first 9 zonals are left identical to those for the reference orbit; the zonals of the "nominal" field are identical to the corresponding ones in the true field (no "commission errors"). "DUP1" and "DUP2" are changes in along-track velocity that cancel the drift due to the displacements "DR1" and "DR2), and are given in the uniformly rotating system of coordinates associated with each satellite. They are transformed to inertial along-track velocities first ("DUP1I" and DUP2I") and then, in combination with DR1 and DR2, to Cartesian, inertial coordinates (lines 62-73). The parameter "DTOUT" in line 49 has been chosen to be a number much larger than the duration of one revolution in seconds, to eliminate the printing of orbit positions and velocities by the integrating subroutines, except for the initial and final conditions.

Lines 74 to 83 arrange the listing of the main parameters that characterize the run (maximum degree of "true" and "nominal" fields, number of integration steps, mean separation between satellites, perturbations in initial conditions, etc.). The reference orbit is initialized according to the starting values obtained with the program of paragraph AIII.1, and then it is integrated by calling subroutine ORBIT which, in turn, invokes a whole package of subroutines described in paragraph AIII.5. The x, y, z coordinates of the orbit are stored in arrays "X1", "Y", "Z", (the orbit being polar and in a zonal field, "Z1" contains only zeroes, and is used only because ORBIT requires it). Then (line 92) the integration step-size is changed temporarily to one quarter of "TSEP", and in two steps from the perigee a point on the reference orbit is found corresponding to a time TSEP/2. Its symmetrical is obtained in lines 107-108, after having shifted the z coordinate by an amount DZCE (line 101). The nominal orbits of both satellites are computed from these initial states (lines 105 and 108) by ORBIT and its associated subroutines. The nominal line-of-sight relative velocity is then computed by SIGNAL, which takes the velocities and positions of both satellites along the orbit (stored by ORBIT in arrays X1, X2, Y1, Y2, Z1, Z2, XD1 XD2, YD1, YD2, ZD1 and ZD2) and puts the result in array "SST" (at regular intervals; "H" has been returned to its original value of TORB/NINT in line 97). In line 109 the maximum degree in the field is made equal to NMAXIM, in preparation for the computation of the "true" orbits.

In line 112 "DTOUT" is changed to 500, thus allowing the integrating subroutines to print out the values of the "true" orbit states every as many seconds, to give a general idea of the shape of each "true" orbit. Next, the initial conditions are changed from "nominal" to "true" in accordance with the specified "orbit estimation errors", and the "true" orbits are integrated with ORBIT (lines 112 to 125). In line 126 SIGNAL is called once more, to obtain the values of the "true" line-of-sight velocity, and substract them from the "nominal" ones, the differences or "residuals" being returned in "SST". From line 127 to 146 the Cartesian components of the gravitational acceleration along the reference orbit are computed by subroutine SECON; these components (FY and FZ, the other one is always zero in a zonal field) are used then to obtain the along-track and radial accelerations AU and AR, and their sum AT, which will be needed later to obtain the analytical lumped coefficients of the line-of-sight relative velocity according to the theory of sections 1, 2 and 3. Before computing the accelerations, the coefficients common to the "nominal" and the "true" fields are set to zero, as the lumped coefficients are supposed to correspond to their differences (see paragraph 4.2) and they are all equal up to degree NREF. Finally, all the information created by the program that will be needed for the subsequent comparison between true and theoretical lumped coefficients is stored on disk in unit 10 (lines 147-150).

Subroutine PCOEFF.

This subroutine creates normalized and dimensionless zonal potential coefficients so that the power spectrum of the corresponding field agrees with that defined by expression (4.1.1) in paragraph (4.1). To prevent the formation of a tremendous spike on the north pole, the signs of the zonal coefficients are made to alternate, two consecutive zonals having the same sign and the two that follow the opposite. This results in a much more uniform appearence of the "gravity disturbance" AR along the orbit than it is the case when all signs are the same. Zonals with $100 < n \leq 218$ are set to 0, and those for $n = 299$ and $n = 230$, to $10^{-5}$.

Subroutine SIGNAL.

This subroutine inputs the coordinates and velocities of both satellites in Cartesian coordinates, stored in arrays "X1", "Y1", etc., and the number of points computed along the orbit, "NINT". The output is the succession of values of the relative line-of-sight velocity, at the times when the orbit points were computed, so they are also "NINT" in number. They are returned to the calling program in array "SST".

Within the main loop (DO 100 ..."), a vector "E" aligned with the line-of-sight, and its modulus "EMI" are computed. Then, the velocity vectors are projected on that line, the difference "S" of their projections is found, and the corresponding component of "SST" is substracted from it. As the initial values of all components of "SST" are 0, a first call in the main program results in the relative "nominal" velocity, while the second call, whose input are the positions and velocities of the "true" orbits, yields the *residual* relative line-of-sight velocities that constitute the simulated signal produced and stored on disk by the calling main program.

```
//GDFGOSCA  JOB   (3114,81),OSCAR,TIME=(0,20),REGION=800K
/*JOBPARM LINES=2,INFORM,Q=F,
//    EXEC FORTXCG,PARM.FORT='OPT=2,'
//FORT.SYSIN DD *
C
C           PROGRAM THAT SIMULATES THE LINE OF SIGHT VELOCITY
C           RESIDUALS WITH RESPECT TO A FIELD OF DEGREE "NREF".
C           THE "TRUE" FIELD HAS DEGREE "NMAXIM".RESULTS ARE
C           SAVED ON DISK IN UNIT 10 .
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X1(2060),SST(2060),AT(2060),
     # Y1(2060),Z1(2060),XD1(2060),YD1(2060),ZD1(2060),X2(2060),
     # Y2(2060),Z2(2060),XD2(2060),YD2(2060),ZD2(2060),CC(310)
     #,XOS(3),XDOS(3),Z(2060),Y(2060),X(3,9),XP(3,9),DELS(3,9)
      COMMON/GEOCON/GM,AE
      COMMON/INTEG/H,EPS
      COMMON/POURCO/FPER(3,9),JJ,KPC,IECO,ITERA,ITX,IJA
      COMMON/IO/IO1,IN1,IN2,IN3,IN4,IN5,IN6,IN7
      COMMON/COEFFS/ CN(310),DSR2I,IR,NMAX,TELEM,DTOUT
      DATA CC/1.D0,0.D0,-484.166D-6,.958475D-6,.541539D-6,.0684389D-6,
     #-.151207D-6,.0933127D-6,.0509491D-6,.027331D-6,300*0.D0/
      IO1 = 6
      PI = 4.D0*DATAN(1.D0)
      DO 1     N = 1,310
    1 CN(N) = CC(N)
      X30 = 6526447.5757058D0
      XP20 = -7812.9831897787D0
      GM = .39860047D15
      AE = 6378139.D0
C ************************************************
      NMAXIM = 60
      NREF = 30
      TELEM = 0.D0
      TORB = 5263.3690684276D0
      HSS = 160016.D0
      NINT = 256
      IECON = 1
      ARCL = TORB
      DZCE = 0.D0
      DX11 = 0.D0
      DX12 = 0.D0
      DR1 = 0.D0
      DR2 = 0.D0
      SEP = 300000.D0
C ************************************************
      NMAX = 9
      IECO = 0
      DTOUT = 10000.D0
      NREFP = NREF+1
      CALL PCOEFF
      W0 = 2.D0*PI/TORB
      NINTP = NINT+1
      TSEP = SEP/(W0*(AE+HS))
      INTS2P = 3
      ARCLL = TSEP/2.D0
      PSI2 = W0*ARCLL
      CPSI2 = DCOS(PSI2)
      SPSI2 = DSIN(PSI2)
      AQ2 = 2.D0*AC
```

```
      CPSI22 = 2.D0*CPSI2
      DUP1 = 2.D0*W0*DR1
      DUP2 = 2.D0*W0*DR2
      DUP1I = DUP1-W0*DR1
      DUP2I = DUP2-W0*DR2
      DX21 =-DR1*SPSI2
      DXP21 = DUP1I*CPSI2
      DX22 = DR2*SPSI2
      DXP22 = DUP2I*CPSI2
      DX31 = DR1*CPSI2
      DXP31 = DUP1I*SPSI2
      DX32 = DR2*CPSI2
      DXP32 =-DUP2I*SPSI2
      WRITE(IO1,3) NMAXIM,NREF,HSS,TORB,SEP,NINT,DZCE,DX11,DX12,DR1,
     #DUP1,DR2,DUP2
    3 FORMAT('1','MAXIMUM DEGREE IN ZONAL FIELD = ',I5,' MAX. DEG'
     #,'. IN REFERENCE FIELD = ',I5,' SATELLITE HEIGHT = ',G14.8/,
     #' ORBITAL PERIOD = ',G20.10,' MEAN SEPARATION = ',G14.8,
     #' NO. INTEGR. STEPS =',I6,
     #/' INITIAL CONDITION ERRORS : DZCE = ',G14.8,' DX11 = ',G12.6,
     #' DX12 = ',G12.6,' DR1 = ',G12.6,' DUP1 = ',G12.6/' DR2 = '
     #,G12.6,' DUP2 = ',G12.6/' UNITS ARE '
     #,'METERS , SECONDS AND METERS PER SECOND.'/)
      DO 15   I = 1,NINTP
   15 SST(I) = 0.D0
      XOS(1) = 0.D0
      XOS(2) = 0.D0
      XOS(3) = X30
      XDOS(1) = 0.D0
      XDOS(2) = XP20
      XDOS(3) = 0.D0
      H = TORB/NINT
      CALL ORBIT(XOS,XDOS,X1,Y,Z,XD1,YD1,ZD1,ARCL)
      H = TSEP/4.D0
      CALL ORBIT(XOS,XDOS,X1,Y1,Z1,XD1,YD1,ZD1,ARCLL)
      NMAX = NREF
      IECO = IECON
      H = TORB/NINT
      XOS(1) = X1 (INTS2P)
      XOS(2) = Y1(INTS2P)
      XOS(3) = Z1(INTS2P)+DZCE
      XDOS(1) =XD1(INTS2P)
      XDOS(2) = YD1(INTS2P)
      XDOS(3) = ZD1(INTS2P)
      CALL ORBIT(XOS,XDOS,X1,Y1,Z1,XD1,YD1,ZD1,ARCL)
      XOS(2) = -XOS(2)
      XDOS(3) = -XDOS(3)
      CALL ORBIT(XOS,XDOS,X2,Y2,Z2,XD2,YD2,ZD2,ARCL)
      CALL SIGNAL(X1,Y1,Z1,XD1,YD1,ZD1,X2,Y2,Z2,XD2,YD2,ZD2,NINTP,SST)
   18 NMAX = NMAXIM
      DTOUT = 500.D0
      XOS(1) = XOS(1)+DX11
      XOS(2) = -XOS(2)+DX21
      XOS(3) = XOS(3)+DX31
      XDOS(1) = XDOS(1)+DXP11
      XDOS(2) = XDOS(2)+DXP21
      XDOS(3) = -XDOS(3)+DXP31
      CALL ORBIT(XOS,XDOS,X1,Y1,Z1,XD1,YD1,ZD1,ARCL)
      XOS(1) = XOS(1)+DX12-DX11
      XOS(2) = -XOS(2)+DX22+DX21
```

```
      XOS(3) = XOS(3)+DX32-DX31
      XDOS(1) = XDOS(1)+DXP12-DXP11
      XDOS(2) = XDOS(2)+DXP22-DXP21
      XDOS(3) = -XDOS(3)+DXP32+DXP31
      CALL ORBIT(XOS,XDOS,X2,Y2,Z2,XD2,YD2,ZD2,ARCL)
      CALL SIGNAL(X1,Y1,Z1,XD1,YD1,ZD1,X2,Y2,Z2,XD2,YD2,ZD2,NINTP,SST)
      NMAXP9 = NMAX+8
      DO 100    N = 1,NREFP
  100 CN(N) = 0.D0
      IECO = 0
      T = 0.D0
      DO 120    I = 1,NINT
      JJ = 5
      X(2,5) = Y(I)
      X(3,5) = Z(I)
      X(1,5) = 0.D0
      CALL SECON(X,XP,T,DELS)
      FY = DELS(2,5)
      FZ = DELS(3,5)
      ZN = DCOS(W0*T)
      YN =-DSIN(W0*T)
      AR = FY*YN+FZ*ZN
      AU = FY*ZN-FZ*YN
      AT(I) = AR+AU
      T = T+H
  120 CONTINUE
      IU = 10
      WRITE(IU)NMAXIM,NMAXP9,NINT,DZCE,DX11,DX12,DR1,DUP1,DR2,DUP2,W0,
     $SEP,SST,AT
     $,HSS,NREF,TORB
      STOP
      END
      SUBROUTINE PCOEFF
C
C     THIS SUBROUTINE GENERATES NORMALIZED ZONAL POTENTIAL COEFFICIENTS
C     EACH EQUAL TO THE SQUARE ROOT OF THE CORRESPONDING DEGREE
C     VARIANCE. THE SIGN OF THE FIRST PAIR (J10, J11) IS NEGATIVE,
C     THAT OF THE FOLLOWING PAIR IS POSITIVE, AND SO ON.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/COEFFS/ CN(310),DS,IR,NM,T,DT
      COMMON/IO/ IO1,IOS(7)
      COMMON/GEOCON/ GM,A
      GM = .39860047D15
      B = 1.D0
      C = 2.D0
      A1 = 3.405D0
      A2 = 140.03D0
      S1 = .998006D0
      S2 = .914232D0
      MM = 1
      AGM = A**4/GM**2*1.D-10
      PS1 = S1**(9+2)
      PS2 = S2**(9+2)
      DO 10    N1 = 11,310
      PS1 = PS1*S1
      PS2 = PS2*S2
      N = N1-1
      CN(N1) = (AGM/(N-2.D0))*((A1*PS1)/(N+B)+(A2*PS2)
     $ /((N-2.D0)*(N+C)))
```

```
      CN(N1) = DSQRT(CN(N1))
      IF(N1.GT.101) CN(N1) = 0.D0
      IF((N1/2)*2.EQ.N1) MM = -MM
   10 CN(N1) = MM*CN(N1)
      CN(300) =  1.D-5
      CN(301) =  1.D-5
      RETURN
      END
```

```
      SUBROUTINE SIGNAL(X1,Y1,Z1,XD1,YD1,ZD1,X2,Y2,Z2,XD2,YD2,
     # ZD2,NINTP,SST)
C
C             THIS SUBROUTINE COMPUTES THE RESIDUAL LINE-OF-SIGHT
C             RELATIVE VELOCITY BETWEEN TWO SATELLITES ("1" AND "2").
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X1(1),Y1(1),Z1(1),XD1(1),YD1(1),ZD1(1),X2(1),
     # Y2(1),Z2(1),XD2(1),YD2(1),ZD2(1),SST(1),E(3)
      T = 0.D0
      DO 100     NI = 1, NINTP
      E(1) = X1(NI)-X2(NI)
      E(2) = Y1(NI)-Y2(NI)
      E(3) = Z1(NI)-Z2(NI)
      EM = DSQRT(E(1)**2+E(2)**2+E(3)**2)
      EMI = 1.D0/EM
      DO 1     I = 1,3
    1 E(I) = E(I)*EMI
      S = (XD1(NI)-XD2(NI))*E(1)+(YD1(NI)-YD2(NI))*E(2)
     # +(ZD1(NI)-ZD2(NI))*E(3)
      SST(NI) = S-SST(NI)
  100 CONTINUE
      RETURN
      END
```

AIII.4 Program for computing the "lumped coefficients" of the signal, both according to the theory and to the Fourier analysis of simulated data, and for comparing them to test the accuracy of the mathematical model.

To explain this program it is necessary to amplify first the formulation of the "lumped coefficients" of the signal given in paragraph (4.2). To obtain these coefficients according to the analytical perturbation theory of section 2 and to the signal equation of section 3, one needs first to know the Fourier coefficients of the forcing terms of the variational equations, $D_{\bar{C}^{\alpha}_{nm}} a_r$ and $D_{\bar{C}^{\alpha}_{nm}} a_u$, in addition to the differences $\Delta\bar{C}^{\alpha}_{nm}$ between the "true" potential coefficients and those used to compute the "nominal" orbits. Assuming that all "nominal" coefficients are correct, then $\Delta\bar{C}^{\alpha}_{nm} = 0$ for $n \leq N_{ref}$, where $N_{ref}$ is the highest degree in the "nominal" field. The perturbations in the radial and along-track accelerations caused by all the $\Delta\bar{C}^{\alpha}_{nm}$ of the same order m should be, according to (2.2.45-46) if $t_0 = 0$,

$$\delta a_{rm}(t) = \sum_{\alpha=0}^{1} \sum_{n=m}^{N_{max}} \Delta\bar{C}^{\alpha}_{nm} D_{\bar{C}^{\alpha}_{nm}} a_r(t) \qquad \text{(AIII.4.1)}$$

$$= \sum_{j=-(N_{max}+3N)}^{N_{max}+3N} \sum_{\alpha n} \Delta\bar{C}^{\alpha}_{nm} a_{rnmj} \cos((jn_0+m\theta')t+\phi_{m\alpha})$$

and, similarly,

$$\delta a_{um}(t) = \sum_{j\alpha n} \Delta\bar{C}^{\alpha}_{nm} a_{unmj} \sin((jn_0+m\theta')t+\phi_{m\alpha}) \qquad \text{(AIII.4.2)}$$

Calling

$$\delta A_m(t) = \delta a_{rm}(t) + \delta a_{um}(t) \qquad \text{(AIII.4.3)}$$

then, for m = 0 (the theory is tested with this program for zonals only)

$$\delta A_0(t) = \sum_{j=0}^{N_{max}+3N} A_{rj} \cos jn_0 t + A_{uj} \sin jn_0 t \qquad \text{(AIII.4.4)}$$

where

$$A_{rj} = \sum_{\alpha n} \Delta\bar{C}^{0}_{mo}(a_{rnoj}+a_{rno(-j)}) \qquad \text{(A.III.4.5a)}$$

$$A_{uj} = \sum_{\alpha n} \Delta\bar{C}^{0}_{n} (a_{unoj}-a_{uno(-j)}) \qquad \text{(A.III.4.5b)}$$

As explained in section 3, the residual signal $\delta s$ consists (in theory) of a periodical part and a non-periodical part. The first, according to expression (3.4.10) is

$$\delta\tilde{s} = \sum_{m=0}^{N_{max}} \sum_{\alpha=0}^{1} \sum_{n=m}^{N_{max}} \sum_{j=(n+3N)}^{n+3N} \Delta\bar{C}_{nm}^{\alpha}\, s_{nmj}\, \sin((jn_0+m\theta')t+\phi_{jm\alpha}) \tag{AIII.4.6}$$

where

$$s_{nmj} = \begin{cases} 2\omega_{mj}[\sin n_0c\; b_{nmj}\cos\omega_{mj}c - \cos n_0c\; g_{nmj}\sin\omega_{mj}c] \\ +b_1\,[\omega_{m(j-1)}b_{nm(j-1)}\sin\omega_{m(j-1)} - \omega_{m(j+1)}b_{nm(j+1)} \\ \sin\omega_{m(j+1)}c] \qquad \text{if } |j| \le n+3N \\ 0 \qquad \text{otherwise.} \end{cases} \tag{AIII.4.7}$$

($\omega_{mj} = jn_0+m\theta'$).

So the contribution of all $\Delta\bar{C}_{nm}^{\alpha}$ of the same order m to $\delta s$ is

$$\delta\tilde{s}_m(t) = \sum_{j=-(N_{max}+3)}^{N_{max}+3N} \sum_{\alpha} \hat{s}_{jm} \sin((jn_0+m\theta')t+\phi_{jm\alpha}) \tag{AIII.4.8}$$

where $\hat{s}_{jm}$, the "*lumped coefficient*", is

$$\hat{s}_{jm} = \sum_{n=m}^{N_{max}} \bar{C}_{nm}\, s_{nmj} \tag{AIII.4.9}$$

For the zonals ($m = 0$, $\alpha = 0$), taking as time-origin the moment when the mid-point between the two satellites is at the "perigee" (i.e., $F' = 0$), the expression for the "lumped coefficient" becomes (as $\Delta\bar{C}_{no}^{0} = 0$ for $n \le N_{ref}$)

$$\hat{s}_{jo} = \sum_{n=N_{ref}+1}^{N_{max}+3N} \Delta\bar{C}_{no}^{0}(s_{noj}+s_{no(-j)}) \tag{AIII.4.10}$$

while the periodical part of the signal takes the form

$$\delta\tilde{s}_0(t) = \sum_{j=0}^{N_{max}+3N} \hat{s}_{jo} \cos jn_0t \tag{AIII.4.11}$$

The aperiodic part is due to secular and very long period effects on the shape of the orbit, mostly caused by the zonals. It has the form $\delta\overset{\ell}{s}(t) = At \sin n_0 t + Bt \cos n_0 t$. Over *one period* of the reference orbit (i.e., in the interval $0 \leq t \leq T$ ) this signal, being continuous and bound, can be represented by a Fourier series as follows:

$$\delta\overset{\ell}{s}(t) = \sum_{j=0}^{\infty} h_c\left(\frac{j}{j^2-1}\right)\cos jn_0 t + h_s\left(\frac{1}{j^2-1}\right)\sin jn_0 t \qquad \text{(AIII.4.12)}$$

where $h_c$ and $h_s$ depend on A, B and $T_0$. So, for $0 \leq t \leq T_0$, the total residual signal for m = 0 should satisfy the relationship (neglecting terms in $\delta\overset{\ell}{s}$ above $j = N_{max}+3N$, as the spectrum decays quickly):

$$\delta s_0(t) = \delta\tilde{s}_0(t) + \delta\overset{\ell}{s}(t)$$
$$= \sum_{j=0}^{N_{max}+3N} h_c\left(\frac{j}{j^2-1}\right)\cos n_0 t + \left(\hat{s}_j + h_s\left(\frac{1}{j^2-1}\right)\right)\sin jn_0 t \qquad \text{(AIII.4.13)}$$

or

$$\delta s_0(t) = \sum_{j=0}^{N_{max}+3N} p_{j0} \cos jn_0 t + q_{j0} \sin jn_0 t \qquad \text{(AIII.4.14)}$$

where $p_{j0}$ and $q_{j0}$ are the Fourier coefficients of the signal over one revolution. Consequently, the "lumped coefficients" corresponding to the frequency $jn_0$ are

$$\hat{s}_{j0} = q_{j0} - h_s\left(\frac{1}{j^2-1}\right) \qquad \text{(AIII.4.15a)}$$

while

$$p_{j0} = h_c\left(\frac{j}{j^2-1}\right) \qquad \text{(AIII.4.15b)}$$

(because of the choice of time-origin, the zonal perturbation in the signal is an odd function of t, so $p_{j0} = 0$ unless there is a perturbation in the initial state as well).

Also according to the analytical theory developed in this work, the "lumped coefficients" for the instantaneous residual velocity must be, for m = 0,

$$\bar{s}_{j0} = 2jn_0[\sin n_0c \sum_{n=N_{ref}+1}^{N_{max}} \Delta\bar{C}^0_{n0}(b_{n0j}-n_{n0(-j)})\cos jn_0c -$$

$$\cos n_0c \sum_{n=N_{ref}+1}^{N_{max}} \Delta\bar{C}^0_{n0}(g_{n0j}+g_{n0(-j)})\sin jn_0c] +$$

$$b_1[(j-1)n_0 \sum_{n=N_{ref}+1}^{N_{max}} \Delta\bar{C}^0_{n0}(b_{n0(j-1)}-b_{n0(j+1)})\sin(j+1)n_0c -$$

$$(j+1)n_0 \sum_{n=N_{ref}+1}^{N_{max}} \Delta\bar{C}^0_{n0}(b_{n0(j+1)}-b_{n0(-j-1)}\sin(j+1)n_0c]$$

$$= \sin n_0c\, B_{j0} \cos jn_0c - \cos n_0c\, G_{j0} \sin jn_0c +$$

$$b_1[B_{(j-1)0} \sin(j-1)n_0c - B_{(j+1)0} \sin(j+1)n_0c] \quad \text{(AIII.4.16)}$$

where

$$B_{j0} = \sum_{n=N_{ref}+1}^{N_{max}} \Delta\bar{C}^0_{n0}\, jn_0(b_{n0j}-b_{n0(-j)}) \quad \text{(AIII.4.17a)}$$

$$G_{j0} = \sum_{n=N_{ref}+1}^{N_{max}} \Delta\bar{C}^0_{n0}\, jn_0(g_{n0j}+g_{n0(-j)}) \quad \text{(AIII.4.17b)}$$

According to the definition of $b_{nmj}$ and $g_{nmj}$ (expressions (2.4.8-9)),

$$B_{j0} = -\frac{(jn_0A_{rj} + 2n_0A_{uj})}{(j^2n_0^2 - n_0^2)} \quad \text{(AIII.4.18a)}$$

$$G_{j0} = -\frac{(2jn_0^2A_{rj} + (j^2+3)n_0^2A_{uj})}{jn_0(j^2n_0^2-n_0^2)} \quad \text{(AIII.4.18b)}$$

where the $A_{rj}$ and $A_{uj}$ are as in (AIII.4.5a-b).

In order to test the agreement between theory and simulation, one needs to know $\delta A(t)$, the sum of $\delta a_r(t)$ and $\delta a_u(t)$, at regular intervals along the reference orbit, do a numerical Fourier analysis of these values to get the $A_{rj}$ and $A_{uj}$, and then obtain with these the $B_{j0}$ and $G_{j0}$, according to (AIII.4.18a-b), to calculate, finally, the lumped coefficients $\bar{s}_{j0}$ using expression (AIII.4.16). Once the theoretical values of the "lumped coefficients" are known, and the residual relative line-of-sight velocity, $\delta s_0(t)$, computed at regular intervals over one period $T_0$, has been analyzed to get the total "true" Fourier coefficients of the signal, $p_{j0}$ and $q_{j0}$ (expression (AIII.4.14)), one should be able to determine the constants $h_c$ and $h_s$ corresponding to the non-periodical part (assuming the theory is correct) by solving for these constants as unknowns in equations (AIII.4.15a-b). Because of possible numerical errors acting as "noise" in the computed results, it is best to calculate $h_c$ and $h_s$ a few times, for different values of j, and then to average the results. Once $h_c$ and $h_s$ are known, one can use them to "correct" the Fourier coefficients of the simulated signal so as to obtain the "lumped coefficients" $\hat{s}_{j0}$ of the periodical part:

$$\hat{s}_{j0} = q_{j0} - h_s\left(\frac{1}{j^2-1}\right)$$

according to expressions (AIII.4.15). Because of their origin, these values of the "lumped coefficients" may be called "empirical", while those obtained from expression (AIII.4.16) as explained before, would be "theoretical". The differences between them would be due to: (a) errors in the simulation of the data (mostly caused by the numerical integrator) and in the subsequent Fourier analysis of the "sampled" signal; (b) errors in the theory. Assuming that the sampling interval, which is also the integration step for the "true" and "nominal" orbits, is small enough to neglect aliasing in the Fourier analysis, and that the integrator is well-chosen, then the differences between empirical and theoretical "lumped coefficients" must be due, mostly, to errors in the theory. As explained in section 4, the shortcomings of the numerical integrator (due mainly to the number of significant figures that the computer can handle) were circumvented mostly by choosing the values of the $\Delta\bar{C}^0_{n0}$ large enough so that their effect will overcome, at most frequencies, the "numerical noise". In addition, it was obvious that the "sine" part of the empirical coefficients, the $p_{j0}$, contained a small constant term unexplained by the theory, but most likely due to the way in which the first point of the orbit is calculated, which is different from that for all the others. This may result in a "spike" at the origin of the data stream, which would have a small, constant effect across the spectrum. This effect was estimated from the value of $p_{j0}$ at a frequency too high for the signal to have a significant component there, and was substracted from all $p_{j0}$, previous to any use of these numbers, to get "corrected" empirical coefficients. It was against these that the theoretical coefficients were tested.

Note: In the development of the theory, the mean value $\overline{\sin \eta_{10}}$, was replaced by $\sin n_0 c$ as an approximation (expression (3.4.5a)), but this substitution had no material effect on the actual derivation that followed (other than to make the resulting equations "look" simpler), so the more accurate value $\overline{\sin \eta_{10}}$ (obtained from a separate analysis of the reference orbit) has been used here, by way of the variable "AQ" initialized in line 53.

In this program, the results of the simulation carried out using the routine described in paragraph AIII.3, including the parameters of the problem (satellite separation, integration step, etc.), are read from disk (line 39) and some of them are printed-out to give a description of the case being studied (lines 42-51). Other parameters defining the gravitation field, etc., are initialized in lines 25-38. The simulated relative line-of-sight residuals are placed in array "SST", and the values of $\delta A(t) = \delta a_r(t) + \delta a_u(t)$ along the reference orbit, in array "AT". They are analyzed by means of the Fast Fourier Transform method in subroutine "FOURIE" (lines 69 and 75) to produce the "empirical" coefficients $p_{j0}$, $q_{j0}$ and $A_{rj}$, $A_{uj}$ which are stored in arrays "PC", "PS", and "ARJ", "AUJ", respectively. A small constant bias is eliminated from the $p_{j0}$, for reasons given earlier in this paragraph, in lines 70-72. All $\Delta\bar{C}^0_{n0}$ with $n \leq N_{ref}$ (here called "NREF") are set to 0 (lines 73-74). The intermediate values $B_{j0}$ and $G_{j0}$ (expressions AIII.4.18) are found and stored in arrays "BJ" and "GJ" (lines 76-84), up to a frequency somewhat higher than $N_{max}n_0$. The "theoretical" values of the "sine" coefficients are determined next, according to equation (AIII.4.16) and stored in array "SSTF" (the "cosine" coefficients are all theoretically 0). Lines 96-98 refer to the printing of the heading for the results to be listed afterwards. These are the "simulated" coefficients $p_{j0}$, $q_{j0}$, the "empirical" lumped coefficients (obtain by correcting the $p_{j0}$, $q_{j0}$ so as to eliminate the effect of the aperiodic part, and, thus, called here "corrected"), the "theoretical" coefficients in "SSTF" ("sine"-type only, under the heading "analytical"), and the percentage error ("empirical"-"analytical")/"empirical" x 100. The constants $h_c$ and $h_s$ (with a change in sign) needed to obtain the non-periodical part and substract it from the $p_{j0}$, $q_{j0}$ are found in lines 99-114. The conversion of the $p_{j0}$, $q_{j0}$ into "empirical lumped-coefficients" takes place between lines 115 and 129. The percentage "error" is calculated in line 130, and the results for the frequency in question ($jn_0$; $n_0$ here is "WO") are printed out according to lines 131-132. Frequencies below $4n_0$ are treated differently from the rest, because they are fully absorbed in an adjustment like the one described in section 3 by the "arc parameters", so the accuracy of the theory regarding these particular frequencies is irrelevant. As explained in section 4, paragraph (4.2), this accuracy is worse than for the rest, because of the effect of the "once-per-revolution" component that is very large and "spreads" over its neighbours after being modulated by the slight changes in geometry along the reference orbit. For this reason, the results up to three cycles per revolution are printed separately, after all the others (lines 143-159). First, the "empirical" coefficients up to j = 2 are given, and for j = 3, also the "analytical" one (lines 151-159). From line 160 to the end, the rms value of the total (i.e. "uncorrected" for non-periodical terms), part of the signal, together with the rms of the periodical (or "corrected") part and the corresponding value, according to the theory (or "analytical") are listed, as well as the percentage error. These values correspond to the added power of all the frequency components within bands of five cycles per revolution, thus producing a short listing summarizing a whole run, like those shown in paragraph (4.4).

Subroutine FOURIE.

This subroutine calls the Fast Fourier subroutine "FFTSC" from the IMSL library, which finds the Fourier coefficients of the time-series stored in "P" and returns them in arrays "PS" (for "sine"-type) and "PC" (for "cosine"-type), multiplied by $(1 + \delta_{j0})/2$ x Number of samples. "FOURIE" corrects this factor, and returns the coefficients to the calling program. "IWK", "WK", and "CWK" are working arrays required by "FFTSC" (see IMSL Handbook).

```
//GDFGOSCA  JOB  (3114,81),OSCAR,TIME=(0,08),REGION=700K
/*JOBPARM LINES=2,INFORM,Q=F,COPIES=5,FORMS=6L20
//    EXEC FORTXCG,PARM.FORT='OPT=2'
//FORT.SYSIN DD *
C
C          PROGRAM TO COMPUTE THE FOURIER COEFFICIENTS OF THE
C          SIMULATED LINE OF SIGHT VELOCITY AND TO COMPARE THEM
C          TO VALUES CALCULATED USING THE ANALYTICAL
C          EXPRESIONS FOR THE PERTURBATIONS AND THE FIRST ORDER
C          MODEL FOR THE RESIDUALS.
C
C ***************************************************
C
C     PROGRAMMER : OSCAR L. COLOMBO, T.H. DELFT, SEP. 1983.
C
C ***************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMPLEX CWK
      DIMENSION PC(1030),PS(1030),IWK(11),WK(1),CWK(2060),
     $ SST(2060),SSTF(2060),SNC(311),CNC(310),
     $GJ(310),BJ(310),AUJ(1030),ARJ(1030),AT(2060)
     $,CC(310),PSC(1030),PCC(1030),PCU(1030)
      COMMON/GEOCON/GM,AE
      COMMON/COEFFS/ CN(310),DSR2I,IR,NMAX,TELEM,DTOUT
      DATA CC/1.D0,0.D0,-484.166D-6,.958475D-6,.541539D-6,.0684389D-6,
     $-.151207D-6,.0933127D-6,.0509491D-6,.027331D-6,300*0.D0/
      IO1 = 6
      PI = 4.D0*DATAN(1.D0)
      DO 1     N = 1,310
    1 CN(N)  = CC(N)
      X30 = 6526447.5757058D0
      XP20 = -7812.9831897787D0
      GM = .39860047D15
      AE = 6378139.D0
      AQ = .2294022357D-1
      BQ =-.1538200005D-2
      CQ = .641643392D-3
      IU = 10
      READ(IU)NMAXIM,NMAXP9,NINT,DZCE,DX11,DX12,DR1,DUP1,DR2,DUP2,W0,SEP
     $,SST,AT
     $,HSS,NREF,TORB
      WRITE(IO1,3) NMAXIM,NREF,HSS,TORB,SEP,NINT,DZCE,DX11,DX12,DR1,DUP1
     $,DR2,DUP2
    3 FORMAT('1','MAXIMUM DEGREE IN ZONAL FIELD = ',I5,' MAX. DEG'
     $,'. IN REFERENCE FIELD = ',I5,' SATELLITE HEIGHT = ',G14.8/,
     $' ORBITAL PERIOD = ',G20.10,' MEAN SEPARATION = ',G14.8,
     $' NO. OF INTEGR. INTERVALS =',I9,
     $/' INITIAL CONDITION ERRORS : DZCE = ',G14.8,' DX11 = ',G12.6,
     $' DX12 = ',G12.6,' DR1 = ',G12.6,' DUP1 = ',G12.6,/' DR2 = ',
     $G12.6,'DUP2 = ',G12.6/' UNITS ARE '
     $,'METERS , SECONDS AND METERS PER SECOND.'/)
      IF(SEP.EQ.300000.D0) GO TO 1234
      AQ = .76473374944391D-2
      BQ =-.15389199119431D-2
      CQ = .642153955D-3
 1234 NMAX = NMAXIM
      NREFP = NREF+1
      NINTP = NINT+1
      TSEP = SEP/(W0*(AE+HS))
      INTS2P = 3
```

```
      ARCLL = TSEP/2.D0
      CPSI2 = DCOS(W0*ARCLL)
      AQ2 = 2.D0*AQ
      CPSI22 = 2.D0*CPSI2
      DO 4    N = 1,310
    4 SSTF(N) = 0.D0
      SNC(311) = 0.D0
      SNC(1) = 0.D0
      CALL FOURIE(SST,NINT,PS,PC,IWK,WK,CWK)
      PCORR = PC(NMAX+50)
      DO 90      J1 = 3,NMAXP9
   90 PC(J1) = PC(J1)-PCORR
      DO 100     N = 1,NREFP
  100 CN(N) = 0.D0
      CALL FOURIE(AT,NINT,AUJ,ARJ,IWK,WK,CWK)
      DO 150     J1 = 3,NMAXP9
      J = J1-1
      DI = (J*W0)**3-J*W0**3
      DII = DI*J*W0
      BJ(J1) =-(J*W0*ARJ(J1)+2.D0*W0*AUJ(J1))/DI
      GJ(J) =-(2.D0*J*W0**2*ARJ(J1)+((J*W0)**2+3.D0*W0**2)*AUJ(J1))/DII
      BJ(J1) =-BJ(J1)*W0*J
      GJ(J) = GJ(J)*W0*J
  150 CONTINUE
      DO 155      N = 1,NMAXP9
      N1 = N+1
      SNC(N1) = DSIN(N*W0*ARCLL)
  155 CNC(N) = DCOS(N*W0*ARCLL)
      DO 200     J = 2,NMAXP9
      J1 = J+1
      SSTF(J) = CPSI22*GJ(J)*SNC(J1)
      CKK = BQ*(BJ(J1-1)*SNC(J1-1)-BJ(J1+1)*SNC(J1+1))
      SSTF(J) = SSTF(J)+CKK+AQ2*CNC(J)*BJ(J1)
  200 CONTINUE
      WRITE(IO1,205)
  205 FORMAT(//,' CYCLES PER REV.',5X,'SIMULATED',27X,'CORRECTED',20X
     *,'ANALYTICAL',5X,'% (SIN) ERROR'/,22X,'(COS/SIN)',27X,'(COS/SIN)',
     *22X,'(SIN)'//)
      HC = 0.D0
      HS = 0.D0
      K1 = 0
      K2 = 0
      NQ = 26
      IF(NMAXIM.LT.26) NQ = NMAXIM-10
      DO 206   I = 6,NQ
      K1 = K1+1
      IF(I.LT.9) K2 = K2+1
      DF = (I**2-1.D0)/I
      HS = HS+(SSTF(I)-PS(I+1))*DF
C     IF(I.LT.9) HC = HC-PC(I+1)*DF*I
  206 CONTINUE
      HS = HS/K1
      HC = HC/K2
      PC(3) = PC(3)+HC/3.D0
      PC(4) = PC(4)+HC/8.D0
      PS(3) = PS(3)+HS*2.D0/3.D0
      PS(4) = PS(4)+HS*3.D0/8.D0
      PCC(3) = PC(3)
      PCC(4) = PC(4)
      PSC(3) = PS(3)
```

```
      PSC(4) = PS(4)
      NMAXI4 = NMAXIM-4
      MAXX = (NMAXI4/10)*10+14
      DO 220    J1 = 5,NMAXP9
      J = J1-1
      DJK = 1.D0/(J**2-1.D0)
      IF(J1.LT.5) GO TO 111
      PCC(J1) = PC(J1)+HC*DJK
      PSC(J1) = PS(J1)+HS*DJK*J
  111 PERRJ = (PSC(J1)-SSTF(J))/PSC(J1)*100.D0
      WRITE(IO1,210) J,PC(J1),PS(J1),PCC(J1),PSC(J1),SSTF(J),PERRJ
  210 FORMAT(3X,I6,3X,6(G12.6,6X))
      IF((J/50)*50.NE.J) GO TO 220
      WRITE(IO1,222)
  222 FORMAT('1')
      WRITE(IO1,205)
  220 CONTINUE
      TRMS = 0.D0
      TERMS = 0.D0
      TPOWER = 0.D0
      WRITE(IO1,3) NMAXIM,NREF,HSS,TORB,SEP,NINT,D2CE,DX11,DX12,DR1,DUP1
     $,DR2,DUP2
      WRITE(IO1,230)
  230 FORMAT(' FROM 0  CY/REV. TO 3 CY/REV :  CY/REV   CORRECTED
     $'(COS'
     $,', SIN)     ANALYTICAL.   '//)
      IY = 0
      IYY = 1
      WRITE(IO1,232) IY,PC(1),IYY,PS(2),PC(2)
  232 FORMAT(30X,I6,2X,G12.6/,30X,I6,2X,2(G12.6,2X))
      DO 240    I1 = 3,4
      I = I1-1
      IF(I.LT.4) GO TO 231
      TPOWER = TPOWER+(PC(I1)**2+PS(I1)**2)
      TRMS = TRMS+(PCC(I1)**2+PSC(I1)**2)
      TERMS = TERMS+(PCC(I1)**2+(PSC(I1)-SSTF(I))**2)
  231 WRITE(IO1,235) I,PCC(I1),PSC(I1),SSTF(I)
  235 FORMAT(30X,I6,2X,3(G12.6,2X))
  240 CONTINUE
      WRITE(IO1,245) MAXX
  245 FORMAT(//' SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM',
     $' 4 TO ',I4,' CY/REV.',//5X,'BAND',5X,'RMS UNCORRECTED',3X,'RMS CO
     $RRECTED',5X,'RMS ANALYTICAL',
     $4X,'% (SIN+COS) ERROR'/)
      DO 300     J1 = 6,NMAXIM,10
      J1P9 = J1+9
      UNRMS = 0.D0
      CRMS = 0.D0
      ARMS = 0.D0
      ERRMS = 0.D0
      JK = J1
      IF(J1.EQ.6) JK = 5
      J = JK-1
      DO 250     I1 = JK,J1P9
      I = I1-1
      UNRMS = UNRMS+(PC(I1)**2+PS(I1)**2)
      CRMS = CRMS+(PCC(I1)**2+PSC(I1)**2)
      ARMS = ARMS+SSTF(I)**2
  250 ERRMS = ERRMS+(PCC(I1)**2+(PSC(I1)-SSTF(I))**2)
      TRMS = TRMS+CRMS
```

```
      TERMS = TERMS+ERRMS
      TPOWER = TPOWER+UNRMS
      UNRMS = DSQRT(UNRMS)
      CRMS = DSQRT(CRMS)
      ARMS = DSQRT(ARMS)
      PERRC = DSQRT(ERRMS)/CRMS*100.D0
      WRITE(IO1,260) J,I,UNRMS,CRMS,ARMS,PERRC
  260 FORMAT(2X,I3,' - ',I3,2X,4(G12.6,6X))
  300 CONTINUE
      TRMS = DSQRT(TRMS)
      TERMS = DSQRT(TERMS)
      TPOWER = DSQRT(TPOWER)
      PEREUN = TERMS/TPOWER*100.D0
      PERECO = TERMS/TRMS*100.D0
      WRITE(IO1,310) MAXX, PEREUN, PERECO
  310 FORMAT(//' TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL'
     #,' UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL.',
     #' (BAND FROM 4 CY/REV TO ',I4,' CY/REV)'/,' (A) ',G20.10,10X,'(B)
     # ',G20.10)
      STOP
      END
      SUBROUTINE FOURIE(P,NINT,PS,PC,IWK,WK,CWK)
C
C             THIS SUBROUTINE CARRIES OUT THE FOURIER ANALYSIS
C             OF THE REAL VECTOR "P". NINT IS A POSITIVE, EVEN
C             INTEGER. PS CONTAINS THE COSINE COEFFICIENTS, AND
C             PS  THE SINE ONES.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMPLEX    CWK
      DIMENSION P(1),PS(1),PC(1),IWK(1),CWK(1),WK(1)
      COMMON/COEFFS/ CN(310),DX,IR,NMAX
      NINT2 = NINT/2
      NMAXP9 = NMAX+9
      IO1 = 6
      CALL FFTSC(P,NINT,PS,PC,IWK,WK,CWK)
      DINT = 1.D0/NINT
      DO 10    I = 1,NINT2
      PC(I) = PC(I)*DINT
   10 PS(I) = PS(I)*DINT
      PC(1) = PC(1)*0.5D0
C     WRITE(IO1,20)
   20 FORMAT(//' FOURIER COMP. NO., COS COEFF.  SINE COEFF.'//)
      DO 30    I = 1,NINT2
      IM = I-1
C     IF(IM.LE.NMAXP9.OR.(IM/10)*10.EQ.IM) WRITE(IO1,25) IM,PC(I),PS(I)
   25 FORMAT(1X,I5,2(2X,G20.14))
   30 CONTINUE
      RETURN
      END
//GO.SYSLIB DD
//          DD
//          DD  DSN=SYS2.IMSLLIBD,DISP=SHR
//          DD  DSN=SYS2.IMSLLIBS,DISP=SHR
//GO.FT10F001 DD UNIT=DISK,DISP=(SHR),VOL=SER=TSVOL5,
//  DSN=GDFGCOL.RUN5
//GO.SYSIN DD *
//
```

AIII.5 Numerical orbit integrator and associated subroutines.

Subroutine ORBIT is called by the program where an orbit integration is required. The arguments, that are defined in that program, are: the initial conditions (in arrays "XOS" and "XDOS" for $\underline{x}_0$ and $\dot{\underline{x}}_0$, respectively), and the length of the integration interval "ARCL". The coordinates of the satellite along the orbit and their derivatives are returned in arrays "XX", "Y", "Z", "XD", "YD", "ZD". Although the orbits are essentially polar, the existence of across-track perturbations in the initial state in some cases makes it necessary to have an array to store the "x" component and its derivative. The integration step "H" and the integration accuracy "EPS" (in significant places) are brought in through the COMMON "INTEG", the remaining parameters that control the integrator are defined through the other common statements, and passed on to the subroutine. The various arguments and parameters are explained in the comments at the beginning of the listing. After initializing the procedure, the orbit is computed over a total of "NINT" steps in the "DO loop" at the end of the program. ORBIT prints out headings, the initial conditions, and values of the coordinates and velocities at regular intervals of "DTOUT" seconds (COMMON "COEFFS"). This subroutine calls the numerical integrator subroutine COWELL.

Subroutine COWELL implements the numerical integrator algorithm, which is a variant of the Cowell predictor-corrector developed by Kulikov (see references in paragraph (4.1)). The order of this particular integrator is 8, and it is self-starting. The starter, which obtains the first point after the initial conditions and opens the way to the ordinary predictor-corrector calculation of all those that follow it, works in the usual "bootstrap" fashion, and is written in the segment of code titled "INITIALISATION". The following points are computed by repeated use of the last segment, called "ROUTINE". If the starting procedure requires more than 50 "bootstrapping" iterations to satisfy the accuracy criterion, the assumption is that it is unable to converge, and the orbit is not integrated. Instead, a warning message is printed and the subroutine reaches a "STOP" statement, so the whole run ends. This subroutine calls SECON.

Subroutine SECON inputs the position and velocity vector at a given time "S", and returns the gravitational accelerations in cartesian coordinates in "DELS". This array contains the accelerations at nine points along the orbit. The particular set computed is determined by the parameter "JJ" in COMMON "POURCO". The purpose of this subroutine is to *organize* the way in which the subroutine FORCES, which actually calculates the accelerations, does work. If the "economizing" feature is "on" (IECO = 1), the accelerations are computed with a very low degree field at the "corrector" stage, as most of the change in their values for small corrections to the coordinates is dominated by the first few zonals. The operation is also different in the start-up, and when the predictor-corrector is iterated (ITERA = 1) at every point to improve accuracy (a feature that turned out to be of little help and was not used, because it doubled the computer time needed). As already mentioned, this subroutine calls FORCES.

Subroutine FORCES calculates the gravitational accelerations in a local geocentric frame. As all the fields considered in this study were zonal, only the radial and North-South components are found ("FR" and "FLAT", respectively) and, converted to quasi-inertial equatorial Cartesian coordinates, they are stored in array "DELS" and returned to SECON and COWELL This subroutine uses the Legendre functions obtained through calls to LEGEND, implementing equations related to (2.3.4-8) to find the accelerations. Because of the zonal field, earth rotation was ignored; drag and radiation pressure are not considered, as the satellites are "drag-free".

Subroutine LEGEND computes all the fully normalized Legendre functions of a given order "M" using the recursive formulas (1.2.6-8). Input arguments and returned values are described in the comments following the "SUBROUTINE" statement.

```
      SUBROUTINE ORBIT(XOS,XDOS,XX,Y,Z,XD,YD,ZD,ARCL)
C
C           THIS SUBROUTINE COMPUTES AN ORBIT STARTING
C           FROM THE INITIAL CONDITIONS  "XOS, XDOS"(BOTH
C           VECTORS OF DIMENSION 3), WHERE  XOS(1) = X0,
C           XOS(2) = Y0, XOS(3) = Z0, XDOS(1) = X(DOT)0,
C           XDOS(2) = Y(DOT)0, AND XDOS(3) = Z(DOT)0.
C
C           "ARCL"  IS THE ARC LENGTH IN SECONDS;
C           "H"  IS THE INTEGRATION STEP;
C           "XX,Y,Z,XD,YD,ZD" ARE ALL VECTORS OF DIMENSION
C           " NINTP"  , WHERE  NINTP = ARCL/H+1 . THESE VECTORS
C           CONTAIN THE COORDINATES AND VELOCITY COMPONENTS
C           ("D" FOR "DOT") OF THE SUCCESSIVE ORBIT POINTS
C           IN AN  INERTIAL, EQUATORIAL, RIGHT-HANDED SYSTEM.
C           XX(1),Y(1),Z(1),XD(1),YD(1), AND ZD(1) ARE SET
C           EQUAL TO THE INITIAL CONDITIONS BY THIS SUBROUTINE.
C
C           ARRAY "CN" AND VARIABLES "TELEM" AND "DTOUT" IN
C           COMMON /COEFFS/ ARE: (A) AN ARRAY OF ZONAL NORMALIZED
C           COEFFICIENTS, WHERE CN(1) CORRESPONDS TO THE ZERO
C           HARMONIC; (B) THE STARTING TIME (USUALLY ZERO SECONDS);
C           (C) THE INTERVAL AT WHICH ORBIT COORDINATES ARE PRINTED
C           OUT. IF  DTOUT  IS LARGER THAN ARCL, ONLY INITIAL AND FINAL
C           STATES ARE PRINTED.
C
C           IF IECO = 1  THE ACCELERATIONS ARE UPDATED DURING THE
C           'CORRECTOR' PHASE (IN "COWELL") WITH A SMALL DEGREE FIELD,
C           TO ECONOMIZE COMPUTING. IECO MUST BE DEFINED IN MAIN PROG.
C
C           THIS SUBROUTINE CALLS SUBROUTINES "COWELL", "SECON",
C           "FORCES", AND "LEGEND".
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/INTEG/ H,EPS
      COMMON/IO/ IO1,IOS(7)
      COMMON/COEFFS/ CN(310),DSR2I,IR,NMAX,TELEM,DTOUT
      COMMON/POURCO/ FPER(3,9),JX,KPX,IECO,ITERA,ITX,III
      DIMENSION XOS(1),XDOS(1),XX(1),Y(1),Z(1),XD(1),YD(1),ZD(1),
     # X(3,9),XP(3,9)
      T = TELEM
      IR = 0
      EPS = 1.D-16
      INIT = 0
      DSR2I = 1.D0/DSQRT(2.D0)
      NINT = ARCL/H+1.D-5
      XX(1) = XOS(1)
      Y(1) = XOS(2)
      Z(1) = XOS(3)
      XD(1) = XDOS(1)
      YD(1) = XDOS(2)
      ZD(1) = XDOS(3)
```

```
      X(1,5)  = XX(1)
      X(2,5)  = Y(1)
      X(3,5)  = Z(1)
      XP(1,5)  = XD(1)
      XP(2,5)  = YD(1)
      XP(3,5)  = ZD(1)
      INT = DTOUT/H+1.D-5
      WRITE(IO1,11)  (X(I,5),I=1,3),(XP(I,5),I=1,3)
   11 FORMAT(//' *** ORBIT : X, Y, Z, XD, YD, ZD,',
     # ' R, V, T = ',//'     INITIAL CONDITIONS : '//3(2X,G20.14))
      IF(IECO.EQ.1) WRITE(IO1,666)
  666 FORMAT(/' #### ECONOMIZING IS ON ####'/)
      DO 100    NI = 1,NINT
      NIP = NI+1
      CALL COWELL(X,XP,TELEM,T,NCA,INIT)
      INIT = 1
      XX(NIP)  = X(1,5)
      Y(NIP)  = X(2,5)
      Z(NIP)  = X(3,5)
      XD(NIP)  = XP(1,5)
      YD(NIP)  = XP(2,5)
      ZD(NIP)  = XP(3,5)
      IF((NI/INT)*INT.NE.NI.AND.NI.NE.NINT)  GO TO 100
      R = DSQRT(X(1,5)**2+X(2,5)**2+X(3,5)**2)
      V = DSQRT(XP(1,5)**2+XP(2,5)**2+XP(3,5)**2)
      WRITE(IO1,12)  (X(I,5),I=1,3),(XP(I,5),I=1,3),R,V,T
   12 FORMAT(//3(2X,G20.14))
  100 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SECON(X,XP,S,DELS)
C
C             THIS SUBROUTINE CALLS SUB. "FORCES" WITH A FULL
C             OR A PARTIAL FIELD, AS AN OPTION TO SAVE COMPUTING.
C
      IMPLICIT REAL*8 (A-H,O-Z)
```

```
      DIMENSION X(3,9),XP(3,9),DELS(3,9),XOLD(3,9),XPOLD(3,9),
     *DELSN(3,9),DELSO(3,9)
      COMMON/POURCO/ FPER(3,9),JJ,KPC,IECO,ITERA,ITX,IJA
      IF(IECO.EQ.1) GO TO 5
      IJA = 1
      CALL FORCES(X,XP,S,DELS)
      RETURN
    5 CONTINUE
      IF(KPC.NE.10) GO TO 20
      IJA = 1
      CALL FORCES(X,XP,S,DELS)
      IJA = 0
      CALL FORCES(X,XP,S,DELSO)
      IF(JJ.NE.8) GO TO 15
      DO 10     N = 1,3
      DO 10     I = 1,8
      XOLD(N,I) = X(N,I)
   10 XPOLD(N,I) = XP(N,I)
   15 IF(JJ.NE.9) RETURN
      DO 18     N = 1,3
      XOLD(N,9) = X(N,9)
      XPOLD(N,9) = XP(N,9)
   18 CONTINUE
      RETURN
   20 IJA = 0
      CALL FORCES(X,XP,S,DELSN)
      DO 30      I = 1,3
   30 DELS(I,JJ) = DELS(I,JJ)+(DELSN(I,JJ)-DELSO(I,JJ))
      IF(JJ.NE.9) GO TO 50
      IF(ITERA.EQ.1.AND.ITX.LT.1) GO TO 60
      DO 40     N = 1,3
      DO 40     I = 1,8
      DELSO(N,I) = DELSN(N,I+1)
      XOLD(N,I) = X(N,I+1)
   40 XPOLD(N,I) = XP(N,I+1)
   50 RETURN
   60 DO 55     N = 1,3
      DO 55     I = 1,8
      DELSO(N,I) = DELSN(N,I)
      XOLD(N,I) = X(N,I)
   55 XPOLD(N,I) = XP(N,I)
      RETURN
      END
      SUBROUTINE FORCES(X,XP,T,DELS)
C
C             COMPUTES ACCELERATIONS IN INERTIAL SPACE
C             FOR A ZONAL FIELD.
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(3,9),XP(3,9),DELS(3,9)
      COMMON/POURCO/ FPER(3,9),JJ,KPC,IECO,ITERA,ITX,IJA
      COMMON/GEOCON/ GM,AE
      COMMON/COEFFS/ CN(310),DSR2I,IR,NMIX
      COMMON/LEG/ DRTS(700)
      DIMENSION RN(310),PNM(310),RLNN(310),PNMP(310)
      NMAX = NMIX
      NMEX = 10
C ***************************************************
      IF(NMAX.GT.NMEX.AND.IJA.EQ.0) NMAX = NMEX
      R = DSQRT(X(1,JJ)**2+X(2,JJ)**2+X(3,JJ)**2)
```

```
      RI = 1.D0/R
      PRO = DSQRT(X(1,JJ)**2+X(2,JJ)**2)
      CPH = PRO*RI
      SPH = X(3,JJ)*RI
      IF(DABS(PRO).LT.1.D-6) GO TO 5
      PROI = 1.D0/PRO
      CL = X(1,JJ)*PPOI
      SL = X(2,JJ)*PROI
      GO TO 7
    5 CL = 0.D0
      SL = 1.D0
    7 NMAXP = NMAX+1
      NMAXPP = NMAX+2
      AOR = AE*RI
      GMA2 = GM/AE**2
      RN(1) = AOR**2
      RN(2) = AOR**3
      CTH = SPH
      STH = CPH
      DO 10   N1 = 3,NMAXP
   10 RN(N1) = AOR*RN(N1-1)
      CALL LEGEND(0,CTH,STH,PNMP,NMAXPP,IR,RLNN)
      DO 15     N1 = 1,NMAXP
      PNMP(N1) = PNMP(N1)*RN(N1)
   15 PNM(N1) = PNMP(N1)
      CALL LEGEND(1,CTH,STH,PNMP,NMAXPP,IR,RLNN)
      FR = 0.D0
      FLAT = 0.D0
      DO 30    N1 = 1,NMAXP
      N = N1-1
      PNMP(N1) = PNMP(N1)*RN(N1)
      PNN = PNM(N1)*N1
      IF(N1.EQ.1) GO TO 35
      FLAT1 = PNMP(N1)*DRTS(N1)*DRTS(N)*DSR2I
      GO TO 36
   35 FLAT1 = 0.D0
   36 FLA = FLAT1
      FLAT = FLAT+FLA*CN(N1)
      FR = FR+PNN*CN(N1)
   30 CONTINUE
      FR = -GMA2*FR
      FLAT = GMA2*FLAT
      FH = FR*CPH-FLAT*SPH
      DELS(1,JJ) = FH*CL
      DELS(2,JJ) = FH*SL
      DELS(3,JJ) = FR*SPH+FLAT*CPH
C****************************************************************
      RETURN
      END
      SUBROUTINE COWELL(X,XP,TELEM,T,NCA,INIT)
C
C     ADAMS-COWELL PREDICTOR/CORRECTOR ALGORITHM
C
C     MUCGIN/COWELL   C .RIZOS' VERSION   OF FEB. 82
C     MODIFIED BY O.L.COLOMBO ON SEP. 83 TO INCLUDE THE OPTION
C     OF ITERATING THE PREDICTOR-CORRECTOR ("ROUTINE") STEP
C     BY SETTING
C                     ITERA = 1
C
C
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(3,9),XP(3,9),DEL(3,9),DELP(3,9),DELS(3,9)
      DIMENSION XPP(27),XPMAX(27)
      COMMON/INTEG/H,EPS
      DIMENSION IALPHA(8,7),IBETA(8,7)
      COMMON/IO/IO1,IN1,IN2,IN3,IN4,IN5,IN6,IN7
      COMMON/POURCO/FPER(3,9),JJ,KPC,IECO,ITERA,ITX,IJA
      DATA D/3628800.D0/,DP/120960.D0/,
     @IALPHA/237671,-9829,1571,-289,-289,1571,-9829,237671,
     #244614,306474,-20826,3594,1734,-11286,70374,-1673526,
     #-401475,38205,339465,-26895,-2475,34725,-217695,5061465,
     #378740,-57460,-16780,349580,-16780,-57460,378740,-8536180,
     #-217695,34725,-2475,-26895,339465,38205,-401475,8697225,
     #70374,-11286,1734,3594,-20826,306474,244614,-5392566,
     #-9829,1571,-289,-289,1571,-9829,237671,1908311/,
     @IBETA/84161,1375,-351,191,-191,351,-1375,36799,
     #-55688,74536,3832,-1688,1528,-2648,9976,-258968,
     #66109,-26813,67165,7843,-5699,8899,-31523,782755,
     #-57024,17984,-14528,60480,14528,-17984,57024,-1319488,
     #31523,-8899,5699,-7843,53795,26813,-66109,1344989,
     #-9976,2648,-1528,1688,-3832,46424,55688,-838888,
     #1375,-351,191,-191,351,-1375,36799,313281/
C
      IF(INIT) 1001,1001,1003
 1001 CONTINUE
      IPRINT = 0
      HH = H/86400
      NPR = 4
      NCA = 3
      JMAX = 5+NPR
      ITERA = 0
      TCOW=TELEM+T/86400.0
      INIT=1
      IBOUCL=0
      IND=0
C
C       ---- INITIALISATION ----   FIRST RUN IN EACH ITERATION...
C
      KPC=10
      JJ=5
C
C     ACCELER AT ORBIT TIME 'TCOW' & PARTIAL VARIATIONS
      CALL SECON(X,XP,TCOW,DELS)
      DO 4 N=1,NCA
      DO 5 I=1,8
    5 DELS(N,I)=DELS(N,5)
      XP(N,8)=0.D0
    4 XPMAX(N)=DABS(XP(N,5))
 1002 DO 11 N=1,NCA
      ZP=0.D0
      Z =0.D0
      DO 12 I=1,7
      ZP=ZP+ IBETA(4,I)*DELS(N,I+1)
   12 Z =Z +IALPHA(4,I)*DELS(N,I+1)
      DELP(N,5)=XP(N,5)/ H    -ZP/DP
      DEL  (N,5)=X  (N,5)/(H*H)-Z /D
      DO 13 I=1,3
      IM=5-I
      IP=5+I
      DELP(N,IM)=DELP(N,IM+1)-DELS( N,IM)
```

```
      DEL  (N,IM)=DEL  (N,IM+1)-DELP(N,IM+1)
      DELP(N,IP)=DELP(N,IP-1)+DELS(N,IP-1)
      DEL  (N,IP)=DEL  (N,IP-1)+DELP(N,IP)
   13 CONTINUE
C
      XPP(N)=XP(N,8)
      DO 14 J=2,8
      IF(J-5) 9,14,9
    9 ZP=0.D0
      Z =0.D0
      DO 15 I=1,7
      ZP=ZP+ IBETA(J-1,I)*DELS(N,I+1)
   15 Z =Z +IALPHA(J-1,I)*DELS(N,I+1)
      XP(N,J)=   H*(DELP(N,J)+ZP/DP)
       X(N,J)=H*H*(DEL  (N,J)+Z /D )
   14 CONTINUE
   11 CONTINUE
C
      DO 16 J=2,8
      S=TCOW+HH*(J-5)
      IF(J-5) 17,16,17
   17 CONTINUE
      JJ=J
C
C     ACCELERS AT OTHER TIMES FOR STARTUP (3 BEFORE, 3 AFTER)
      CALL SECON(X,XP,S,DELS)
   16 CONTINUE
C
      DO 18 N=1,NCA
      XPMAX(N)=DMAX1(XPMAX(N),DABS(XP(N,8)))
      IF(DABS(XP(N,8)-XPP(N)).GT.XPMAX(N)*EPS) GO TO 19
   18 CONTINUE
      IF(IPRINT.GE.3) WRITE(IO1,2000) IBOUCL
 2000 FORMAT(/,' NO. OF ITERATIONS FOR START UP ',I3,/)
      GO TO 1003
   19 IBOUCL=IBOUCL+1
      IF(IBOUCL-50) 1002,1002,1000
 1000 WRITE(IO1,3663)
 3663 FORMAT(//' +++ STOP||  MORE THAN 50 ITERATIONS FOR STARTUP')
      STOP
C
C       ---- ROUTINE ----
C
 1003 ITX = 0
 1004 IND=1
      DO 61 N=1,NCA
      ZP=0.D0
      Z =0.D0
      DO 62 I=1,7
      ZP=ZP+ IBETA(4,I)*DELS(N,I+1)
   62 Z =Z +IALPHA(4,I)*DELS(N,I+1)
      DELP(N,5)=XP(N,5)/ H    -ZP/DP
      DEL  (N,5)=X  (N,5)/(H*H)-Z /D
      DO 63 I=1,4
      IP=5+I
      DELP(N,IP)=DELP(N,IP-1)+DELS(N,IP-1)
   63 DEL  (N,IP)=DEL  (N,IP-1)+DELP(N,IP)
      ZP=0.D0
      Z =0.D0
      DO 55 I=1,7
```

```
      ZP=ZP+ IBETA(8,I)*DELS(N,I+1)
   55 Z =Z +IALPHA(8,I)*DELS(N,I+1)
      XP(N,9)=  H*(DELP(N,9)+ZP/DP)
      X (N,9)=H*H*(DEL  (N,9)+Z /D )
   61 CONTINUE
C
      S=TCOW+HH*4.D0
      KPC=10
      JJ=9
C
C    ACCELER ONE PT. BEYOND RANGE (7 PTS, 3 BEFORE, 3 AFTER & ORBIT)
      CALL SECON(X,XP,S,DELS)
      DO 54 N=1,NCA
      DO 64 J=6,JMAX
      ZP=0.D0
      Z =0.D0
      DO 65 I=1,7
      ZP=ZP+ IBETA(J-2,I)*DELS(N,I+2)
   65 Z =Z +IALPHA(J-2,I)*DELS(N,I+2)
      XP(N,J)=  H*(DELP(N,J)+ZP/DP)
   64 X (N,J)=H*H*(DEL  (N,J)+Z /D )
   54 CONTINUE
C
      DO 51 J=6,JMAX
      S=TCOW+HH*(J-5)
      KPC=J
      JJ=J
C
C    EVALUATION (PARTIAL AS AN OPTION) OF ACCELS. AT UPDATED
C    4 'FORWARD' ELEMENTS.
      CALL SECON(X,XP,S,DELS)
   51 CONTINUE
      ITX = ITX+1
      IF(ITERA.EQ.1.AND.ITX.LT.2) GO TO 1004
C
C    SHUFFLE ALL ELEMENTS TO NEXT TIME PT.
      T=T+H
      DO 200 N=1,NCA
      DO 201 I=1,8
      XP(N,I)=XP(N,I+1)
      X (N,I)=X (N,I+1)
      FPER(N,I)=FPER(N,I+1)
  201 DELS(N,I)=DELS(N,I+1)
  200 CONTINUE
      RETURN
      END
      SUBROUTINE LEGEND(M,COTHET,SITHET,RLEG,NMX,IR,RLNN)
C
C            THIS SUBROUTINE COMPUTES  ALL NORMALIZED LEGENDRE FUNCTIONS
C            IN 'RLEG' . ORDER IS ALWAYS
C            M , AND COLATITUDE IS ALWAYS THETA  (RADIANS). MAXIMUM DEGR
C            IS  NMX  .
C            IR  MUST BE SET TO ZERO BEFORE THE FIRST CALL TO THIS SUB.
C            THE DIMENSIONS OF ARRAYS  RLEG, AND RLNN  MUST BE
C            AT LEAST EQUAL TO  NMX+1  .
C
C
C       PROGRAMMER :  OSCAR L. COLOMBO, DEPT. OF GEODETIC SCIENCE,
C       THE OHIO STATE UNIVERSITY, AUGUST 1980 . **********************
C
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION RLEG(1),RLNN(1)
     2, DIRT(700)
      COMMON/LEG/DRTS(700)
      NMX1 = NMX+1
      NMX2P = 2*NMX+1
      DO 123     N = 1,35
      RLEG(N) = 0.D0
  123 RLNN(N) = 0.D0
      M1 = M+1
      M2 = M+2
      M3 = M+3
      IF(IR.EQ.1) GO TO 10
      IR = 1
      DO 5      N = 1,NMX2P
      DRTS(N) = DSQRT(N*1.D0)
    5 DIRT(N) = 1.D0/DRTS(N)
   10 CONTINUE
C
C             COMPUTE THE LEGENDRE FUNCTIONS .
C
      RLNN(1) = 1.D0
      RLNN(2) = SITHET*DRTS(3)
      DO 15     N1 = 3,M1
      N = N1-1
      N2 = N+N
   15 RLNN(N1) = DRTS(N2+1)*DIRT(N2)*SITHET*RLNN(N1-1)
      IF(M.GT.1) GO TO 20
      IF(M.EQ.0) GO TO 16
      RLEG(2) = RLNN(2)
      RLEG(3) = DRTS(5)*COTHET*RLEG(2)
      GO TO 20
   16 RLEG(1) = 1.D0
      RLEG(2) = COTHET*DRTS(3)
   20 CONTINUE
      RLEG(M1) = RLNN(M1)
      RLEG(M2) = DRTS(M1*2+1)*COTHET*RLEG(M1)
      DO 30     N1 = M3,NMX1
      N = N1-1
      IF(M.EQ.0.AND.N.LT.2.OR.M.EQ.1.AND.N.LT.3) GO TO 30
      N2 = N+N
      RLEG(N1) = DRTS(N2+1)*DIRT(N+M)*DIRT(N-M)*(DRTS(N2-1)*COTHET*
     2 RLEG(N1-1)-DRTS(N+M-1)*DRTS(N-M-1)*DIRT(N2-3)*RLEG(N1-2))
      GO TO 30
   30 CONTINUE
       RETURN
      END
//GO.FT10F001 DD UNIT=DISK,DISP=(NEW,CATLG),VOL=SER=DISK10,
//    DCB=(RECFM=VBS,LRECL=16500,BLKSIZE=16504),
//   SPACE=(TRK,(1,1)),DSN=GDFGCOL.SSTPY5
//GO.SYSIN DD *
//
```

## APPENDIX IV.

### Detailed listings.

In this Appendix, Table AIV-1 contains the detailed listing of the percentage errors of the lumped coefficients of the model, compared to those obtained by simulation and correction for nonperiodic effects, for the case where the separation between satellites is 100 km with no errors in the initial states of the nominal orbits. Table AIV-2 shows similar results for 300 km separation, also without initial state errors. The absorbtion bands, where integrator errors and nonlinear effects dominate the very small first order perturbations, can be seen quite clearly.

Table AIV-1

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 100000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = .0 DX11 = .0 DX12 = .0 DR1 = .0 DUP1 = .0
DR2 = .0 DUP2 = .0
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 4 | .331896D-08 | .106742D-04 | .331896D-08 | .721135D-05 | .710599D-05 | 1.46104 |
| 5 | .225890D-08 | .925684D-05 | .225890D-08 | .655151D-05 | .659245D-05 | -.624962 |
| 6 | .612056D-09 | .843878D-05 | .612056D-09 | .621267D-05 | .624859D-05 | -.578093 |
| 7 | .151886D-08 | .841565D-05 | .151886D-08 | .652191D-05 | .651891D-05 | .460451D-01 |
| 8 | .351827D-08 | .808417D-05 | .351827D-08 | .643520D-05 | .642472D-05 | .162848 |
| 9 | .334884D-09 | .846540D-05 | .334884D-09 | .700452D-05 | .706211D-05 | -.822231 |
| 10 | .662453D-08 | .839880D-05 | .662453D-08 | .708712D-05 | .702492D-05 | .877631 |
| 11 | -.476278D-08 | .899746D-05 | -.476278D-08 | .780711D-05 | .793119D-05 | -1.58935 |
| 12 | -.679630D-08 | .897408D-05 | -.679630D-08 | .788438D-05 | .788716D-05 | -.352690D-01 |
| 13 | .640714D-08 | .101475D-04 | .640714D-08 | .914269D-05 | .904646D-05 | 1.05251 |
| 14 | .869005D-08 | .989020D-05 | .869005D-08 | .895790D-05 | .896755D-05 | -.107693 |
| 15 | -.430958D-08 | .111789D-04 | -.430958D-08 | .103093D-04 | .104026D-04 | -.905363 |
| 16 | -.653833D-08 | .110874D-04 | -.653833D-08 | .102726D-04 | .102733D-04 | -.650137D-02 |
| 17 | .700842D-08 | .128843D-04 | .700842D-08 | .121178D-04 | .120390D-04 | .650355 |
| 18 | .954808D-08 | .125662D-04 | .954808D-08 | .118425D-04 | .118481D-04 | -.467122D-01 |
| 19 | -.354148D-08 | .146521D-04 | -.354148D-08 | .139667D-04 | .140409D-04 | -.531256 |
| 20 | -.593375D-08 | .144309D-04 | -.593375D-08 | .137800D-04 | .137779D-04 | .151262D-01 |
| 21 | .806151D-08 | .172540D-04 | .806151D-08 | .166342D-04 | .165646D-04 | .418487 |
| 22 | .108783D-07 | .168060D-04 | .108783D-07 | .162145D-04 | .162167D-04 | -.133589D-01 |
| 23 | -.215553D-08 | .204076D-04 | -.215553D-08 | .198419D-04 | .199024D-04 | -.304899 |
| 24 | -.467284D-08 | .199974D-04 | -.467284D-08 | .194554D-04 | .194502D-04 | .266832D-01 |
| 25 | .100042D-07 | .252500D-04 | .100042D-07 | .247298D-04 | .246647D-04 | .262876 |
| 26 | .131770D-07 | .245724D-04 | .131770D-07 | .240723D-04 | .240706D-04 | .702471D-02 |
| 27 | .859640D-09 | .328276D-04 | .859640D-09 | .323460D-04 | .323953D-04 | -.152416 |
| 28 | -.155763D-08 | .320139D-04 | -.155763D-08 | .315495D-04 | .315408D-04 | .276358D-01 |
| 29 | .155358D-07 | .492837D-04 | .155358D-07 | .488354D-04 | .487828D-04 | .107542 |
| 30 | .199391D-07 | .471054D-04 | .199391D-07 | .466721D-04 | .466801D-04 | -.170837D-01 |
| 31 | .200026D-07 | .120013D-03 | .200026D-07 | .119593D-03 | .119525D-03 | .574469D-01 |
| 32 | .188631D-07 | .127292D-03 | .188631D-07 | .126886D-03 | .126784D-03 | .797860D-01 |
| 33 | -.120503D-07 | -.103416D-03 | -.120503D-07 | -.103810D-03 | -.103739D-03 | .685837D-01 |
| 34 | -.192340D-07 | -.123588D-03 | -.192340D-07 | -.123970D-03 | -.123881D-03 | .717501D-01 |
| 35 | .109473D-07 | .923127D-04 | .109473D-07 | .919414D-04 | .919270D-04 | .156557D-01 |
| 36 | .181691D-07 | .112394D-03 | .181691D-07 | .112033D-03 | .112003D-03 | .271059D-01 |
| 37 | -.119762D-07 | -.913208D-04 | -.119762D-07 | -.916721D-04 | -.916135D-04 | .638314D-01 |
| 38 | -.194124D-07 | -.111776D-03 | -.194124D-07 | -.112118D-03 | -.112047D-03 | .632220D-01 |
| 39 | .104644D-07 | .800224D-04 | .104644D-07 | .796892D-04 | .796808D-04 | .105175D-01 |
| 40 | .179437D-07 | .100302D-03 | .179437D-07 | .999769D-04 | .999557D-04 | .212132D-01 |
| 41 | -.117895D-07 | -.810240D-04 | -.117895D-07 | -.813409D-04 | -.812927D-04 | .592226D-01 |
| 42 | -.194611D-07 | -.101517D-03 | -.194611D-07 | -.101827D-03 | -.101768D-03 | .573081D-01 |
| 43 | .988502D-08 | .694092D-04 | .988502D-08 | .691071D-04 | .691034D-04 | .528323D-02 |
| 44 | .175650D-07 | .896273D-04 | .175650D-07 | .893320D-04 | .893183D-04 | .154250D-01 |
| 45 | -.115440D-07 | -.721560D-04 | -.115440D-07 | -.724447D-04 | -.724036D-04 | .567436D-01 |
| 46 | -.193930D-07 | -.924602D-04 | -.193930D-07 | -.927426D-04 | -.926935D-04 | .528720D-01 |
| 47 | .922634D-08 | .601335D-04 | .922634D-08 | .598571D-04 | .598569D-04 | .306722D-03 |
| 48 | .170485D-07 | .800854D-04 | .170485D-07 | .798148D-04 | .798069D-04 | .991160D-02 |
| 49 | -.112560D-07 | -.644489D-04 | -.112560D-07 | -.647140D-04 | -.646780D-04 | .556277D-01 |
| 50 | -.192233D-07 | -.843840D-04 | -.192233D-07 | -.846439D-04 | -.846020D-04 | .494754D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 51 | .850482D-08 | .519668D-04 | .850482D-08 | .517121D-04 | .517145D-04 | -.456978D-02 |
| 52 | .164119D-07 | .714901D-04 | .164119D-07 | .712402D-04 | .712370D-04 | .459653D-02 |
| 53 | -.109393D-07 | -.577157D-04 | -.109393D-07 | -.579608D-04 | -.579285D-04 | .555979D-01 |
| 54 | -.189690D-07 | -.771392D-04 | -.189690D-07 | -.773797D-04 | -.773434D-04 | .469101D-01 |
| 55 | .773637D-08 | .447451D-04 | .773637D-08 | .445089D-04 | .445132D-04 | -.952793D-02 |
| 56 | .156739D-07 | .637119D-04 | .156739D-07 | .634799D-04 | .634803D-04 | -.631222D-03 |
| 57 | -.106083D-07 | -.518157D-04 | -.106083D-07 | -.520436D-04 | -.520142D-04 | .564761D-01 |
| 58 | -.186472D-07 | -.706165D-04 | -.186472D-07 | -.708405D-04 | -.708085D-04 | .450435D-01 |
| 59 | .693594D-08 | .383433D-04 | .693594D-08 | .381232D-04 | .381288D-04 | -.147928D-01 |
| 60 | .148533D-07 | .566551D-04 | .148533D-07 | .564386D-04 | .564420D-04 | -.589247D-02 |
| 61 | -.102737D-07 | -.466374D-04 | -.102737D-07 | -.468503D-04 | -.468231D-04 | .581352D-01 |
| 62 | -.182730D-07 | -.647311D-04 | -.182730D-07 | -.649406D-04 | -.649121D-04 | .437833D-01 |
| 63 | .611676D-08 | .326621D-04 | .611676D-08 | .324559D-04 | .324626D-04 | -.206307D-01 |
| 64 | .139680D-07 | .502454D-04 | .139680D-07 | .500425D-04 | .500481D-04 | -.113039D-01 |
| 65 | -.994398D-08 | -.420874D-04 | -.994398D-08 | -.422872D-04 | -.422617D-04 | .604599D-01 |
| 66 | -.178598D-07 | -.594125D-04 | -.178598D-07 | -.596093D-04 | -.595836D-04 | .430615D-01 |
| 67 | .529012D-08 | .276201D-04 | .529012D-08 | .274262D-04 | .274338D-04 | -.273986D-01 |
| 68 | .130345D-07 | .444227D-04 | .130345D-07 | .442317D-04 | .442392D-04 | -.170167D-01 |
| 69 | -.962627D-08 | -.380852D-04 | -.962627D-08 | -.382734D-04 | -.382491D-04 | .633956D-01 |
| 70 | -.174184D-07 | -.545995D-04 | -.174184D-07 | -.547850D-04 | -.547616D-04 | .428309D-01 |
| 71 | .446760D-08 | .231504D-04 | .446760D-08 | .229675D-04 | .229756D-04 | -.355150D-01 |
| 72 | .120691D-07 | .391379D-04 | .120691D-07 | .389575D-04 | .389665D-04 | -.231670D-01 |
| 73 | -.932288D-08 | -.345582D-04 | -.932288D-08 | -.347362D-04 | -.347129D-04 | .668791D-01 |
| 74 | -.169574D-07 | -.502360D-04 | -.169574D-07 | -.504115D-04 | -.503898D-04 | .430553D-01 |
| 75 | .366011D-08 | .191984D-04 | .366011D-08 | .190252D-04 | .190339D-04 | -.455913D-01 |
| 76 | .110879D-07 | .343514D-04 | .110879D-07 | .341805D-04 | .341907D-04 | -.299253D-01 |
| 77 | -.903368D-08 | -.314387D-04 | -.903368D-08 | -.316074D-04 | -.315850D-04 | .708719D-01 |
| 78 | -.164806D-07 | -.462677D-04 | -.164806D-07 | -.464342D-04 | -.464139D-04 | .437088D-01 |
| 79 | .287953D-08 | .157223D-04 | .287953D-08 | .155579D-04 | .155670D-04 | -.585021D-01 |
| 80 | .101081D-07 | .300341D-04 | .101081D-07 | .298717D-04 | .298829D-04 | -.374849D-01 |
| 81 | -.875403D-08 | -.286586D-04 | -.875403D-08 | -.288190D-04 | -.287973D-04 | .753772D-01 |
| 82 | -.159875D-07 | -.426380D-04 | -.159875D-07 | -.427964D-04 | -.427773D-04 | .447782D-01 |
| 83 | .214198D-08 | .126966D-04 | .214198D-08 | .125401D-04 | .125496D-04 | -.754910D-01 |
| 84 | .915053D-08 | .261708D-04 | .915053D-08 | .260161D-04 | .260281D-04 | -.460644D-01 |
| 85 | -.847026D-08 | -.261417D-04 | -.847026D-08 | -.262945D-04 | -.262734D-04 | .804859D-01 |
| 86 | -.154676D-07 | -.392798D-04 | -.154676D-07 | -.394308D-04 | -.394125D-04 | .462679D-01 |
| 87 | .147266D-08 | .101228D-04 | .147266D-08 | .997348D-05 | .998327D-05 | -.981653D-01 |
| 88 | .824711D-08 | .227715D-04 | .824711D-08 | .226239D-04 | .226366D-04 | -.558828D-01 |
| 89 | -.815085D-08 | -.237827D-04 | -.815085D-08 | -.239286D-04 | -.239079D-04 | .865335D-01 |
| 90 | -.148905D-07 | -.360947D-04 | -.148905D-07 | -.362390D-04 | -.362215D-04 | .482008D-01 |
| 91 | .924886D-09 | .806399D-05 | .924886D-09 | .792128D-05 | .793138D-05 | -.127565 |
| 92 | .745934D-08 | .199066D-04 | .745934D-08 | .197654D-04 | .197787D-04 | -.669936D-01 |
| 93 | -.771279D-08 | -.213781D-04 | -.771279D-08 | -.215177D-04 | -.214974D-04 | .946883D-01 |
| 94 | -.141681D-07 | -.328853D-04 | -.141681D-07 | -.330234D-04 | -.330065D-04 | .513212D-01 |
| 95 | .651096D-09 | .679366D-05 | .651096D-09 | .665695D-05 | .666749D-05 | -.158294 |
| 96 | .696290D-08 | .178519D-04 | .696290D-08 | .177166D-04 | .177303D-04 | -.775508D-01 |
| 97 | -.684023D-08 | -.182415D-04 | -.684023D-08 | -.183753D-04 | -.183548D-04 | .111862 |
| 98 | -.129412D-07 | -.290030D-04 | -.129412D-07 | -.291355D-04 | -.291186D-04 | .580154D-01 |
| 99 | .153059D-08 | .816293D-05 | .153059D-08 | .803174D-05 | .804530D-05 | -.168788 |
| 100 | .778361D-08 | .181194D-04 | .778361D-08 | .179895D-04 | .180076D-04 | -.100335 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 101 | -.135548D-08 | -.549721D-05 | -.135548D-08 | -.562580D-05 | -.561226D-05 | .240673 |
| 102 | -.550458D-08 | -.110844D-04 | -.550458D-08 | -.112117D-04 | -.111996D-04 | .108641 |
| 103 | -.643980D-08 | -.121721D-04 | -.643980D-08 | -.122982D-04 | -.122798D-04 | .149431 |
| 104 | -.661892D-08 | -.122397D-04 | -.661892D-08 | -.123645D-04 | -.123452D-04 | .156260 |
| 105 | -.679989D-08 | -.125053D-04 | -.679989D-08 | -.126290D-04 | -.126099D-04 | .151282 |
| 106 | -.687862D-08 | -.124779D-04 | -.687862D-08 | -.126004D-04 | -.125817D-04 | .148424 |
| 107 | -.705311D-08 | -.127372D-04 | -.705311D-08 | -.128586D-04 | -.128402D-04 | .143283 |
| 108 | -.713110D-08 | -.127059D-04 | -.713110D-08 | -.128261D-04 | -.128079D-04 | .142001 |
| 109 | -.730919D-08 | -.129672D-04 | -.730919D-08 | -.130864D-04 | -.130682D-04 | .138743 |
| 110 | -.738732D-08 | -.129324D-04 | -.738732D-08 | -.130505D-04 | -.130325D-04 | .137634 |
| 111 | -.756916D-08 | -.131956D-04 | -.756916D-08 | -.133126D-04 | -.132947D-04 | .134008 |
| 112 | -.764733D-08 | -.131574D-04 | -.764733D-08 | -.132733D-04 | -.132556D-04 | .133602 |
| 113 | -.783307D-08 | -.134225D-04 | -.783307D-08 | -.135374D-04 | -.135197D-04 | .131076 |
| 114 | -.791106D-08 | -.133808D-04 | -.791106D-08 | -.134948D-04 | -.134771D-04 | .130537 |
| 115 | -.810053D-08 | -.136477D-04 | -.810053D-08 | -.137607D-04 | -.137431D-04 | .127550 |
| 116 | -.817840D-08 | -.136027D-04 | -.817840D-08 | -.137146D-04 | -.136972D-04 | .127430 |
| 117 | -.837162D-08 | -.138715D-04 | -.837162D-08 | -.139825D-04 | -.139650D-04 | .125142 |
| 118 | -.844950D-08 | -.138230D-04 | -.844950D-08 | -.139331D-04 | -.139156D-04 | .124899 |
| 119 | -.864622D-08 | -.140936D-04 | -.864622D-08 | -.142027D-04 | -.141854D-04 | .122265 |
| 120 | -.872381D-08 | -.140417D-04 | -.872381D-08 | -.141499D-04 | -.141326D-04 | .122293 |
| 121 | -.892452D-08 | -.143142D-04 | -.892452D-08 | -.144215D-04 | -.144042D-04 | .120151 |
| 122 | -.900182D-08 | -.142589D-04 | -.900182D-08 | -.143653D-04 | -.143481D-04 | .120083 |
| 123 | -.920613D-08 | -.145331D-04 | -.920613D-08 | -.146387D-04 | -.146215D-04 | .117676 |
| 124 | -.928315D-08 | -.144744D-04 | -.928315D-08 | -.145792D-04 | -.145620D-04 | .117798 |
| 125 | -.949136D-08 | -.147506D-04 | -.949136D-08 | -.148544D-04 | -.148373D-04 | .115757 |
| 126 | -.956798D-08 | -.146885D-04 | -.956798D-08 | -.147915D-04 | -.147744D-04 | .115804 |
| 127 | -.977981D-08 | -.149664D-04 | -.977981D-08 | -.150686D-04 | -.150515D-04 | .113561 |
| 128 | -.985606D-08 | -.149009D-04 | -.985606D-08 | -.150024D-04 | -.149853D-04 | .113746 |
| 129 | -.100718D-07 | -.151807D-04 | -.100718D-07 | -.152813D-04 | -.152643D-04 | .111785 |
| 130 | -.101474D-07 | -.151118D-04 | -.101474D-07 | -.152117D-04 | -.151947D-04 | .111911 |
| 131 | -.103669D-07 | -.153934D-04 | -.103669D-07 | -.154925D-04 | -.154755D-04 | .109795 |
| 132 | -.104420D-07 | -.153212D-04 | -.104420D-07 | -.154196D-04 | -.154026D-04 | .110025 |
| 133 | -.106655D-07 | -.156046D-04 | -.106655D-07 | -.157022D-04 | -.156852D-04 | .108128 |
| 134 | -.107400D-07 | -.155290D-04 | -.107400D-07 | -.156260D-04 | -.156090D-04 | .108313 |
| 135 | -.109671D-07 | -.158142D-04 | -.109671D-07 | -.159104D-04 | -.158935D-04 | .106298 |
| 136 | -.110410D-07 | -.157354D-04 | -.110410D-07 | -.158309D-04 | -.158140D-04 | .106558 |
| 137 | -.112720D-07 | -.160224D-04 | -.112720D-07 | -.161172D-04 | -.161003D-04 | .104719 |
| 138 | -.113453D-07 | -.159402D-04 | -.113453D-07 | -.160343D-04 | -.160175D-04 | .104943 |
| 139 | -.115801D-07 | -.162290D-04 | -.115801D-07 | -.163224D-04 | -.163056D-04 | .103013 |
| 140 | -.116526D-07 | -.161435D-04 | -.116526D-07 | -.162363D-04 | -.162195D-04 | .103293 |
| 141 | -.118912D-07 | -.164341D-04 | -.118912D-07 | -.165262D-04 | -.165095D-04 | .101504 |
| 142 | -.119629D-07 | -.163453D-04 | -.119629D-07 | -.164368D-04 | -.164201D-04 | .101756 |
| 143 | -.122055D-07 | -.166378D-04 | -.122055D-07 | -.167286D-04 | -.167119D-04 | .998980D-01 |
| 144 | -.122762D-07 | -.165457D-04 | -.122762D-07 | -.166359D-04 | -.166192D-04 | .100191 |
| 145 | -.125226D-07 | -.168400D-04 | -.125226D-07 | -.169296D-04 | -.169129D-04 | .984456D-01 |
| 146 | -.125927D-07 | -.167446D-04 | -.125927D-07 | -.168336D-04 | -.168169D-04 | .987172D-01 |
| 147 | -.128430D-07 | -.170407D-04 | -.128430D-07 | -.171291D-04 | -.171125D-04 | .969214D-01 |
| 148 | -.129120D-07 | -.169421D-04 | -.129120D-07 | -.170298D-04 | -.170133D-04 | .972206D-01 |
| 149 | -.131662D-07 | -.172401D-04 | -.131662D-07 | -.173272D-04 | -.173107D-04 | .955165D-01 |
| 150 | -.132343D-07 | -.171382D-04 | -.132343D-07 | -.172247D-04 | -.172082D-04 | .957974D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 151 | -.134924D-07 | -.174380D-04 | -.134924D-07 | -.175240D-04 | -.175075D-04 | .940566D-01 |
| 152 | -.135595D-07 | -.173328D-04 | -.135595D-07 | -.174183D-04 | -.174018D-04 | .943560D-01 |
| 153 | -.138216D-07 | -.176345D-04 | -.138216D-07 | -.177194D-04 | -.177030D-04 | .926902D-01 |
| 154 | -.138876D-07 | -.175261D-04 | -.138876D-07 | -.176105D-04 | -.175941D-04 | .929740D-01 |
| 155 | -.141537D-07 | -.178297D-04 | -.141537D-07 | -.179135D-04 | -.178971D-04 | .912820D-01 |
| 156 | -.142186D-07 | -.177181D-04 | -.142186D-07 | -.178013D-04 | -.177850D-04 | .915771D-01 |
| 157 | -.144887D-07 | -.180236D-04 | -.144887D-07 | -.181063D-04 | -.180900D-04 | .899467D-01 |
| 158 | -.145524D-07 | -.179087D-04 | -.145524D-07 | -.179909D-04 | -.179747D-04 | .902270D-01 |
| 159 | -.148265D-07 | -.182161D-04 | -.148265D-07 | -.182978D-04 | -.182816D-04 | .885799D-01 |
| 160 | -.148890D-07 | -.180981D-04 | -.148890D-07 | -.181792D-04 | -.181631D-04 | .888654D-01 |
| 161 | -.151672D-07 | -.184074D-04 | -.151672D-07 | -.184881D-04 | -.184719D-04 | .872691D-01 |
| 162 | -.152285D-07 | -.182862D-04 | -.152285D-07 | -.183664D-04 | -.183503D-04 | .875406D-01 |
| 163 | -.155108D-07 | -.185975D-04 | -.155108D-07 | -.186771D-04 | -.186611D-04 | .859337D-01 |
| 164 | -.155708D-07 | -.184731D-04 | -.155708D-07 | -.185523D-04 | -.185363D-04 | .862060D-01 |
| 165 | -.158572D-07 | -.187863D-04 | -.158572D-07 | -.188651D-04 | -.188491D-04 | .846423D-01 |
| 166 | -.159159D-07 | -.186588D-04 | -.159159D-07 | -.187370D-04 | -.187211D-04 | .849003D-01 |
| 167 | -.162064D-07 | -.189741D-04 | -.162064D-07 | -.190518D-04 | -.190360D-04 | .833309D-01 |
| 168 | -.162639D-07 | -.188434D-04 | -.162639D-07 | -.189207D-04 | -.189048D-04 | .835855D-01 |
| 169 | -.165585D-07 | -.191607D-04 | -.165585D-07 | -.192375D-04 | -.192218D-04 | .820527D-01 |
| 170 | -.166147D-07 | -.190268D-04 | -.166147D-07 | -.191032D-04 | -.190875D-04 | .822923D-01 |
| 171 | -.169134D-07 | -.193463D-04 | -.169134D-07 | -.194222D-04 | -.194065D-04 | .807588D-01 |
| 172 | -.169681D-07 | -.192093D-04 | -.169681D-07 | -.192848D-04 | -.192692D-04 | .809920D-01 |
| 173 | -.172713D-07 | -.195308D-04 | -.172713D-07 | -.196059D-04 | -.195903D-04 | .794884D-01 |
| 174 | -.173245D-07 | -.193907D-04 | -.173245D-07 | -.194654D-04 | -.194498D-04 | .797068D-01 |
| 175 | -.176319D-07 | -.197144D-04 | -.176319D-07 | -.197886D-04 | -.197732D-04 | .782059D-01 |
| 176 | -.176837D-07 | -.195712D-04 | -.176837D-07 | -.196450D-04 | -.196296D-04 | .784140D-01 |
| 177 | -.179956D-07 | -.198972D-04 | -.179956D-07 | -.199705D-04 | -.199552D-04 | .769394D-01 |
| 178 | -.180458D-07 | -.197509D-04 | -.180458D-07 | -.198238D-04 | -.198085D-04 | .771319D-01 |
| 179 | -.183622D-07 | -.200791D-04 | -.183622D-07 | -.201516D-04 | -.201364D-04 | .756619D-01 |
| 180 | -.184109D-07 | -.199297D-04 | -.184109D-07 | -.200019D-04 | -.199867D-04 | .758420D-01 |
| 181 | -.187317D-07 | -.202602D-04 | -.187317D-07 | -.203320D-04 | -.203168D-04 | .743949D-01 |
| 182 | -.187788D-07 | -.201078D-04 | -.187788D-07 | -.201792D-04 | -.201641D-04 | .745580D-01 |
| 183 | -.191043D-07 | -.204407D-04 | -.191043D-07 | -.205116D-04 | -.204966D-04 | .731185D-01 |
| 184 | -.191498D-07 | -.202852D-04 | -.191498D-07 | -.203558D-04 | -.203409D-04 | .732661D-01 |
| 185 | -.194799D-07 | -.206206D-04 | -.194799D-07 | -.206908D-04 | -.206759D-04 | .718462D-01 |
| 186 | -.195239D-07 | -.204621D-04 | -.195239D-07 | -.205319D-04 | -.205171D-04 | .719759D-01 |
| 187 | -.198588D-07 | -.207999D-04 | -.198588D-07 | -.208694D-04 | -.208547D-04 | .705648D-01 |
| 188 | -.199010D-07 | -.206385D-04 | -.199010D-07 | -.207075D-04 | -.206929D-04 | .706780D-01 |
| 189 | -.202407D-07 | -.209789D-04 | -.202407D-07 | -.210476D-04 | -.210330D-04 | .692846D-01 |
| 190 | -.202815D-07 | -.208145D-04 | -.202815D-07 | -.208828D-04 | -.208683D-04 | .693777D-01 |
| 191 | -.206260D-07 | -.211576D-04 | -.206260D-07 | -.212256D-04 | -.212111D-04 | .679948D-01 |
| 192 | -.206651D-07 | -.209901D-04 | -.206651D-07 | -.210578D-04 | -.210434D-04 | .680689D-01 |
| 193 | -.210148D-07 | -.213360D-04 | -.210148D-07 | -.214033D-04 | -.213890D-04 | .667016D-01 |
| 194 | -.210523D-07 | -.211656D-04 | -.210523D-07 | -.212326D-04 | -.212184D-04 | .667548D-01 |
| 195 | -.214071D-07 | -.215144D-04 | -.214071D-07 | -.215810D-04 | -.215669D-04 | .653994D-01 |
| 196 | -.214429D-07 | -.213411D-04 | -.214429D-07 | -.214073D-04 | -.213933D-04 | .654314D-01 |
| 197 | -.218030D-07 | -.216929D-04 | -.218030D-07 | -.217588D-04 | -.217448D-04 | .640904D-01 |
| 198 | -.218371D-07 | -.215166D-04 | -.218371D-07 | -.215822D-04 | -.215684D-04 | .641013D-01 |
| 199 | -.222026D-07 | -.218715D-04 | -.222026D-07 | -.219367D-04 | -.219230D-04 | .627741D-01 |
| 200 | -.222352D-07 | -.216924D-04 | -.222352D-07 | -.217573D-04 | -.217436D-04 | .627643D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 201 | -.226063D-07 | -.220505D-04 | -.226063D-07 | -.221151D-04 | -.221015D-04 | .614424D-01 |
| 202 | -.226372D-07 | -.218684D-04 | -.226372D-07 | -.219327D-04 | -.219193D-04 | .613931D-01 |
| 203 | -.230140D-07 | -.222299D-04 | -.230140D-07 | -.222939D-04 | -.222805D-04 | .600891D-01 |
| 204 | -.230433D-07 | -.220451D-04 | -.230433D-07 | -.221087D-04 | -.220955D-04 | .600330D-01 |
| 205 | -.234261D-07 | -.224101D-04 | -.234261D-07 | -.224735D-04 | -.224603D-04 | .587381D-01 |
| 206 | -.234538D-07 | -.222224D-04 | -.234538D-07 | -.222855D-04 | -.222724D-04 | .586375D-01 |
| 207 | -.238427D-07 | -.225911D-04 | -.238427D-07 | -.226539D-04 | -.226409D-04 | .573594D-01 |
| 208 | -.238688D-07 | -.224007D-04 | -.238688D-07 | -.224631D-04 | -.224502D-04 | .572520D-01 |
| 209 | -.242641D-07 | -.227733D-04 | -.242641D-07 | -.228354D-04 | -.228226D-04 | .559824D-01 |
| 210 | -.242887D-07 | -.225800D-04 | -.242887D-07 | -.226418D-04 | -.226292D-04 | .558245D-01 |
| 211 | -.246905D-07 | -.229567D-04 | -.246905D-07 | -.230183D-04 | -.230057D-04 | .545719D-01 |
| 212 | -.247136D-07 | -.227607D-04 | -.247136D-07 | -.228219D-04 | -.228095D-04 | .544095D-01 |
| 213 | -.251223D-07 | -.231417D-04 | -.251223D-07 | -.232027D-04 | -.231903D-04 | .531645D-01 |
| 214 | -.251439D-07 | -.229430D-04 | -.251439D-07 | -.230036D-04 | -.229915D-04 | .529457D-01 |
| 215 | -.255598D-07 | -.233285D-04 | -.255598D-07 | -.233889D-04 | -.233768D-04 | .517177D-01 |
| 216 | -.255801D-07 | -.231271D-04 | -.255801D-07 | -.231872D-04 | -.231753D-04 | .514961D-01 |
| 217 | -.260034D-07 | -.235175D-04 | -.260034D-07 | -.235773D-04 | -.235654D-04 | .502757D-01 |
| 218 | -.260223D-07 | -.233134D-04 | -.260223D-07 | -.233729D-04 | -.233612D-04 | .499908D-01 |
| 219 | -.264534D-07 | -.237088D-04 | -.264534D-07 | -.237681D-04 | -.237565D-04 | .487879D-01 |
| 220 | -.264711D-07 | -.235021D-04 | -.264711D-07 | -.235612D-04 | -.235497D-04 | .485029D-01 |
| 221 | -.269104D-07 | -.239030D-04 | -.269104D-07 | -.239617D-04 | -.239504D-04 | .473068D-01 |
| 222 | -.269270D-07 | -.236937D-04 | -.269270D-07 | -.237522D-04 | -.237410D-04 | .469508D-01 |
| 223 | -.273749D-07 | -.241003D-04 | -.273749D-07 | -.241585D-04 | -.241475D-04 | .457729D-01 |
| 224 | -.273904D-07 | -.238885D-04 | -.273904D-07 | -.239465D-04 | -.239356D-04 | .454205D-01 |
| 225 | -.278473D-07 | -.243012D-04 | -.278473D-07 | -.243590D-04 | -.243482D-04 | .442491D-01 |
| 226 | -.278620D-07 | -.240870D-04 | -.278620D-07 | -.241444D-04 | -.241338D-04 | .438164D-01 |
| 227 | -.283284D-07 | -.245062D-04 | -.283284D-07 | -.245634D-04 | -.245529D-04 | .426633D-01 |
| 228 | -.283424D-07 | -.242895D-04 | -.283424D-07 | -.243465D-04 | -.243362D-04 | .422397D-01 |
| 229 | -.288190D-07 | -.247158D-04 | -.288190D-07 | -.247725D-04 | -.247623D-04 | .410926D-01 |
| 230 | -.288324D-07 | -.244967D-04 | -.288324D-07 | -.245532D-04 | -.245432D-04 | .405773D-01 |
| 231 | -.293196D-07 | -.249305D-04 | -.293196D-07 | -.249867D-04 | -.249768D-04 | .394494D-01 |
| 232 | -.293327D-07 | -.247091D-04 | -.293327D-07 | -.247651D-04 | -.247554D-04 | .389494D-01 |
| 233 | -.298314D-07 | -.251509D-04 | -.298314D-07 | -.252066D-04 | -.251971D-04 | .378274D-01 |
| 234 | -.298443D-07 | -.249273D-04 | -.298443D-07 | -.249828D-04 | -.249735D-04 | .372227D-01 |
| 235 | -.303552D-07 | -.253778D-04 | -.303552D-07 | -.254331D-04 | -.254239D-04 | .361207D-01 |
| 236 | -.303682D-07 | -.251521D-04 | -.303682D-07 | -.252072D-04 | -.251982D-04 | .355395D-01 |
| 237 | -.308922D-07 | -.256120D-04 | -.308922D-07 | -.256668D-04 | -.256579D-04 | .344427D-01 |
| 238 | -.309056D-07 | -.253843D-04 | -.309056D-07 | -.254388D-04 | -.254302D-04 | .337412D-01 |
| 239 | -.314436D-07 | -.258543D-04 | -.314436D-07 | -.259086D-04 | -.259001D-04 | .326655D-01 |
| 240 | -.314578D-07 | -.256246D-04 | -.314578D-07 | -.256787D-04 | -.256705D-04 | .319981D-01 |
| 241 | -.320110D-07 | -.261056D-04 | -.320110D-07 | -.261595D-04 | -.261514D-04 | .309275D-01 |
| 242 | -.320263D-07 | -.258742D-04 | -.320263D-07 | -.259278D-04 | -.259200D-04 | .301204D-01 |
| 243 | -.325958D-07 | -.263672D-04 | -.325958D-07 | -.264206D-04 | -.264129D-04 | .290719D-01 |
| 244 | -.326126D-07 | -.261341D-04 | -.326126D-07 | -.261873D-04 | -.261799D-04 | .283128D-01 |
| 245 | -.331999D-07 | -.266402D-04 | -.331999D-07 | -.266932D-04 | -.266859D-04 | .272695D-01 |
| 246 | -.332188D-07 | -.264056D-04 | -.332188D-07 | -.264583D-04 | -.264514D-04 | .263469D-01 |
| 247 | -.338254D-07 | -.269260D-04 | -.338254D-07 | -.269786D-04 | -.269717D-04 | .253266D-01 |
| 248 | -.338470D-07 | -.266901D-04 | -.338470D-07 | -.267425D-04 | -.267359D-04 | .244702D-01 |
| 249 | -.344747D-07 | -.272263D-04 | -.344747D-07 | -.272784D-04 | -.272721D-04 | .234551D-01 |
| 250 | -.344997D-07 | -.269893D-04 | -.344997D-07 | -.270412D-04 | -.270352D-04 | .224051D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 251 | -.351505D-07 | -.275429D-04 | -.351505D-07 | -.275946D-04 | -.275887D-04 | .214147D-01 |
| 252 | -.351798D-07 | -.273051D-04 | -.351798D-07 | -.273566D-04 | -.273510D-04 | .204553D-01 |
| 253 | -.358561D-07 | -.278779D-04 | -.358561D-07 | -.279292D-04 | -.279237D-04 | .194705D-01 |
| 254 | -.358904D-07 | -.276395D-04 | -.358904D-07 | -.276906D-04 | -.276856D-04 | .182794D-01 |
| 255 | -.365950D-07 | -.282337D-04 | -.365950D-07 | -.282846D-04 | -.282797D-04 | .173207D-01 |
| 256 | -.366356D-07 | -.279952D-04 | -.366356D-07 | -.280459D-04 | -.280414D-04 | .162521D-01 |
| 257 | -.373716D-07 | -.286132D-04 | -.373716D-07 | -.286638D-04 | -.286594D-04 | .152997D-01 |
| 258 | -.374197D-07 | -.283750D-04 | -.374197D-07 | -.284253D-04 | -.284214D-04 | .139496D-01 |
| 259 | -.381909D-07 | -.290198D-04 | -.381909D-07 | -.290700D-04 | -.290662D-04 | .130252D-01 |
| 260 | -.382482D-07 | -.287824D-04 | -.382482D-07 | -.288324D-04 | -.288290D-04 | .118414D-01 |
| 261 | -.390590D-07 | -.294575D-04 | -.390590D-07 | -.295072D-04 | -.295040D-04 | .109247D-01 |
| 262 | -.391270D-07 | -.292214D-04 | -.391270D-07 | -.292710D-04 | -.292682D-04 | .939288D-02 |
| 263 | -.399827D-07 | -.299307D-04 | -.399827D-07 | -.299801D-04 | -.299775D-04 | .850682D-02 |
| 264 | -.400641D-07 | -.296969D-04 | -.400641D-07 | -.297461D-04 | -.297439D-04 | .720177D-02 |
| 265 | -.409708D-07 | -.304453D-04 | -.409708D-07 | -.304944D-04 | -.304924D-04 | .632516D-02 |
| 266 | -.410679D-07 | -.302145D-04 | -.410679D-07 | -.302634D-04 | -.302620D-04 | .458111D-02 |
| 267 | -.420333D-07 | -.310060D-04 | -.420333D-07 | -.310567D-04 | -.310555D-04 | .373779D-02 |
| 268 | -.421498D-07 | -.307816D-04 | -.421498D-07 | -.308300D-04 | -.308293D-04 | .230757D-02 |
| 269 | -.431831D-07 | -.316273D-04 | -.431831D-07 | -.316756D-04 | -.316751D-04 | .147657D-02 |
| 270 | -.433227D-07 | -.314064D-04 | -.433227D-07 | -.314545D-04 | -.314546D-04 | -.522667D-03 |
| 271 | -.444355D-07 | -.323132D-04 | -.444355D-07 | -.323611D-04 | -.323616D-04 | -.131666D-02 |
| 272 | -.446039D-07 | -.321000D-04 | -.446039D-07 | -.321477D-04 | -.321486D-04 | -.287295D-02 |
| 273 | -.458106D-07 | -.330791D-04 | -.458106D-07 | -.331267D-04 | -.331279D-04 | -.365110D-02 |
| 274 | -.460138D-07 | -.328755D-04 | -.460138D-07 | -.329228D-04 | -.329248D-04 | -.597062D-02 |
| 275 | -.473328D-07 | -.339412D-04 | -.473328D-07 | -.339884D-04 | -.339907D-04 | -.670808D-02 |
| 276 | -.475797D-07 | -.337506D-04 | -.475797D-07 | -.337976D-04 | -.338005D-04 | -.837967D-02 |
| 277 | -.490350D-07 | -.349214D-04 | -.490350D-07 | -.349683D-04 | -.349714D-04 | -.909751D-02 |
| 278 | -.493361D-07 | -.347472D-04 | -.493361D-07 | -.347939D-04 | -.347980D-04 | -.118443D-01 |
| 279 | -.509594D-07 | -.360477D-04 | -.509594D-07 | -.360942D-04 | -.360988D-04 | -.125169D-01 |
| 280 | -.513303D-07 | -.358959D-04 | -.513303D-07 | -.359423D-04 | -.359475D-04 | -.142645D-01 |
| 281 | -.531650D-07 | -.373597D-04 | -.531650D-07 | -.374059D-04 | -.374115D-04 | -.149188D-01 |
| 282 | -.536249D-07 | -.372366D-04 | -.536249D-07 | -.372826D-04 | -.372894D-04 | -.182885D-01 |
| 283 | -.557317D-07 | -.389106D-04 | -.557317D-07 | -.389565D-04 | -.389639D-04 | -.188958D-01 |
| 284 | -.563105D-07 | -.388274D-04 | -.563105D-07 | -.388732D-04 | -.388812D-04 | -.205836D-01 |
| 285 | -.587782D-07 | -.407808D-04 | -.587782D-07 | -.408263D-04 | -.408350D-04 | -.212073D-01 |
| 286 | -.595152D-07 | -.407498D-04 | -.595152D-07 | -.407952D-04 | -.408057D-04 | -.255877D-01 |
| 287 | -.624792D-07 | -.430881D-04 | -.624792D-07 | -.431333D-04 | -.431447D-04 | -.262229D-01 |
| 288 | -.634400D-07 | -.431324D-04 | -.634400D-07 | -.431775D-04 | -.431892D-04 | -.272020D-01 |
| 289 | -.671212D-07 | -.460283D-04 | -.671212D-07 | -.460732D-04 | -.460862D-04 | -.282203D-01 |
| 290 | -.684006D-07 | -.461733D-04 | -.684006D-07 | -.462181D-04 | -.462339D-04 | -.341521D-01 |
| 291 | -.731900D-07 | -.499359D-04 | -.731900D-07 | -.499805D-04 | -.499976D-04 | -.343245D-01 |
| 292 | -.749561D-07 | -.502225D-04 | -.749561D-07 | -.502670D-04 | -.502865D-04 | -.388531D-01 |
| 293 | -.816417D-07 | -.554810D-04 | -.816417D-07 | -.555254D-04 | -.555470D-04 | -.390074D-01 |
| 294 | -.841729D-07 | -.559201D-04 | -.841729D-07 | -.559643D-04 | -.559892D-04 | -.445955D-01 |
| 295 | -.947257D-07 | -.642913D-04 | -.947257D-07 | -.643353D-04 | -.643651D-04 | -.463089D-01 |
| 296 | -.983687D-07 | -.645117D-04 | -.983687D-07 | -.645556D-04 | -.645906D-04 | -.542479D-01 |
| 297 | -.119857D-06 | -.818472D-04 | -.119857D-06 | -.818909D-04 | -.819385D-04 | -.581201D-01 |
| 298 | -.125936D-06 | -.809277D-04 | -.125936D-06 | -.809713D-04 | -.810490D-04 | -.959138D-01 |
| 299 | -.181094D-06 | -.122584D-03 | -.181094D-06 | -.122627D-03 | -.122690D-03 | -.514678D-01 |
| 300 | -.246384D-06 | -.172409D-03 | -.246384D-06 | -.172452D-03 | -.172575D-03 | -.713156D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 301 | -.156946D-06 | -.945851D-04 | -.156946D-06 | -.946283D-04 | -.948365D-04 | -.220016 |
| 302 | -.612361D-07 | -.322216D-04 | -.612361D-07 | -.322646D-04 | -.324394D-04 | -.541705 |
| 303 | -.171921D-07 | -.802013D-05 | -.171921D-07 | -.806299D-05 | -.820482D-05 | -1.75899 |
| 304 | -.372012D-08 | -.150671D-05 | -.372012D-08 | -.154942D-05 | -.167024D-05 | -7.79723 |
| 305 | -.580458D-09 | -.140251D-06 | -.580458D-09 | -.182827D-06 | -.287209D-06 | -57.0931 |
| 306 | .698134D-11 | .915684D-07 | .698134D-11 | .491313D-07 | -.429734D-07 | 187.467 |
| 307 | .899444D-10 | .119382D-06 | .899444D-10 | .770827D-07 | -.563336D-08 | 107.308 |

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MAXIMUM DEGREE IN ZONAL FIELD =   300 MAX. DEG. IN REFERENCE FIELD =    30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD =      5263.369068      MEAN SEPARATION =  100000.00       NO. OF INTEGR. INTERVALS =       2048
INITIAL CONDITION ERRORS : DZCE =   .0             DX11 =   .0            DX12 =  .0            DR1 =  .0          DUP1 =  .0
DR2 =  .0          DUP2 =  .0
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

FROM 0  CY/REV. TO 3 CY/REV :  CY/REV   CORRECTED   (COS, SIN)    ANALYTICAL.

                                  0  -.264407D-08
                                  1  -.204751D-01   -.203807D-04
                                  2  -.414886D-06   -.388638D-04    .138998D-04
                                  3  -.863398D-07    .326021D-04    .868815D-05
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SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM 4 TO 304 CY/REV.

| BAND | RMS UNCORRECTED | RMS CORRECTED | RMS ANALYTICAL | % (SIN+COS) ERROR |
|---|---|---|---|---|
| 4 - 14 | .301910D-04 | .245705D-04 | .245635D-04 | .876589 |
| 15 - 24 | .489005D-04 | .468581D-04 | .468811D-04 | .366296 |
| 25 - 34 | .254100D-03 | .253660D-03 | .253495D-03 | .785435D-01 |
| 35 - 44 | .296939D-03 | .296759D-03 | .296650D-03 | .456506D-01 |
| 45 - 54 | .228369D-03 | .228586D-03 | .228504D-03 | .474843D-01 |
| 55 - 64 | .168432D-03 | .168363D-03 | .168329D-03 | .431452D-01 |
| 65 - 74 | .135110D-03 | .135313D-03 | .135271D-03 | .536785D-01 |
| 75 - 84 | .965006D-04 | .965192D-04 | .965019D-04 | .603655D-01 |
| 85 - 94 | .819482D-04 | .821604D-04 | .821278D-04 | .743961D-01 |
| 95 - 104 | .488341D-04 | .489319D-04 | .489156D-04 | .112007 |
| 105 - 114 | .409586D-04 | .413335D-04 | .412762D-04 | .149882 |
| 115 - 124 | .444918D-04 | .448354D-04 | .447805D-04 | .136936 |
| 125 - 134 | .479012D-04 | .482183D-04 | .481644D-04 | .130029 |
| 135 - 144 | .511890D-04 | .514834D-04 | .514303D-04 | .125472 |
| 145 - 154 | .543597D-04 | .546344D-04 | .545821D-04 | .122440 |
| 155 - 164 | .574204D-04 | .576779D-04 | .576267D-04 | .120474 |
| 165 - 174 | .603825D-04 | .606248D-04 | .605750D-04 | .119342 |
| 175 - 184 | .632627D-04 | .634914D-04 | .634433D-04 | .118872 |
| 185 - 194 | .660855D-04 | .663022D-04 | .662562D-04 | .118959 |
| 195 - 204 | .688868D-04 | .690926D-04 | .690493D-04 | .119532 |
| 205 - 214 | .717193D-04 | .719153D-04 | .718751D-04 | .120539 |
| 215 - 224 | .746616D-04 | .748487D-04 | .748123D-04 | .121973 |
| 225 - 234 | .778347D-04 | .780136D-04 | .779818D-04 | .123842 |
| 235 - 244 | .814326D-04 | .816040D-04 | .815777D-04 | .126189 |
| 245 - 254 | .857865D-04 | .859510D-04 | .859314D-04 | .129093 |
| 255 - 264 | .915134D-04 | .916716D-04 | .916603D-04 | .132684 |
| 265 - 274 | .999254D-04 | .100078D-03 | .100077D-03 | .137168 |
| 275 - 284 | .114498D-03 | .114645D-03 | .114661D-03 | .142891 |
| 285 - 294 | .150041D-03 | .150182D-03 | .150232D-03 | .151064 |
| 295 - 304 | .276308D-03 | .276422D-03 | .276675D-03 | .201159 |

TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL. (BAND FROM 4 CY/REV TO 304 CY/REV)
(A) .1212166476 (B) .1212191840

Table AIV-2

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 300000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = .0 DX11 = .0 DX12 = .0 DR1 = .0 DUP1 = .0
DR2 = .0 DUP2 = .0
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 4 | -.782233D-08 | .992307D-05 | -.782233D-08 | .212703D-04 | .213209D-04 | -.238141 |
| 5 | -.245408D-08 | .108252D-04 | -.245408D-08 | .196902D-04 | .197758D-04 | -.434832 |
| 6 | -.179100D-08 | .113696D-04 | -.179100D-08 | .186642D-04 | .187396D-04 | -.403689 |
| 7 | -.965402D-09 | .132865D-04 | -.965402D-09 | .194920D-04 | .195340D-04 | -.215593 |
| 8 | .129916D-09 | .137995D-04 | .129916D-09 | .192029D-04 | .192389D-04 | -.187599 |
| 9 | .265384D-08 | .162754D-04 | .265384D-08 | .210625D-04 | .211169D-04 | -.258219 |
| 10 | -.215503D-09 | .166829D-04 | -.215503D-09 | .209811D-04 | .209835D-04 | -.113044D-01 |
| 11 | .232186D-08 | .196672D-04 | .232186D-08 | .235678D-04 | .236429D-04 | -.318857 |
| 12 | -.279907D-08 | .198895D-04 | -.279907D-08 | .234603D-04 | .234778D-04 | -.745394D-01 |
| 13 | .303753D-09 | .236002D-04 | .303753D-09 | .268929D-04 | .268612D-04 | .118086 |
| 14 | .611993D-08 | .235056D-04 | .611993D-08 | .265606D-04 | .265787D-04 | -.678790D-01 |
| 15 | .420357D-08 | .278439D-04 | .420357D-08 | .306934D-04 | .307404D-04 | -.153303 |
| 16 | -.899560D-09 | .276192D-04 | -.899560D-09 | .302892D-04 | .302920D-04 | -.937268D-02 |
| 17 | .236683D-08 | .329016D-04 | .236683D-08 | .354133D-04 | .353773D-04 | .101706 |
| 18 | .815657D-08 | .323511D-04 | .815657D-08 | .347224D-04 | .347273D-04 | -.140159D-01 |
| 19 | .662432D-08 | .387239D-04 | .662432D-08 | .409697D-04 | .409969D-04 | -.663176D-01 |
| 20 | .163017D-08 | .379872D-04 | .163017D-08 | .401201D-04 | .401106D-04 | .235719D-01 |
| 21 | .534180D-08 | .460289D-04 | .534180D-08 | .480598D-04 | .480178D-04 | .873065D-01 |
| 22 | .112412D-07 | .449204D-04 | .112412D-07 | .468586D-04 | .468526D-04 | .127330D-01 |
| 23 | .104987D-07 | .553674D-04 | .104987D-07 | .572210D-04 | .572315D-04 | -.183954D-01 |
| 24 | .571300D-08 | .539670D-04 | .571300D-08 | .557431D-04 | .557225D-04 | .370363D-01 |
| 25 | .105828D-07 | .686421D-04 | .105828D-07 | .703469D-04 | .702974D-04 | .704479D-01 |
| 26 | .168277D-07 | .666977D-04 | .168277D-07 | .683367D-04 | .683198D-04 | .247055D-01 |
| 27 | .184710D-07 | .898526D-04 | .184710D-07 | .914307D-04 | .914275D-04 | .355315D-02 |
| 28 | .144896D-07 | .871184D-04 | .144896D-07 | .886401D-04 | .886101D-04 | .338059D-01 |
| 29 | .252517D-07 | .134742D-03 | .252517D-07 | .136211D-03 | .136186D-03 | .182521D-01 |
| 30 | .342255D-07 | .128236D-03 | .342255D-07 | .129655D-03 | .129671D-03 | -.117335D-01 |
| 31 | .685846D-07 | .328536D-03 | .685846D-07 | .329910D-03 | .329623D-03 | .869803D-01 |
| 32 | .680950D-07 | .346611D-03 | .680950D-07 | .347943D-03 | .347685D-03 | .740325D-01 |
| 33 | -.467677D-07 | -.283451D-03 | -.467677D-07 | -.282160D-03 | -.281923D-03 | .840406D-01 |
| 34 | -.688376D-07 | -.336175D-03 | -.688376D-07 | -.334922D-03 | -.334752D-03 | .507722D-01 |
| 35 | .426644D-07 | .246739D-03 | .426644D-07 | .247956D-03 | .247829D-03 | .512113D-01 |
| 36 | .647196D-07 | .298813D-03 | .647196D-07 | .299996D-03 | .299934D-03 | .208469D-01 |
| 37 | -.452055D-07 | -.244058D-03 | -.452055D-07 | -.242907D-03 | -.242722D-03 | .761485D-01 |
| 38 | -.676919D-07 | -.296218D-03 | -.676919D-07 | -.295097D-03 | -.294983D-03 | .386979D-01 |
| 39 | .398988D-07 | .208338D-03 | .398988D-07 | .209430D-03 | .209327D-03 | .490294D-01 |
| 40 | .623227D-07 | .259558D-03 | .623227D-07 | .260622D-03 | .260582D-03 | .153650D-01 |
| 41 | -.431872D-07 | -.210473D-03 | -.431872D-07 | -.209434D-03 | -.209293D-03 | .677285D-01 |
| 42 | -.659065D-07 | -.261270D-03 | -.659065D-07 | -.260256D-03 | -.260181D-03 | .288365D-01 |
| 43 | .367829D-07 | .175453D-03 | .367829D-07 | .176443D-03 | .176359D-03 | .476601D-01 |
| 44 | .592936D-07 | .225020D-03 | .592936D-07 | .225987D-03 | .225964D-03 | .103886D-01 |
| 45 | -.409000D-07 | -.181582D-03 | -.409000D-07 | -.180636D-03 | -.180526D-03 | .607747D-01 |
| 46 | -.635655D-07 | -.230326D-03 | -.635655D-07 | -.229400D-03 | -.229355D-03 | .200221D-01 |
| 47 | .334244D-07 | .147092D-03 | .334244D-07 | .147998D-03 | .147927D-03 | .476672D-01 |
| 48 | .557465D-07 | .194406D-03 | .557465D-07 | .195293D-03 | .195281D-03 | .616744D-02 |
| 49 | -.384259D-07 | -.156572D-03 | -.384259D-07 | -.155704D-03 | -.155619D-03 | .543900D-01 |
| 50 | -.607645D-07 | -.202749D-03 | -.607645D-07 | -.201898D-03 | -.201874D-03 | .117436D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 51 | .299280D-07 | .122577D-03 | .299280D-07 | .123411D-03 | .123351D-03 | .492027D-01 |
| 52 | .518052D-07 | .167196D-03 | .518052D-07 | .168015D-03 | .168010D-03 | .274267D-02 |
| 53 | -.358412D-07 | -.134860D-03 | -.358412D-07 | -.134057D-03 | -.133992D-03 | .481467D-01 |
| 54 | -.576034D-07 | -.178100D-03 | -.576034D-07 | -.177312D-03 | -.177305D-03 | .370255D-02 |
| 55 | .263902D-07 | .101399D-03 | .263902D-07 | .102173D-03 | .102120D-03 | .525194D-01 |
| 56 | .475905D-07 | .143017D-03 | .475905D-07 | .143777D-03 | .143777D-03 | .158433D-03 |
| 57 | -.332123D-07 | -.115991D-03 | -.332123D-07 | -.115244D-03 | -.115196D-03 | .416895D-01 |
| 58 | -.541761D-07 | -.156048D-03 | -.541761D-07 | -.155314D-03 | -.155321D-03 | -.434360D-02 |
| 59 | .228954D-07 | .831568D-04 | .228954D-07 | .838782D-04 | .838296D-04 | .579963D-01 |
| 60 | .432186D-07 | .121576D-03 | .432186D-07 | .122286D-03 | .122287D-03 | -.150069D-02 |
| 61 | -.305932D-07 | -.995926D-04 | -.305932D-07 | -.988948D-04 | -.988605D-04 | .346821D-01 |
| 62 | -.505686D-07 | -.136325D-03 | -.505686D-07 | -.135638D-03 | -.135655D-03 | -.126193D-01 |
| 63 | .195161D-07 | .675112D-04 | .195161D-07 | .681868D-04 | .681416D-04 | .662154D-01 |
| 64 | .387937D-07 | .102633D-03 | .387937D-07 | .103298D-03 | .103300D-03 | -.207907D-02 |
| 65 | -.280240D-07 | -.853472D-04 | -.280240D-07 | -.846924D-04 | -.846698D-04 | .267584D-01 |
| 66 | -.468552D-07 | -.118699D-03 | -.468552D-07 | -.118054D-03 | -.118079D-03 | -.213569D-01 |
| 67 | .163127D-07 | .541713D-04 | .163127D-07 | .548066D-04 | .547638D-04 | .780345D-01 |
| 68 | .344103D-07 | .859744D-04 | .344103D-07 | .866003D-04 | .866014D-04 | -.136687D-02 |
| 69 | -.255332D-07 | -.729752D-04 | -.255332D-07 | -.723584D-04 | -.723457D-04 | .175694D-01 |
| 70 | -.430974D-07 | -.102963D-03 | -.430974D-07 | -.102355D-03 | -.102387D-03 | -.308065D-01 |
| 71 | .133346D-07 | .428818D-04 | .133346D-07 | .434812D-04 | .434400D-04 | .948200D-01 |
| 72 | .301503D-07 | .714101D-04 | .301503D-07 | .720012D-04 | .720005D-04 | .989819D-03 |
| 73 | -.231372D-07 | -.622275D-04 | -.231372D-07 | -.616444D-04 | -.616403D-04 | .668596D-02 |
| 74 | -.393446D-07 | -.889275D-04 | -.393446D-07 | -.883524D-04 | -.883889D-04 | -.412653D-01 |
| 75 | .106220D-07 | .334167D-04 | .106220D-07 | .339842D-04 | .339439D-04 | .118612 |
| 76 | .260856D-07 | .587660D-04 | .260856D-07 | .593260D-04 | .593227D-04 | .548521D-02 |
| 77 | -.208410D-07 | -.528787D-04 | -.208410D-07 | -.523260D-04 | -.523293D-04 | -.640124D-02 |
| 78 | -.356325D-07 | -.764136D-04 | -.356325D-07 | -.758680D-04 | -.759083D-04 | -.531014D-01 |
| 79 | .820761D-08 | .255780D-04 | .820761D-08 | .261167D-04 | .260769D-04 | .152528 |
| 80 | .222786D-07 | .478840D-04 | .222786D-07 | .484160D-04 | .484098D-04 | .128165D-01 |
| 81 | -.186386D-07 | -.447218D-04 | -.186386D-07 | -.441964D-04 | -.442062D-04 | -.223313D-01 |
| 82 | -.319833D-07 | -.652501D-04 | -.319833D-07 | -.647311D-04 | -.647743D-04 | -.667983D-01 |
| 83 | .612181D-08 | .191962D-04 | .612181D-08 | .197089D-04 | .196692D-04 | .201243 |
| 84 | .187864D-07 | .386244D-04 | .187864D-07 | .391310D-04 | .391216D-04 | .239331D-01 |
| 85 | -.165096D-07 | -.375590D-04 | -.165096D-07 | -.370584D-04 | -.370739D-04 | -.419632D-01 |
| 86 | -.284022D-07 | -.552660D-04 | -.284022D-07 | -.547712D-04 | -.548166D-04 | -.830372D-01 |
| 87 | .439920D-08 | .141386D-04 | .439920D-08 | .146278D-04 | .145881D-04 | .271129 |
| 88 | .156686D-07 | .308748D-04 | .156686D-07 | .313584D-04 | .313458D-04 | .399973D-01 |
| 89 | -.144101D-07 | -.311831D-04 | -.144101D-07 | -.307050D-04 | -.307254D-04 | -.665701D-01 |
| 90 | -.248660D-07 | -.462735D-04 | -.248660D-07 | -.458006D-04 | -.458478D-04 | -.102894 |
| 91 | .310041D-08 | .103387D-04 | .310041D-08 | .108064D-04 | .107667D-04 | .367034 |
| 92 | .130091D-07 | .245797D-04 | .130091D-07 | .250423D-04 | .250267D-04 | .619506D-01 |
| 93 | -.122348D-07 | -.253147D-04 | -.122348D-07 | -.248571D-04 | -.248816D-04 | -.984307D-01 |
| 94 | -.212839D-07 | -.380093D-04 | -.212839D-07 | -.375566D-04 | -.376045D-04 | -.127722 |
| 95 | .239643D-08 | .792762D-05 | .239643D-08 | .837559D-05 | .833629D-05 | .469191 |
| 96 | .110076D-07 | .198647D-04 | .110076D-07 | .203080D-04 | .202899D-04 | .889457D-01 |
| 97 | -.960357D-08 | -.192546D-04 | -.960357D-08 | -.188158D-04 | -.188427D-04 | -.143001 |
| 98 | -.172666D-07 | -.298346D-04 | -.172666D-07 | -.294004D-04 | -.294478D-04 | -.161177 |
| 99 | .332811D-08 | .848056D-05 | .332811D-08 | .891042D-05 | .887531D-05 | .394035 |
| 100 | .107645D-07 | .179141D-04 | .107645D-07 | .183396D-04 | .183246D-04 | .821398D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 101 | -.152103D-08 | -.491833D-05 | -.152103D-08 | -.449698D-05 | -.452613D-05 | -.648193 |
| 102 | -.600310D-08 | -.983585D-05 | -.600310D-08 | -.941863D-05 | -.945824D-05 | -.420533 |
| 103 | -.675457D-08 | -.104338D-04 | -.675457D-08 | -.100207D-04 | -.100570D-04 | -.362814 |
| 104 | -.667300D-08 | -.101135D-04 | -.667300D-08 | -.970428D-05 | -.974053D-05 | -.373562 |
| 105 | -.660020D-08 | -.993796D-05 | -.660020D-08 | -.953266D-05 | -.956995D-05 | -.391178 |
| 106 | -.640327D-08 | -.953004D-05 | -.640327D-08 | -.912857D-05 | -.916666D-05 | -.417224 |
| 107 | -.630157D-08 | -.932774D-05 | -.630157D-08 | -.893002D-05 | -.896913D-05 | -.437933 |
| 108 | -.608943D-08 | -.891352D-05 | -.608943D-08 | -.851948D-05 | -.855910D-05 | -.464944 |
| 109 | -.596824D-08 | -.869193D-05 | -.596824D-08 | -.830151D-05 | -.834183D-05 | -.485661 |
| 110 | -.574167D-08 | -.827257D-05 | -.574167D-08 | -.788570D-05 | -.792648D-05 | -.517066 |
| 111 | -.560045D-08 | -.803168D-05 | -.560045D-08 | -.764830D-05 | -.768984D-05 | -.543119 |
| 112 | -.535962D-08 | -.760820D-05 | -.535962D-08 | -.722824D-05 | -.727009D-05 | -.578944 |
| 113 | -.519847D-08 | -.734862D-05 | -.519847D-08 | -.697203D-05 | -.701444D-05 | -.608371 |
| 114 | -.494383D-08 | -.692178D-05 | -.494383D-08 | -.654849D-05 | -.659121D-05 | -.652365 |
| 115 | -.476222D-08 | -.664366D-05 | -.476222D-08 | -.627362D-05 | -.631697D-05 | -.690995 |
| 116 | -.449408D-08 | -.621450D-05 | -.449408D-08 | -.584765D-05 | -.589120D-05 | -.744777 |
| 117 | -.429174D-08 | -.591846D-05 | -.429174D-08 | -.555475D-05 | -.559879D-05 | -.792886 |
| 118 | -.401059D-08 | -.548782D-05 | -.401059D-08 | -.512718D-05 | -.517142D-05 | -.862802 |
| 119 | -.378736D-08 | -.517415D-05 | -.378736D-08 | -.481655D-05 | -.486132D-05 | -.929481 |
| 120 | -.349351D-08 | -.474303D-05 | -.349351D-08 | -.438841D-05 | -.443329D-05 | -1.02279 |
| 121 | -.324965D-08 | -.441239D-05 | -.324965D-08 | -.406070D-05 | -.410601D-05 | -1.11574 |
| 122 | -.294368D-08 | -.398168D-05 | -.294368D-08 | -.363287D-05 | -.367828D-05 | -1.24996 |
| 123 | -.267907D-08 | -.363447D-05 | -.267907D-08 | -.328850D-05 | -.333435D-05 | -1.39442 |
| 124 | -.236144D-08 | -.320516D-05 | -.236144D-08 | -.286197D-05 | -.290787D-05 | -1.60351 |
| 125 | -.207613D-08 | -.284206D-05 | -.207613D-08 | -.250162D-05 | -.254787D-05 | -1.84871 |
| 126 | -.174767D-08 | -.241505D-05 | -.174767D-08 | -.207732D-05 | -.212358D-05 | -2.22723 |
| 127 | -.144153D-08 | -.203657D-05 | -.144153D-08 | -.170149D-05 | -.174812D-05 | -2.74064 |
| 128 | -.110256D-08 | -.161284D-05 | -.110256D-08 | -.128039D-05 | -.132698D-05 | -3.63901 |
| 129 | -.776125D-09 | -.121969D-05 | -.776125D-09 | -.889811D-06 | -.936693D-06 | -5.26866 |
| 130 | -.427683D-09 | -.800161D-06 | -.427683D-09 | -.472819D-06 | -.519640D-06 | -9.90247 |
| 131 | -.807823D-10 | -.392928D-06 | -.807823D-10 | -.680854D-07 | -.115193D-06 | -69.1884 |
| 132 | .276430D-09 | .214425D-07 | .276430D-09 | .343824D-06 | .296829D-06 | 13.6684 |
| 133 | .643282D-09 | .441997D-06 | .643282D-09 | .761955D-06 | .714741D-06 | 6.19633 |
| 134 | .100863D-08 | .850304D-06 | .100863D-08 | .116787D-05 | .112080D-05 | 4.03092 |
| 135 | .139503D-08 | .128352D-05 | .139503D-08 | .159874D-05 | .155145D-05 | 2.95759 |
| 136 | .176748D-08 | .168482D-05 | .176748D-08 | .199772D-05 | .195062D-05 | 2.35768 |
| 137 | .217331D-08 | .212989D-05 | .217331D-08 | .244051D-05 | .239326D-05 | 1.93590 |
| 138 | .255208D-08 | .252329D-05 | .255208D-08 | .283165D-05 | .278462D-05 | 1.66100 |
| 139 | .297662D-08 | .297950D-05 | .297662D-08 | .328565D-05 | .323848D-05 | 1.43542 |
| 140 | .336077D-08 | .336408D-05 | .336077D-08 | .366803D-05 | .362113D-05 | 1.27880 |
| 141 | .380361D-08 | .383059D-05 | .380361D-08 | .413239D-05 | .408541D-05 | 1.13685 |
| 142 | .419217D-08 | .420547D-05 | .419217D-08 | .450514D-05 | .445845D-05 | 1.03645 |
| 143 | .465274D-08 | .468149D-05 | .465274D-08 | .497907D-05 | .493233D-05 | .938791 |
| 144 | .504469D-08 | .504579D-05 | .504469D-08 | .534130D-05 | .529488D-05 | .869002 |
| 145 | .552223D-08 | .553045D-05 | .552223D-08 | .582392D-05 | .577751D-05 | .796927 |
| 146 | .591662D-08 | .588332D-05 | .591662D-08 | .617478D-05 | .612873D-05 | .745848 |
| 147 | .641052D-08 | .637578D-05 | .641052D-08 | .666527D-05 | .661924D-05 | .690503 |
| 148 | .680622D-08 | .671637D-05 | .680622D-08 | .700390D-05 | .695827D-05 | .651445 |
| 149 | .731570D-08 | .721574D-05 | .731570D-08 | .750133D-05 | .745579D-05 | .607166 |
| 150 | .771160D-08 | .754323D-05 | .771160D-08 | .782692D-05 | .778181D-05 | .576426 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 151 | .823591D-08 | .804862D-05 | .823591D-08 | .833043D-05 | .828543D-05 | .540252 |
| 152 | .863086D-08 | .836221D-05 | .863086D-08 | .864217D-05 | .859763D-05 | .515358 |
| 153 | .916907D-08 | .887269D-05 | .916907D-08 | .915082D-05 | .910644D-05 | .484974 |
| 154 | .956182D-08 | .917160D-05 | .956182D-08 | .944793D-05 | .940405D-05 | .464438 |
| 155 | .101131D-07 | .968625D-05 | .101131D-07 | .996079D-05 | .991710D-05 | .438597 |
| 156 | .105027D-07 | .996973D-05 | .105027D-07 | .102425D-04 | .101994D-04 | .421303 |
| 157 | .110660D-07 | .104876D-04 | .110660D-07 | .107586D-04 | .107157D-04 | .398879 |
| 158 | .114509D-07 | .107549D-04 | .114509D-07 | .110242D-04 | .109819D-04 | .384116 |
| 159 | .120252D-07 | .112750D-04 | .120252D-07 | .115427D-04 | .115006D-04 | .364502 |
| 160 | .124046D-07 | .115255D-04 | .124046D-07 | .117914D-04 | .117500D-04 | .351695 |
| 161 | .129886D-07 | .120469D-04 | .129886D-07 | .123112D-04 | .122701D-04 | .334271 |
| 162 | .133609D-07 | .122798D-04 | .133609D-07 | .125425D-04 | .125020D-04 | .323042 |
| 163 | .139538D-07 | .128016D-04 | .139538D-07 | .130626D-04 | .130225D-04 | .307477 |
| 164 | .143180D-07 | .130163D-04 | .143180D-07 | .132757D-04 | .132362D-04 | .297499 |
| 165 | .149184D-07 | .135374D-04 | .149184D-07 | .137953D-04 | .137562D-04 | .283427 |
| 166 | .152731D-07 | .137332D-04 | .152731D-07 | .139896D-04 | .139512D-04 | .274479 |
| 167 | .158797D-07 | .142527D-04 | .158797D-07 | .145075D-04 | .144695D-04 | .261702 |
| 168 | .162238D-07 | .144291D-04 | .162238D-07 | .146824D-04 | .146452D-04 | .253587 |
| 169 | .168354D-07 | .149459D-04 | .168354D-07 | .151977D-04 | .151610D-04 | .241872 |
| 170 | .171674D-07 | .151024D-04 | .171674D-07 | .153528D-04 | .153168D-04 | .234451 |
| 171 | .177828D-07 | .156156D-04 | .177828D-07 | .158645D-04 | .158290D-04 | .223673 |
| 172 | .181014D-07 | .157516D-04 | .181014D-07 | .159990D-04 | .159643D-04 | .216819 |
| 173 | .187191D-07 | .162601D-04 | .187191D-07 | .165061D-04 | .164720D-04 | .206824 |
| 174 | .190231D-07 | .163752D-04 | .190231D-07 | .166197D-04 | .165864D-04 | .200443 |
| 175 | .196417D-07 | .168780D-04 | .196417D-07 | .171212D-04 | .170885D-04 | .191147 |
| 176 | .199298D-07 | .169716D-04 | .199298D-07 | .172134D-04 | .171815D-04 | .185154 |
| 177 | .205478D-07 | .174679D-04 | .205478D-07 | .177083D-04 | .176770D-04 | .176451 |
| 178 | .208187D-07 | .175396D-04 | .208187D-07 | .177787D-04 | .177483D-04 | .170781 |
| 179 | .214347D-07 | .180282D-04 | .214347D-07 | .182659D-04 | .182362D-04 | .162609 |
| 180 | .216872D-07 | .180777D-04 | .216872D-07 | .183141D-04 | .182853D-04 | .157201 |
| 181 | .222996D-07 | .185577D-04 | .222996D-07 | .187928D-04 | .187647D-04 | .149488 |
| 182 | .225324D-07 | .185845D-04 | .225324D-07 | .188183D-04 | .187912D-04 | .144293 |
| 183 | .231397D-07 | .190549D-04 | .231397D-07 | .192874D-04 | .192610D-04 | .136991 |
| 184 | .233515D-07 | .190589D-04 | .233515D-07 | .192901D-04 | .192647D-04 | .131965 |
| 185 | .239521D-07 | .195186D-04 | .239521D-07 | .197486D-04 | .197239D-04 | .125020 |
| 186 | .241416D-07 | .194994D-04 | .241416D-07 | .197282D-04 | .197045D-04 | .120125 |
| 187 | .247339D-07 | .199475D-04 | .247339D-07 | .201751D-04 | .201522D-04 | .113502 |
| 188 | .248999D-07 | .199049D-04 | .248999D-07 | .201312D-04 | .201094D-04 | .108703 |
| 189 | .254823D-07 | .203405D-04 | .254823D-07 | .205657D-04 | .205446D-04 | .102360 |
| 190 | .256237D-07 | .202742D-04 | .256237D-07 | .204982D-04 | .204782D-04 | .976254D-01 |
| 191 | .261943D-07 | .206963D-04 | .261943D-07 | .209191D-04 | .209000D-04 | .915329D-01 |
| 192 | .263100D-07 | .206063D-04 | .263100D-07 | .208279D-04 | .208098D-04 | .868351D-01 |
| 193 | .268672D-07 | .210139D-04 | .268672D-07 | .212344D-04 | .212172D-04 | .809619D-01 |
| 194 | .269560D-07 | .208999D-04 | .269560D-07 | .211193D-04 | .211032D-04 | .762735D-01 |
| 195 | .274982D-07 | .212922D-04 | .274982D-07 | .215104D-04 | .214952D-04 | .705934D-01 |
| 196 | .275588D-07 | .211542D-04 | .275588D-07 | .213713D-04 | .213572D-04 | .658891D-01 |
| 197 | .280842D-07 | .215301D-04 | .280842D-07 | .217461D-04 | .217329D-04 | .603776D-01 |
| 198 | .281156D-07 | .213681D-04 | .281156D-07 | .215830D-04 | .215710D-04 | .556332D-01 |
| 199 | .286224D-07 | .217267D-04 | .286224D-07 | .219405D-04 | .219295D-04 | .502689D-01 |
| 200 | .286236D-07 | .215406D-04 | .286236D-07 | .217534D-04 | .217435D-04 | .454625D-01 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 201 | .291101D-07 | .218811D-04 | .291101D-07 | .220928D-04 | .220839D-04 | .402120D-01 |
| 202 | .290800D-07 | .216710D-04 | .290800D-07 | .218816D-04 | .218739D-04 | .352995D-01 |
| 203 | .295443D-07 | .219924D-04 | .295443D-07 | .222020D-04 | .221953D-04 | .301560D-01 |
| 204 | .294820D-07 | .217583D-04 | .294820D-07 | .219669D-04 | .219614D-04 | .251491D-01 |
| 205 | .299225D-07 | .220598D-04 | .299225D-07 | .222674D-04 | .222629D-04 | .200827D-01 |
| 206 | .298269D-07 | .218019D-04 | .298269D-07 | .220084D-04 | .220052D-04 | .149169D-01 |
| 207 | .302416D-07 | .220825D-04 | .302416D-07 | .222881D-04 | .222859D-04 | .991472D-02 |
| 208 | .301118D-07 | .218009D-04 | .301118D-07 | .220055D-04 | .220044D-04 | .460491D-02 |
| 209 | .304990D-07 | .220599D-04 | .304990D-07 | .222635D-04 | .222635D-04 | -.362576D-03 |
| 210 | .303342D-07 | .217546D-04 | .303342D-07 | .219573D-04 | .219586D-04 | -.589080D-02 |
| 211 | .306919D-07 | .219911D-04 | .306919D-07 | .221928D-04 | .221952D-04 | -.108367D-01 |
| 212 | .304913D-07 | .216626D-04 | .304913D-07 | .218633D-04 | .218669D-04 | -.165638D-01 |
| 213 | .308177D-07 | .218756D-04 | .308177D-07 | .220754D-04 | .220802D-04 | -.215204D-01 |
| 214 | .305805D-07 | .215240D-04 | .305805D-07 | .217229D-04 | .217288D-04 | -.275345D-01 |
| 215 | .308738D-07 | .217129D-04 | .308738D-07 | .219108D-04 | .219179D-04 | -.325170D-01 |
| 216 | .305990D-07 | .213384D-04 | .305990D-07 | .215354D-04 | .215438D-04 | -.387957D-01 |
| 217 | .308574D-07 | .215023D-04 | .308574D-07 | .216984D-04 | .217079D-04 | -.438414D-01 |
| 218 | .305444D-07 | .211053D-04 | .305444D-07 | .213005D-04 | .213113D-04 | -.504909D-01 |
| 219 | .307658D-07 | .212433D-04 | .307658D-07 | .214376D-04 | .214496D-04 | -.556192D-01 |
| 220 | .304141D-07 | .208242D-04 | .304141D-07 | .210176D-04 | .210307D-04 | -.626168D-01 |
| 221 | .305967D-07 | .209355D-04 | .305967D-07 | .211281D-04 | .211424D-04 | -.678734D-01 |
| 222 | .302055D-07 | .204945D-04 | .302055D-07 | .206862D-04 | .207018D-04 | -.753498D-01 |
| 223 | .303473D-07 | .205784D-04 | .303473D-07 | .207692D-04 | .207860D-04 | -.807618D-01 |
| 224 | .299160D-07 | .201160D-04 | .299160D-07 | .203060D-04 | .203240D-04 | -.886971D-01 |
| 225 | .300151D-07 | .201716D-04 | .300151D-07 | .203607D-04 | .203799D-04 | -.943234D-01 |
| 226 | .295431D-07 | .196882D-04 | .295431D-07 | .198764D-04 | .198969D-04 | -.102883 |
| 227 | .295975D-07 | .197147D-04 | .295975D-07 | .199021D-04 | .199238D-04 | -.108764 |
| 228 | .290843D-07 | .192106D-04 | .290843D-07 | .193972D-04 | .194201D-04 | -.117937 |
| 229 | .290920D-07 | .192072D-04 | .290920D-07 | .193930D-04 | .194171D-04 | -.124155 |
| 230 | .285371D-07 | .186830D-04 | .285371D-07 | .188680D-04 | .188933D-04 | -.134154 |
| 231 | .284959D-07 | .186488D-04 | .284959D-07 | .188330D-04 | .188595D-04 | -.140774 |
| 232 | .278988D-07 | .181049D-04 | .278988D-07 | .182883D-04 | .183160D-04 | -.151612 |
| 233 | .278067D-07 | .180390D-04 | .278067D-07 | .182216D-04 | .182506D-04 | -.158750 |
| 234 | .271669D-07 | .174759D-04 | .271669D-07 | .176578D-04 | .176879D-04 | -.170716 |
| 235 | .270216D-07 | .173775D-04 | .270216D-07 | .175586D-04 | .175899D-04 | -.178483 |
| 236 | .263387D-07 | .167957D-04 | .263387D-07 | .169760D-04 | .170085D-04 | -.191634 |
| 237 | .261380D-07 | .166638D-04 | .261380D-07 | .168433D-04 | .168770D-04 | -.200209 |
| 238 | .254115D-07 | .160637D-04 | .254115D-07 | .162425D-04 | .162774D-04 | -.214959 |
| 239 | .251531D-07 | .158973D-04 | .251531D-07 | .160753D-04 | .161114D-04 | -.224540 |
| 240 | .243825D-07 | .152795D-04 | .243825D-07 | .154568D-04 | .154941D-04 | -.241044 |
| 241 | .240637D-07 | .150775D-04 | .240637D-07 | .152541D-04 | .152925D-04 | -.251931 |
| 242 | .232486D-07 | .144425D-04 | .232486D-07 | .146183D-04 | .146579D-04 | -.270823 |
| 243 | .228668D-07 | .142038D-04 | .228668D-07 | .143789D-04 | .144197D-04 | -.283388 |
| 244 | .220067D-07 | .135520D-04 | .220067D-07 | .137264D-04 | .137683D-04 | -.305047 |
| 245 | .215592D-07 | .132754D-04 | .215592D-07 | .134491D-04 | .134921D-04 | -.319853 |
| 246 | .206534D-07 | .126072D-04 | .206534D-07 | .127802D-04 | .128243D-04 | -.345347 |
| 247 | .201370D-07 | .122913D-04 | .201370D-07 | .124636D-04 | .125088D-04 | -.363162 |
| 248 | .191850D-07 | .116072D-04 | .191850D-07 | .117788D-04 | .118251D-04 | -.393420 |
| 249 | .185965D-07 | .112505D-04 | .185965D-07 | .114214D-04 | .114689D-04 | -.415445 |
| 250 | .175975D-07 | .105508D-04 | .175975D-07 | .107210D-04 | .107696D-04 | -.452555 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 251 | .169332D-07 | .101517D-04 | .169332D-07 | .103212D-04 | .103708D-04 | -.480615 |
| 252 | .158863D-07 | .943677D-05 | .158863D-07 | .960563D-05 | .965625D-05 | -.527063 |
| 253 | .151423D-07 | .899327D-05 | .151423D-07 | .916146D-05 | .921315D-05 | -.564237 |
| 254 | .140465D-07 | .826329D-05 | .140465D-07 | .843082D-05 | .848353D-05 | -.625119 |
| 255 | .132182D-07 | .777326D-05 | .132182D-07 | .794014D-05 | .799388D-05 | -.676812 |
| 256 | .120722D-07 | .702843D-05 | .120722D-07 | .719465D-05 | .724935D-05 | -.760278 |
| 257 | .111546D-07 | .648936D-05 | .111546D-07 | .665493D-05 | .671064D-05 | -.837115 |
| 258 | .995674D-08 | .572966D-05 | .995674D-08 | .589459D-05 | .595125D-05 | -.961084 |
| 259 | .894402D-08 | .513862D-05 | .894402D-08 | .530292D-05 | .536055D-05 | -1.08676 |
| 260 | .769257D-08 | .436398D-05 | .769257D-08 | .452765D-05 | .458617D-05 | -1.29248 |
| 261 | .657790D-08 | .371753D-05 | .657790D-08 | .388057D-05 | .394003D-05 | -1.53237 |
| 262 | .527034D-08 | .292762D-05 | .527034D-08 | .309004D-05 | .315037D-05 | -1.95240 |
| 263 | .404594D-08 | .222168D-05 | .404594D-08 | .238347D-05 | .244471D-05 | -2.56928 |
| 264 | .267942D-08 | .141599D-05 | .267942D-08 | .157717D-05 | .163922D-05 | -3.93425 |
| 265 | .133588D-08 | .645635D-06 | .133588D-08 | .806210D-06 | .869131D-06 | -7.80452 |
| 266 | -.932669D-10 | -.176679D-06 | -.932669D-10 | -.167072D-07 | .469907D-07 | 381.260 |
| 267 | -.156711D-08 | -.101739D-05 | -.156711D-08 | -.858016D-06 | -.793491D-06 | 7.52034 |
| 268 | -.306365D-08 | -.185752D-05 | -.306365D-08 | -.169875D-05 | -.163349D-05 | 3.84135 |
| 269 | -.468139D-08 | -.277590D-05 | -.468139D-08 | -.261771D-05 | -.255167D-05 | 2.52281 |
| 270 | -.625111D-08 | -.363555D-05 | -.625111D-08 | -.347794D-05 | -.341123D-05 | 1.91827 |
| 271 | -.802947D-08 | -.464067D-05 | -.802947D-08 | -.448365D-05 | -.441620D-05 | 1.50439 |
| 272 | -.968010D-08 | -.552217D-05 | -.968010D-08 | -.536573D-05 | -.529764D-05 | 1.26908 |
| 273 | -.116400D-07 | -.662549D-05 | -.116400D-07 | -.646962D-05 | -.640084D-05 | 1.06315 |
| 274 | -.133812D-07 | -.753197D-05 | -.133812D-07 | -.737667D-05 | -.730733D-05 | .940021 |
| 275 | -.155493D-07 | -.874812D-05 | -.155493D-07 | -.859339D-05 | -.852342D-05 | .814223 |
| 276 | -.173938D-07 | -.968397D-05 | -.173938D-07 | -.952979D-05 | -.945926D-05 | .740132 |
| 277 | -.198046D-07 | -.110320D-04 | -.198046D-07 | -.108784D-04 | -.108073D-04 | .653620 |
| 278 | -.217692D-07 | -.120029D-04 | -.217692D-07 | -.118499D-04 | -.117784D-04 | .603588 |
| 279 | -.244683D-07 | -.135083D-04 | -.244683D-07 | -.133558D-04 | -.132838D-04 | .539252 |
| 280 | -.265761D-07 | -.145225D-04 | -.265761D-07 | -.143705D-04 | -.142980D-04 | .504692 |
| 281 | -.296255D-07 | -.162199D-04 | -.296255D-07 | -.160685D-04 | -.159955D-04 | .454029 |
| 282 | -.319071D-07 | -.172880D-04 | -.319071D-07 | -.171371D-04 | -.170639D-04 | .427106 |
| 283 | -.353935D-07 | -.192262D-04 | -.353935D-07 | -.190758D-04 | -.190023D-04 | .385411 |
| 284 | -.378935D-07 | -.203643D-04 | -.378935D-07 | -.202145D-04 | -.201405D-04 | .366284 |
| 285 | -.419433D-07 | -.226144D-04 | -.419433D-07 | -.224651D-04 | -.223908D-04 | .330605 |
| 286 | -.447237D-07 | -.238440D-04 | -.447237D-07 | -.236953D-04 | -.236210D-04 | .313272 |
| 287 | -.495290D-07 | -.265151D-04 | -.495290D-07 | -.263668D-04 | -.262925D-04 | .281674 |
| 288 | -.526871D-07 | -.278703D-04 | -.526871D-07 | -.277225D-04 | -.276474D-04 | .271007 |
| 289 | -.585598D-07 | -.311398D-04 | -.585598D-07 | -.309926D-04 | -.309177D-04 | .241678 |
| 290 | -.622407D-07 | -.326672D-04 | -.622407D-07 | -.325204D-04 | -.324460D-04 | .228894 |
| 291 | -.697302D-07 | -.368512D-04 | -.697302D-07 | -.367049D-04 | -.366309D-04 | .201760 |
| 292 | -.741808D-07 | -.386242D-04 | -.741808D-07 | -.384784D-04 | -.384042D-04 | .192969 |
| 293 | -.843875D-07 | -.443617D-04 | -.843875D-07 | -.442165D-04 | -.441432D-04 | .165679 |
| 294 | -.900105D-07 | -.464582D-04 | -.900105D-07 | -.463135D-04 | -.462398D-04 | .159138 |
| 295 | -.105670D-06 | -.553720D-04 | -.105670D-06 | -.552278D-04 | -.551576D-04 | .127174 |
| 296 | -.112929D-06 | -.575855D-04 | -.112929D-06 | -.574417D-04 | -.573698D-04 | .125206 |
| 297 | -.143797D-06 | -.755322D-04 | -.143797D-06 | -.753889D-04 | -.753284D-04 | .803025D-01 |
| 298 | -.154205D-06 | -.772380D-04 | -.154205D-06 | -.770952D-04 | -.770495D-04 | .593023D-01 |
| 299 | -.231782D-06 | -.120676D-03 | -.231782D-06 | -.120533D-03 | -.120458D-03 | .627035D-01 |
| 300 | -.328145D-06 | -.174965D-03 | -.328145D-06 | -.174823D-03 | -.174835D-03 | -.675274D-02 |

| CYCLES PER REV. | SIMULATED (COS/SIN) | | CORRECTED (COS/SIN) | | ANALYTICAL (SIN) | % (SIN) ERROR |
|---|---|---|---|---|---|---|
| 301 | -.208636D-06 | -.990539D-04 | -.208636D-06 | -.989126D-04 | -.990555D-04 | -.144487 |
| 302 | -.816894D-07 | -.348845D-04 | -.816894D-07 | -.347436D-04 | -.349009D-04 | -.452765 |
| 303 | -.231401D-07 | -.908226D-05 | -.231401D-07 | -.894182D-05 | -.908479D-05 | -1.59890 |
| 304 | -.507486D-08 | -.191735D-05 | -.507486D-08 | -.177737D-05 | -.190172D-05 | -6.99617 |
| 305 | -.813246D-09 | -.369720D-06 | -.813246D-09 | -.230204D-06 | -.336006D-06 | -45.9601 |
| 306 | -.520699D-11 | -.994182D-07 | -.520699D-11 | .396419D-07 | -.516182D-07 | 230.211 |
| 307 | .110239D-09 | -.653705D-07 | .110239D-09 | .732367D-07 | -.694245D-08 | 109.479 |

MAXIMUM DEGREE IN ZONAL FIELD = 300 MAX. DEG. IN REFERENCE FIELD = 30 SATELLITE HEIGHT = 160016.00
ORBITAL PERIOD = 5263.369068 MEAN SEPARATION = 300000.00 NO. OF INTEGR. INTERVALS = 2048
INITIAL CONDITION ERRORS : DZCE = .0 DX11 = .0 DX12 = .0 DR1 = .0 DUP1 = .0
DR2 = .0 DUP2 = .0
UNITS ARE METERS , SECONDS AND METERS PER SECOND.

FROM 0 CY/REV. TO 3 CY/REV : CY/REV CORRECTED (COS, SIN) ANALYTICAL.

| CY/REV | CORRECTED | (COS, SIN) | ANALYTICAL |
|---|---|---|---|
| 0 | .188049D-07 | | |
| 1 | -.894372D-02 | .654939D-04 | |
| 2 | .231722D-06 | .181704D-04 | .416936D-04 |
| 3 | -.423704D-06 | .364512D-04 | .260653D-04 |

SIGNAL STRENGTH OVER BANDS 10 CYCLES/SEC. WIDE, FROM 4 TO 304 CY/REV.

| BAND | RMS UNCORRECTED | RMS CORRECTED | RMS ANALYTICAL | % (SIN+COS) ERROR |
|---|---|---|---|---|
| 4 - 14 | .560978D-04 | .731615D-04 | .733002D-04 | .232767 |
| 15 - 24 | .129322D-03 | .135994D-03 | .135982D-03 | .621289D-01 |
| 25 - 34 | .693438D-03 | .694904D-03 | .694447D-03 | .730045D-01 |
| 35 - 44 | .776036D-03 | .776627D-03 | .776330D-03 | .485214D-01 |
| 45 - 54 | .551326D-03 | .550612D-03 | .550468D-03 | .438234D-01 |
| 55 - 64 | .365829D-03 | .366063D-03 | .366014D-03 | .439519D-01 |
| 65 - 74 | .257575D-03 | .256917D-03 | .256925D-03 | .511845D-01 |
| 75 - 84 | .155790D-03 | .155749D-03 | .155765D-03 | .741701D-01 |
| 85 - 94 | .107370D-03 | .106692D-03 | .106753D-03 | .112255 |
| 95 - 104 | .494289D-04 | .491960D-04 | .492333D-04 | .223844 |
| 105 - 114 | .269134D-04 | .256878D-04 | .258140D-04 | .502619 |
| 115 - 124 | .159993D-04 | .148945D-04 | .150316D-04 | .953066 |
| 125 - 134 | .488465D-05 | .426503D-05 | .433980D-05 | 3.46983 |
| 135 - 144 | .107387D-04 | .116389D-04 | .114990D-04 | 1.27916 |
| 145 - 154 | .236110D-04 | .244978D-04 | .243567D-04 | .592562 |
| 155 - 164 | .361672D-04 | .370059D-04 | .368750D-04 | .371651 |
| 165 - 174 | .475307D-04 | .483224D-04 | .482082D-04 | .262753 |
| 175 - 184 | .570403D-04 | .577890D-04 | .576970D-04 | .198984 |
| 185 - 194 | .641205D-04 | .648303D-04 | .647658D-04 | .160210 |
| 195 - 204 | .682829D-04 | .689574D-04 | .689245D-04 | .140205 |
| 205 - 214 | .691338D-04 | .697763D-04 | .697784D-04 | .138422 |
| 215 - 224 | .663773D-04 | .669904D-04 | .670298D-04 | .156611 |
| 225 - 234 | .598046D-04 | .603908D-04 | .604682D-04 | .199219 |
| 235 - 244 | .492678D-04 | .498286D-04 | .499433D-04 | .281338 |
| 245 - 254 | .346432D-04 | .351780D-04 | .353267D-04 | .461530 |
| 255 - 264 | .161104D-04 | .165904D-04 | .167567D-04 | 1.11945 |
| 265 - 274 | .133609D-04 | .129764D-04 | .128117D-04 | 1.62608 |
| 275 - 284 | .466453D-04 | .461613D-04 | .459598D-04 | .527555 |
| 285 - 294 | .107577D-03 | .107126D-03 | .106897D-03 | .290752 |
| 295 - 304 | .272660D-03 | .272291D-03 | .272283D-03 | .227380 |

TOTAL PERCENTAGE ERROR RESPECT TO: (A) TOTAL UNCORRECTED SIGNAL; (B) TOTAL CORRECTED SIGNAL. (BAND FROM 4 CY/REV TO 304 CY/REV)
(A) .9046809412D-01 (B) .9031086767D-01