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A QUALITY INVESTIGATION OF  
GLOBAL VERTICAL DATUM CONNECTION

by

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**Abstract.**

Vertical datum connection is investigated from the view point of quality, i.e. accuracy, reliability and detectability, based on the method originally proposed by Rummel and Teunissen (1988). Three simulations are carried out. The levelling accuracy is assumed to be 1 mm/km and the position accuracy of space stations 5 cm. The three error models of the geoid used are the Rapp-81 model, a tailored gravitational field and the GEM-T1-related variance-covariance matrix. The results show that the quality of the vertical datum connection depends on the relative accuracy of levelling and space stations, but in particular on that of the geoid. It is also related to the number of space stations inside each datum zone and their geographical distribution. Only when the absolute determination of vertical datum is required, the absolute accuracy of the space stations plays an important role.

Finally, we discuss an alternative datum connection model using terrestrial gravity anomalies and satellite derived geopotential coefficients.

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## Introduction.

The sea surface is continuously changing. Main factors, causing these variations are climate changes and tectonic processes. Current sea level rise and major tectonic readjustments are still a consequence of the last de-glaciation, but awareness is growing that man-made climate changes might lead to a significant change of the rate of present day sea level rise, with far reaching consequences for all low land countries (Wind, 1987). Accurate monitoring of relative sea level changes can be done with tide gauges. However as these changes are not uniform it is highly desirable, in order to attain a better understanding of the involved processes, to be able to compare sea level changes in different parts of the world or inside larger geographical regions. This requires connection of tide gauges into one uniform vertical datum. The datum connection can be achieved, in principle, by means of space positioning in combination with levelling and precise geoid computation.

By datum we mean a computational basis in geodesy. The vertical datum is needed to compute "absolute" heights in levelling networks, because only height increments are measurable. It is good tradition to use some average value of the observed sea level changes at a tide gauge as height reference or datum for a certain geographical area. Tide gauges are set up along coastal lines (or large rivers). They provide, after some computational procedure is applied, mean sea level values averaged over certain time spans (month, year). Because of the sea surface topography, i.e. the deviation of the actual ocean surface from an equipotential surface, and its variation in space, different gauges, even located along the same coast line, result in different height datum values, as can be seen from the height differences between tide gauges being not equal to zero. Different vertical datums refer to different equipotential surfaces. As a result, constant off-sets exist among the different height zones. As a secondary consequence also constant off-sets result between gravity anomaly data of different datum zones.

Several methods have been developed over the years for the determination of these off-sets between the datum zones. Heck and Rummel (1990) recently gave a review on strategies for solving the vertical datum problem using terrestrial and satellite geodetic data. For example, Mather (1976) proposed the use of an oceanographic approach and combination with satellite altimetry

and dynamic levelling. The starting point of the oceanographic method is Newton's second theorem. With some approximations and the assumption that the reference level is an isobaric surface, the potential difference between the ocean surface and a reference level can be computed by using pressure, temperature and salinity profiles. Another assumption for doing such a computation is that the sea surface is also isobaric. On the other hand, the important modelling of frictional forces will affect the precision of extrapolated sea surface topography. There also exist difficulties in the combination of satellite altimetry and dynamic levelling. Therefore, at present time, the method cannot be used to establish vertical datum connections between continents with sufficient precision.

The purpose of the method proposed by Colombo (1980) is to determine a set of potential differences among benchmarks located in various continents. For simplicity, in the case of two benchmarks A and B, the quantity of interest is the potential difference  $\Delta W(A,B)$ . If we have a set of potential differences between the benchmark and some point in the vertical datum zone of each continent, respectively, it is easy to form the observational equations (Colombo, 1980; Hajela, 1983). A set of precise space stations will play an important role in the computation of the normal gravitational potential  $V'$  and the centrifugal potential  $\Phi$ . The potential differences are obtained from levelling and gravity data. To obtain precisely the value of the normal gravitational potential, it is suggested to use a spherical harmonic expansion series up to a high degree. Because the disturbing potential is involved in the observational equations, gravity anomalies in a cap with an optimum density and accuracy around every station are needed. One can make a datum connection and calculate an accuracy estimation by using the least squares adjustment method.

Rummel and Teunissen (1988) recently proposed a new approach to the vertical datum connection by the geodetic boundary value problem. It can be viewed as a modification of Colombo's method. The method consists of three main steps. The first one is to linearize potential numbers and scalar gravity with respect to some fixed normal potential and approximate boundary surface, and derive the fundamental equations of physical geodesy. Secondly, one will solve the geodetic boundary value problem and determine geoid height  $N$  or disturbing

potential  $T$ . Disturbing potential or geoid height results from the well known Stokes' formula.  $\Delta W_0$  could also be derived, in principle, by using an additional precise distance and a precise value of  $GM$ . Neglect of  $\Delta W_0$  and the datum off-sets  $C_{Q_1,0}$  cause biases in  $T$  and  $N$ . One can e.g. assume that there are  $(I + 1)$  vertical datums over the world. Then we can easily formulate the datum connection problem as an ordinary least squares adjustment problem. The estimated parameters are the potential differences  $C_{Q_1,0}$  between benchmarks  $Q_1$  and  $O$ , and  $\Delta W_0$  from the reference normal gravitational potential. For more details of the method, see chapter 2 or (Rummel and Teunissen, 1988; Heck and Rummel, 1990). The basic requirements of the method are threefold: (1) In each of the  $(I+1)$  vertical datum zones, precise geocentric coordinates of at least one station are available, derived by space methods such as VLBI or SLR. It is easy to compute the geometric height  $h$  above the adopted global reference system and the geodetic coordinates. (2) Precise orthometric (or normal) heights or potential differences are available for all stations. They can be obtained by levelling. (3) Gravity anomalies referring to the  $(I+1)$  datum zones are known globally.

Hajela (1983) looked into the accuracy of Colombo's method. The purpose of this report is to investigate the accuracy, reliability and detectability of the method by Rummel and Teunissen (1988). This includes the computation of the effects of errors of the geometric heights  $h$  above a chosen reference surface, orthometric heights  $H$  and geoid heights  $N$  on the estimated parameters. In chapter 2, we will briefly discuss the mathematical model for the datum connection with some formulae of the accuracy, reliability and detectability of the method. Chapters 3 and 4 will focus on the a priori precision matrices of geometric (ellipsoidal) heights  $h$ , orthometric heights and the term  $S(\Delta g^i)$ . The precision of  $h$  depends on the characteristics of the space techniques. The connection of space stations to tide gauges can be carried out by either levelling operations or GPS. Finally, we will carry out an error analysis for a number of examples. Related problems will also be investigated. The question is whether a global height datum correction with sufficient accuracy can be achieved at present time and what the expected trend is for the future. Despite the global character of the problem special attention shall be given to the height datum problem around the North Sea.

**Mathematical model for the vertical datum connection.**

We will briefly summarize the method of the vertical datum connection, proposed by Rummel and Teunissen (1988). The vertical datum connection problem is a least squares adjustment by geodetic boundary value techniques, if precise geocentric coordinates (or only vertical components) of space stations, orthometric heights and gravity anomalies referring to their own vertical datum zones are available in each of worldwide (I+1) datum zones. The parameters to be estimated are the potential differences  $C_{Q_i0}$  and  $\Delta W_0$ . In this chapter, we discuss the observational equations of the vertical datum connection, including the computation of the coefficients of the observational equations. The theoretical aspects on accuracy, reliability and detectability of the method, and on the effects of errors of  $h$ ,  $H$  and  $S(\Delta g^i)$  on the estimated parameters will also be investigated here. The introduction of prior precision matrices of  $h$ ,  $H$  and  $S(\Delta g)$  is left to the next chapter.

1. The observation equations.

Suppose that there are (I+1) vertical datums in the world. The potential differences between the fundamental benchmark 0 and the benchmark  $Q_i$  of all other datums are denoted below.

$$C_{Q_i0} = W(0) - W(Q_i) \quad i = 1, 2, \dots, I. \quad (1)$$

Linearizing the potential number with respect to the normal potential  $U$  and the approximate point, we obtain

$$\Delta W_0 - C_{Q_i0} = -\gamma N + T \quad (2)$$

where the height anomaly  $N$  is the vertical distance from the adopted reference surface to the equipotential surface through datum point  $Q_i$ ;  $\Delta W_0$  and  $C_{Q_i0}$  are explained in Figure 1.

Bruns' equation is then from (2)



$$N = \frac{T - \Delta W_0 + C_{Q_i O}}{\gamma} \quad (3)$$

Similarly, linearization of scalar gravity in combination with (2) gives the fundamental equation of physical geodesy

$$\begin{aligned} \Delta g^{(i)} &= g(p) - \frac{\partial \gamma}{\partial n} \cdot \frac{1}{g} C_{Q_i O} - \gamma(p_0') \\ &= \frac{2}{r} \Delta W_0 - \frac{2}{r} C_{Q_i O} - \left(\frac{2}{r} + \frac{\partial}{\partial r}\right) T \end{aligned} \quad (4)$$

with approximations,  $\frac{\partial \gamma}{\partial n} = -\frac{2\gamma}{r}$ , where the upperindex (i) of  $\Delta g$  indicates that the observed gravity is reduced to the level surface passing through  $Q_i$ .

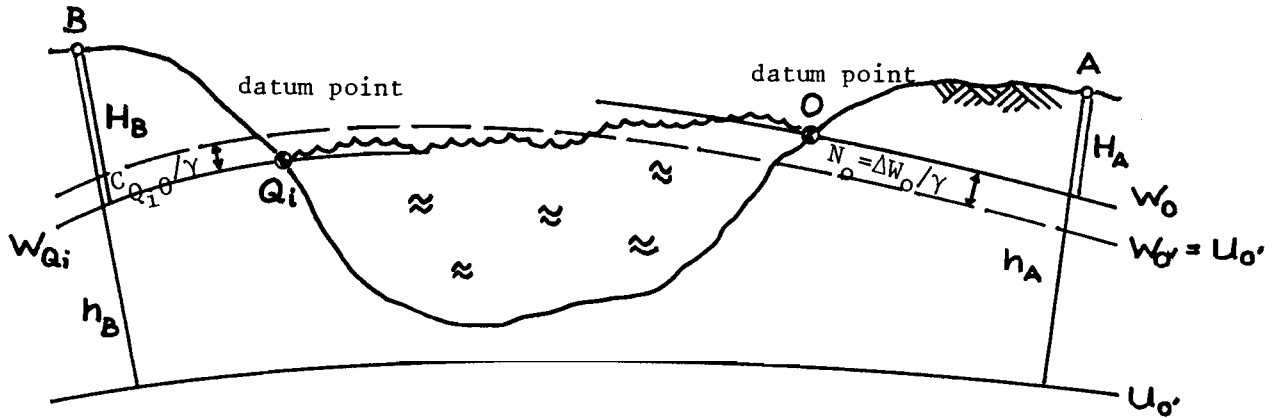


Figure 1: Geometrical representation of related quantities.

By defining  $C_{Q_i O} = 0$  when  $Q_i = 0$ , we obtain the relationship between  $\Delta g$  and  $\Delta g^{(i)}$ ,

$$\Delta g = \Delta g^{(i)} + \frac{2}{r} C_{Q_i O} \quad (5)$$

Therefore, the general solution of the geodetic boundary value problem is

$$T(P) = \frac{\delta(GM)}{R} + \frac{R}{4\pi} \int_{\sigma} \text{St}(\psi_{PQ}) \{ \Delta g^{(j)} + \frac{2}{R} C_{Q_j 0} \} d\sigma_j \quad (6)$$

Inserting (6) into (3), we have

$$N^{(i)}(P) = \frac{\delta(GM)}{R\gamma} - \frac{\Delta W_0}{\gamma} + \frac{C_{Q_1 0}}{\gamma} + \frac{R}{4\pi\gamma} \int_{\sigma} \text{St}(\psi_{PQ}) \{ \Delta g^{(j)} + \frac{2}{R} C_{Q_j 0} \} d\sigma_j \quad (7)$$

where  $\delta(GM) = GM - GM'$ ,  $G$  is the gravitational constant,  $M'$  is the approximate mass of the earth and  $M$  its actual mass. The assumption of  $\delta(GM)$  being equal to zero is acceptable, because the value of  $GM'$  can accurately be determined (see e.g. Rummel and Teunissen 1988). Otherwise, it would not be possible to separate between  $\frac{\delta(GM)}{R\gamma}$  and  $\frac{\Delta W_0}{\gamma}$ . Equation (7) then becomes

$$N^{(i)}(P) = - \frac{\Delta W_0}{\gamma} + \frac{C_{Q_1 0}}{\gamma} + \frac{R}{4\pi\gamma} \int_{\sigma} \text{St}(\psi) \{ \Delta g^{(j)} + \frac{2}{R} C_{Q_j 0} \} d\sigma_j \quad (8)$$

The geoidal height  $N^{(i)}$  can also be obtained by the combination of the geocentric coordinates and the orthometric height of a space station

$$N^{(i)} = h - H^{(i)} \quad (9)$$

where the upperindex indicates again that  $N$  and  $H$  belong to datum zone  $i$ .

Inserting (9) in (8), we get

$$h - H^{(i)} = - \frac{\Delta W_0}{\gamma} + \frac{C_{Q_1 0}}{\gamma} + \frac{R}{4\pi\gamma} \int_{\sigma} \text{St}(\psi) \{ \Delta g^{(j)} + \frac{2}{R} C_{Q_j 0} \} d\sigma_j \quad (10)$$

Rearranging (10) leads to (Rummel and Teunissen, 1988):

$$h - H^{(i)} - \frac{R}{\gamma} S(\Delta g^{(j)}) = - \frac{\Delta W_0}{\gamma} + \frac{C_{Q_1 0}}{\gamma} + \frac{2}{\gamma} \sum_{j=1}^I C_{Q_j 0} IS_{PQ_j} \quad (11)$$

where

$$\frac{R}{\gamma} S(\Delta g^{(j)}) = \frac{R}{4\pi\gamma} \int_{\sigma} St(\psi)\Delta g^{(j)} d\sigma_j \quad (12)$$

$$IS_{PQ_j} = \frac{1}{4\pi} \int_{\sigma_j} St(\psi) d\sigma_j \quad (13)$$

where  $\sigma_j$  is the  $j$ -th datum zone. Denoting

$$y_{P_k} = h - H^{(i)} - \frac{R}{\gamma} S(\Delta g^{(j)}) \quad (14)$$

the observational equation of station  $P_k$  in zone  $i$  becomes

$$y_{P_k} = -\frac{\Delta W_0}{\gamma} + (1 + 2IS_{PQ_i}) \frac{C_{Q_iO}}{\gamma} + 2 \sum_{j=1(j \neq i)}^I IS_{P_kQ_j} \frac{C_{Q_jO}}{\gamma} + \epsilon_{P_k} \quad (15)$$

where  $\epsilon_{P_k}$  is the error of observation  $y_{P_k}$  in the geocentric coordinate system.

It is clear from equation (15) that the parameters  $\Delta W_0$ ,  $C_{Q_iO}$  can be uniquely determined by a least squares adjustment, if there is at least one station in each datum zone and orthometric heights and gravity anomalies are available. The vertical datum connection becomes simply a least squares adjustment problem.

In an ideal case, if there would be no sea surface topography, all  $C_{Q_jO}$  would be equal to zero, and (15) would become

$$y_{P_k} = -\frac{\Delta W_0}{\gamma} + \epsilon_{P_k} \quad (16)$$

which means that we could accurately determine  $\Delta W_0$  by space techniques, levelling operations and suitable gravity anomalies.

If only a vertical datum connection between two zones is of interest, equation (15) becomes

$$y_{P_k} = -\frac{\Delta W_0}{\gamma} + 2IS_{P_kQ_1} \frac{C_{Q_1O}}{\gamma} \quad ; \text{ for the stations in datum O} \quad (17)$$

$$y_{P_k} = -\frac{\Delta W_0}{\gamma} + (1 + 2IS_{P_kQ_1}) \frac{C_{Q_1O}}{\gamma} \quad ; \text{ for the stations in datum Q} \quad (18)$$

The general structure of the coefficient matrix for the vertical datum connection is shown in Figure 2.

	$\Delta W_0$	$c_{10}$	$c_{20}$	$c_{30}$	$c_{40}$	$c_{50}$	$c_{60}$	$c_{70}$	$c_{80}$
$y_1$ 0	a	$c_{11}$	$c_{12}$	$c_{13}$	$c_{14}$	$c_{15}$	$c_{16}$	$c_{17}$	$c_{18}$
$y_2$ 0	a	$c_{21}$	$c_{22}$	$c_{23}$	$c_{24}$	$c_{25}$	$c_{26}$	$c_{27}$	$c_{28}$
$y_3$ 1	a	$b_{31}$	$c_{32}$	$c_{33}$	.	.	.	.	.
$y_4$ 2	a	.	$b_{42}$	.	.	.	.	.	.
$y_5$ 3	a			$b_{53}$					
$y_6$ 3	a			$b_{63}$					
$y_7$ 4	a				$b_{74}$				
$y_8$ 5	a					$b_{85}$			
$y_9$ 6	a						$b_{96}$		
$y_{10}$ 7	a							$b_{107}$	
$y_{11}$ 8	a								$b_{118}$

Figure 2: Structure of the coefficients matrix for an example with 11 stations and 9 vertical datums. The types of elements appearing in the coefficient matrix are (compare eq. (15))  $a = -1$ ,  $b_{ki} = (1 + 2IS_{P_kQ_i})$  and  $C_{kj} = 2IS_{P_kQ_j}$ , adopted from Rummel and Teunissen (1988).

## 2. Coefficient computations of the observation equations.

The integral element  $d\sigma_j$  in equation (13) can be rewritten as

$$d\sigma_j = \sin \psi \, d\psi \, d\alpha \quad (19)$$

where  $\alpha$  is an azimuth angle of  $Q_j$  relative to P. Then equation (13) becomes

$$IS_{PQ_j} = \frac{1}{4\pi} \int_{\sigma_j} St(\psi) \sin \psi \, d\psi \, d\alpha \quad (20)$$

The function  $St(\psi) \sin \psi$  may be denoted  $F(\psi)$ . Its integral over a given cap with spherical radius  $\psi$ , denoted (Lambert and Darling, 1936)

$$\begin{aligned} \Phi(\psi) &= \int_0^\psi F(\psi) d\psi \\ &= \frac{1}{2} \left\{ 1 + 4 \sin \frac{\psi}{2} - \cos \psi - 6 \sin^3 \frac{\psi}{2} - \frac{7}{4} \sin^2 \psi - \right. \\ &\quad \left. - \frac{3}{2} \sin^2 \psi \log_e \left( \sin \frac{\psi}{2} + \sin^2 \frac{\psi}{2} \right) \right\} \end{aligned} \quad (21)$$

has been tabulated in (Lambert and Darling, 1936). These tables can be used approximately for the computations of the right hand side of equation (20). A computation program would be more simple and convenient because the integral of  $F(\psi)$  is an analytical expression, as long as the boundary lines of the considered vertical datum zone are a set of great circles. But the situation in general is more complicated for the boundary line of  $\sigma_j$  is usually irregular.

With the station P located in the datum zone  $Q_j$  (Fig. 3), furtheron with P supposed to be roughly in the center and the boundary line roughly a circle, equation (20) can be approximately rewritten as

$$\begin{aligned} IS_{PQ_j} &\approx \frac{1}{4\pi} \int_0^{\psi_j} d\alpha \int_0^{\psi_j} St(\psi) \sin \psi \, d\psi \\ &= \frac{1}{2} \int_0^{\psi_j} St(\psi) \sin \psi \, d\psi \\ &= \frac{1}{2} \Phi(\psi_j) \end{aligned} \quad (22)$$

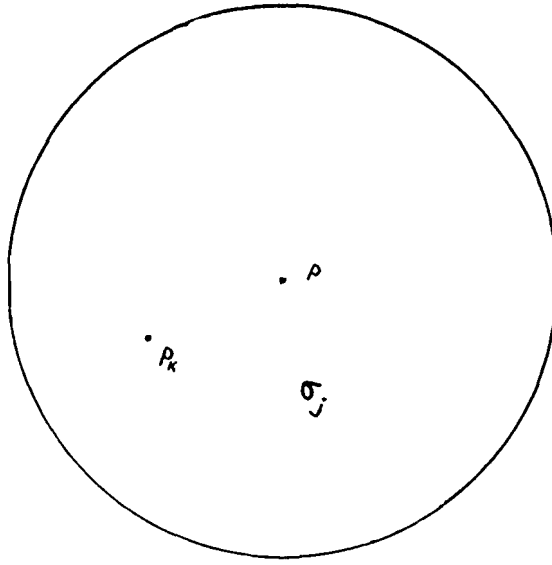


Figure 3: The positions of the stations P and P<sub>k</sub> and datum zone Q<sub>j</sub>.

On the other hand, with the station P not inside the datum zone Q<sub>j</sub> (Fig. 4), and with the actual boundary line of the datum zone approximated by an area indicated by the dashed lines, the value of IS<sub>PQ<sub>j</sub></sub> becomes

$$\begin{aligned}
 IS_{PQ_j} &\approx \frac{1}{4\pi} \int_{\alpha_1}^{\alpha_2} d\alpha \int_{\psi_1}^{\psi_2} St(\psi) \sin \psi \, d\psi \\
 &= \frac{1}{4\pi} (\alpha_2 - \alpha_1) \int_{\psi_1}^{\psi_2} F(\psi) d\psi \\
 &= \frac{\alpha_2 - \alpha_1}{4\pi} \{ \Phi(\psi_2) - \Phi(\psi_1) \} \quad . \quad (23)
 \end{aligned}$$

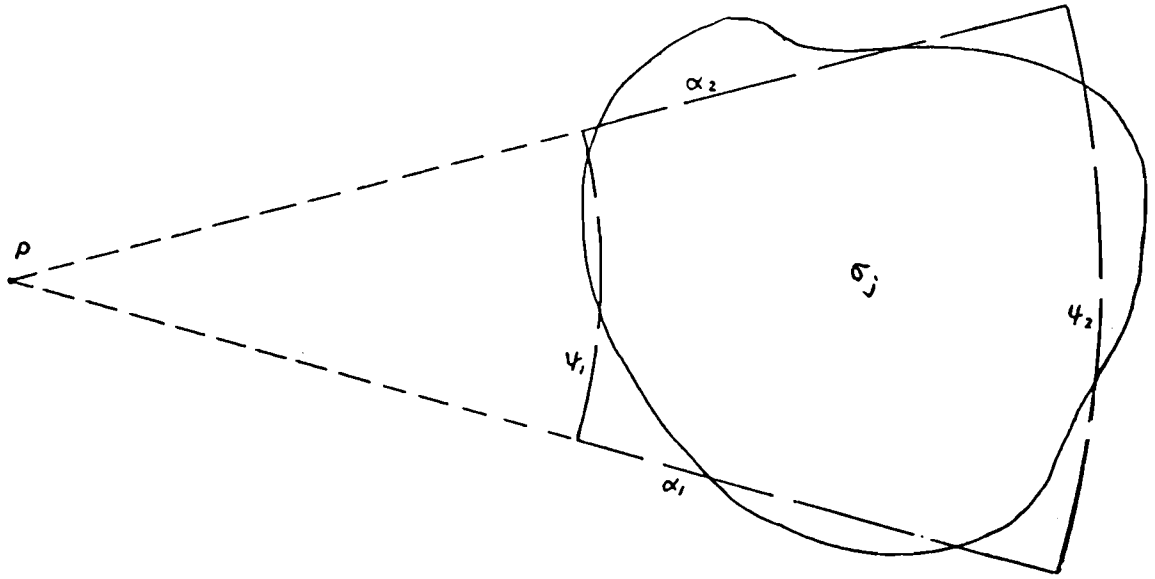


Figure 4: The position of station P and the datum zone  $\sigma_j$ .

It is clear that for an accurate value of  $IS_{PQ_j}$ , the combination of (22) and (23) is very useful, only if the  $\sigma_j$  is divided into a large number of small regular integral zones. In principle, other numerical integral techniques serve the purpose too. Because the boundary lines of the datum zones for an arbitrary space station generally are not great circles, one would have to make a lot of coordinate transformations and solve for  $\psi$  or  $\Delta\alpha$  by iteration if using equation (21) to compute a coefficient  $IS_{P_kQ_j}$ . Therefore, we determine the values of  $IS_{P_kQ_j}$  by a numerical integral method, gridding the datum zone into 30' \* 30' small elements.

### 3. Accuracy of the method.

The matrix form of the observation equations (15) is

$$Y = AX + \varepsilon \quad (24)$$

where Y is the observation vector with elements (14)

A is the design matrix whose elements are either  $-1$ ,  $(1 + 2IS_{P_kQ_i})$ , or  $2IS_{P_kQ_j}$ ,

X is the parameter vector with elements  $\frac{\Delta W_0}{\gamma}$ ,  $\frac{C_{Q_1 0}}{\gamma}$ ,  $i = 1, 2, \dots, I$ ,

$\varepsilon$  is the error vector of Y.

Assume that the expectation of  $\epsilon$  is equal to zero. The linear Gauss-Markov model is then

$$E(Y) = AX \tag{25a}$$

$$D(Y) = \Sigma_y \tag{25b}$$

where  $\Sigma_y$  is the variance-covariance matrix of  $Y$ , which can be computed by

$$\Sigma_y = \Sigma_h + \Sigma_H + \Sigma_N \tag{26}$$

if the observations  $h$ ,  $H$  and  $N$  related to  $S(\Delta g^{(j)})$  are assumed to be uncorrelated among each other. More details on  $\Sigma_h$ ,  $\Sigma_H$  and  $\Sigma_N$  are left for chapters 3 and 4.

By using a least squares adjustment, we have the parameter estimate

$$X = (A^T \Sigma_y^{-1} A)^{-1} A^T \Sigma_y^{-1} Y \tag{27}$$

and the variance-covariance matrix

$$D(X) = \Sigma_x = (A^T \Sigma_y^{-1} A)^{-1} = \begin{pmatrix} d_{00} & d_{01} & d_{02} & \dots & d_{0I} \\ d_{10} & d_{11} & d_{12} & \dots & d_{1I} \\ d_{20} & d_{21} & d_{22} & \dots & d_{2I} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{I0} & d_{I1} & d_{I2} & \vdots & d_{II} \end{pmatrix} \tag{28}$$

in which  $d_{00}$  is the precision of  $\Delta W_0/\gamma$ ,  $d_{0j}$  are the covariance values between  $\Delta W_0/\gamma$  and  $C_{Q_j 0}/\gamma$ , and  $d_{jj}$  are the precisions of  $C_{Q_j 0}/\gamma$ .



#### 4. Effects of errors of h, H and N on the estimated parameters.

The effects of errors of h, H and N on the estimated parameters can be investigated based on either the true errors of the observations or their variance-covariance matrix. Suppose that the observations h, H and N have the errors  $\epsilon_h$ ,  $\epsilon_H$  and  $\epsilon_N$  respectively, then their total effect formula is, from equation (27)

$$\epsilon_X = (A^T \Sigma_y^{-1} A)^{-1} A^T \Sigma_y^{-1} (\epsilon_h - \epsilon_H - \epsilon_N) \quad (29)$$

from which we can write the effect expressions of h, H and N on X, respectively:

$$\epsilon_{Xh} = (A^T \Sigma_y^{-1} A)^{-1} A^T \Sigma_y^{-1} \epsilon_h \quad (30)$$

$$\epsilon_{XH} = -(A^T \Sigma_y^{-1} A)^{-1} A^T \Sigma_y^{-1} \epsilon_H \quad (31)$$

$$\epsilon_{XN} = -(A^T \Sigma_y^{-1} A)^{-1} A^T \Sigma_y^{-1} \epsilon_N \quad (32)$$

In general cases, we do not know the size of the errors  $\epsilon_h$ ,  $\epsilon_H$  and  $\epsilon_N$ . Therefore, only in a simulation it is possible to use (30)~(32). The results will vary from one simulation to another. In other words, equations (30)~(32) are not computable in actual cases. Instead, we will discuss this problem based on their variance-covariance matrix. Application of the variance-covariance propagation law to equations (30)~(32) gives respectively

$$\Sigma_{Xh} = Q_x A^T \Sigma_y^{-1} \Sigma_h \Sigma_y^{-1} A Q_x \quad (33)$$

$$\Sigma_{XH} = Q_x A^T \Sigma_y^{-1} \Sigma_H \Sigma_y^{-1} A Q_x \quad (34)$$

$$\Sigma_{XN} = Q_x A^T \Sigma_y^{-1} \Sigma_N \Sigma_y^{-1} A Q_x \quad (35)$$

where

$$Q_x = (A^T \Sigma_y^{-1} A)^{-1}$$

The summation of the right hand sides of (33)~(35) yields

$$\begin{aligned} D(X) &= \Sigma_{Xh} + \Sigma_{XH} + \Sigma_{XN} \\ &= (A^T \Sigma_y^{-1} A)^{-1} \end{aligned} \quad (36)$$

We therefore can compute the percentages of the effects of h, H and N on  $\Delta W_0/\gamma$ ,

$$\rho_{Xh}(\Delta W_0) = \frac{d_{00}^{Xh}}{d_{00}^{00}} \quad (37)$$

$$\rho_{XH}(\Delta W_0) = \frac{d_{00}^{XH}}{d_{00}^{00}} \quad (38)$$

$$\rho_{XN}(\Delta W_0) = \frac{d_{00}^{XN}}{d_{00}^{00}} = 1 - \rho_{Xh}(\Delta W_0) - \rho_{XH}(\Delta W_0) \quad (39)$$

and the percentage of their effects on  $C_{Q_j0}/\gamma$ , respectively,

$$\rho_{Xh}(C_{j0}) = \frac{d_{jj}^{Xh}}{d_{jj}^{00}} \quad (40)$$

$$\rho_{XH}(C_{j0}) = \frac{d_{jj}^{XH}}{d_{jj}^{00}} \quad (41)$$

$$\rho_{XN}(C_{j0}) = \frac{d_{jj}^{XN}}{d_{jj}^{00}} = 1 - \rho_{Xh}(C_{j0}) - \rho_{XH}(C_{j0}) \quad (42)$$

where  $d_{00}^{Xh}$ ,  $d_{jj}^{Xh}$  are the diagonal elements of matrix  $\Sigma_{Xh}$ ,  $d_{00}^{XH}$ ,  $d_{jj}^{XH}$  the diagonal elements of matrix  $\Sigma_{XH}$ ,  $d_{00}^{XN}$  and  $d_{jj}^{XN}$  the diagonal elements of matrix  $\Sigma_{XN}$ .

### 5. Reliability of the method.

The concept of reliability in geodesy was first presented by Baarda (1968). The reliability measures of geodetic networks consist of the internal reliability and external reliability. The internal reliability expresses the robustness of geodetic networks against gross errors, and the external reliability the distortion degree of the geodetic coordinates due to undetected gross errors. In the case of the vertical datum connection, we are interested in the robustness of the method against gross errors in the observations  $h$ ,  $H$  and  $N$ , and the distortion of the potential differences.

The test statistic is, based on the data snooping theory,

$$W = \frac{v_i}{\sigma_{0v} \sqrt{q_i}} \quad (43)$$

where  $v_i$  is the correction of observation  $Y_i$  and  $q_i$  the  $i$ -th main diagonal element of  $Q_v$ .

Since

$$E(V) = (AQ_X A^T P - I) \Delta Y \quad (44)$$

$$Q_v = P^{-1} - AQ_X A^T \quad (45)$$

where  $\Delta Y$  denotes an outlier vector and

$$P = \sigma_0^2 \Sigma_y^{-1} \quad (46)$$

Then

$$E(V) = -Q_v P \Delta Y \quad (47)$$

Thus the bound value of undetected gross error is

$$\overline{\Delta y_i} = K_0 \sigma_0 / P_i \sqrt{q_i} \quad (48)$$

if the matrix P is diagonal, where the parameter  $K_0$  is selected with the significant level  $\alpha$  and power  $\beta$  and

$$\overline{\Delta y}_i = K_0 \sigma_0 \sqrt{q}_i / \sum_{j=1}^n q_{ij} P_{ij} \quad (49)$$

if the matrix P is nondiagonal.

There exist several other test statistics for the purpose of gross error detection, which will lead to slightly different bounds of detectable gross errors.

One of the measures of internal reliability is the so-called redundant quantity, that is

$$r_i = (Q_v \Sigma_y^{-1}) \quad (50)$$

and the effect of undetected gross errors on X (external reliability measure) is

$$\overline{\Delta X}_i = Q_x A^T \Sigma_y^{-1} \overline{\Delta Y}_i \quad (51)$$

or

$$\overline{\Delta X} = Q_x A^T \Sigma_y^{-1} \overline{\Delta Y} = \Sigma \overline{\Delta X}_i \quad (52)$$

where

$$\overline{\Delta Y}_i = (0, 0, \dots, \overline{\Delta y}_i, 0, \dots, 0)^T \quad (53)$$

$$\overline{\Delta Y} = \Sigma \overline{\Delta Y}_i \quad (54)$$

## 6. Detectability of the method.

As indicated in the previous sections, the potential differences  $\Delta W_0$  and  $C_{Q_j 0}$  can be computed by a least squares adjustment for the purpose of the vertical datum connection. Theoretically speaking, it is not a problem, if the

basic requirements are satisfied. Problems here are whether or not the potential differences are significantly different from zero, how we pick up the signals from the measurements. This shall be investigated through hypothesis testings in the following.

The null hypothesis first of all is

$$H_0: E(X) = 0 \quad , \quad (55)$$

then the derived multidimensional  $\chi^2$  test statistic is after the null hypothesis,

$$\chi^2 = X^T \Sigma_X^{-1} X \sim \chi^2(I+1) \quad . \quad (56)$$

When a significant level  $\alpha$  is given, we can test the null hypothesis using (56). If the null hypothesis is accepted with the confidence level  $(1-\alpha)$ , it will mean that the potential differences are globally undetected. Some aspects of the method (e.g. the requirements of the method) should be improved. On the other hand, if the null hypothesis is rejected, the parameters can be globally detectable. Further tests are needed. The corresponding null hypothesis is then,

$$H_0: E(x_i) = 0 \quad (57)$$

the statistic is

$$u_i = \frac{x_i}{\sqrt{d_{ii}}} \quad . \quad (58)$$

Thus the measure of detectability of this method is as follows

$$\bar{x}_i = k_1 \sqrt{d_{ii}} \quad (59)$$

where  $k_1$  is a constant based on the significant level  $\alpha$ ,  $d_{ii}$  is the  $i$ -th main diagonal element of the matrix  $\Sigma_X$ .

## Prior precision matrices of geometrical and normal heights.

### 1. Determination of geometric heights and their accuracy.

A set of precise station coordinates is absolutely necessary for the vertical datum connection. It is obtained by combination of very long baseline interferometry (VLBI), satellite laser ranging (SLR) and the use of the global positioning system (GPS). The VLBI technique began in Radio Astronomy in the late 1960's. The observations are time delay and time delay rate from radio sources in deep space, from which baselines and orientations between receiver stations can be derived. The VLBI coordinate system is therefore an inertial (quasi) celestial reference frame (Brouwer, 1985; Mueller, 1989). In principle, the VLBI technique is relative. At present time, there are more than 22 VLBI stations around the world for geodetic purposes (Clark et al., p. 151, 1989):

Westford, MA;	Fairbanks, AK;	Mojave, CA;
Richmond, FL;	Wettzell, FRG;	Haystack, MA;
Medicina, Italy;	Owens Valley, CA;	Hartebeesthoek, S.Afr.;
Madrid, Spain;	Kashima, Japan;	Tidbinbilla, Australia;
Ft. Davis, TX;	Vandenberg AFB, CA;	Maryland Point, MD;
Shanghai, China;	Goldstone, CA;	Onsala, Sweden;
Hat Creek, CA;	Kauai, Hawaii;	Roi Namor, Kwajalein;
Pie Town, NW.		

There are some other VLBI stations to be put into use or under development in many countries. The quality of VLBI measurements can be assessed from the repeatability in baseline lengths. The present repeatability level is 1~2 cm within a distance of 4000 km (Fig. 5). The accuracy of VLBI networks can therefore be expected to reach easily a few centimeters or better.

Satellite laser ranging uses short pulse lasers to make range measurements between ground stations and retro-reflectors on artificial satellites. The observable is the travel time of the laser pulses. Because SLR measurements involve orbit dynamics, it is possible to determine absolute positions of SLR stations with respect to the earth's center of mass. Most of the measurements

now available are from the laser geodynamics satellite (Lageos). There are approximately 30 fixed SLR stations worldwide. Additionally, there are some mobile SLR sites. Though not all of them provide data each year, they could be very useful in research of vertical datum connection. The density of SLR coverage is different from place to place. Most of them being located in Europe and the USA. Japan, China and USSR, etc. are developing their own SLR systems. Figure 6 gives fixed and mobile SLR stations in Europe and in the Mediterranean. For the Topex/Poseidon mission, starting in 1992 an almost global coverage by SLR is envisaged. Worldwide SLR networks form a control coordinate frame.

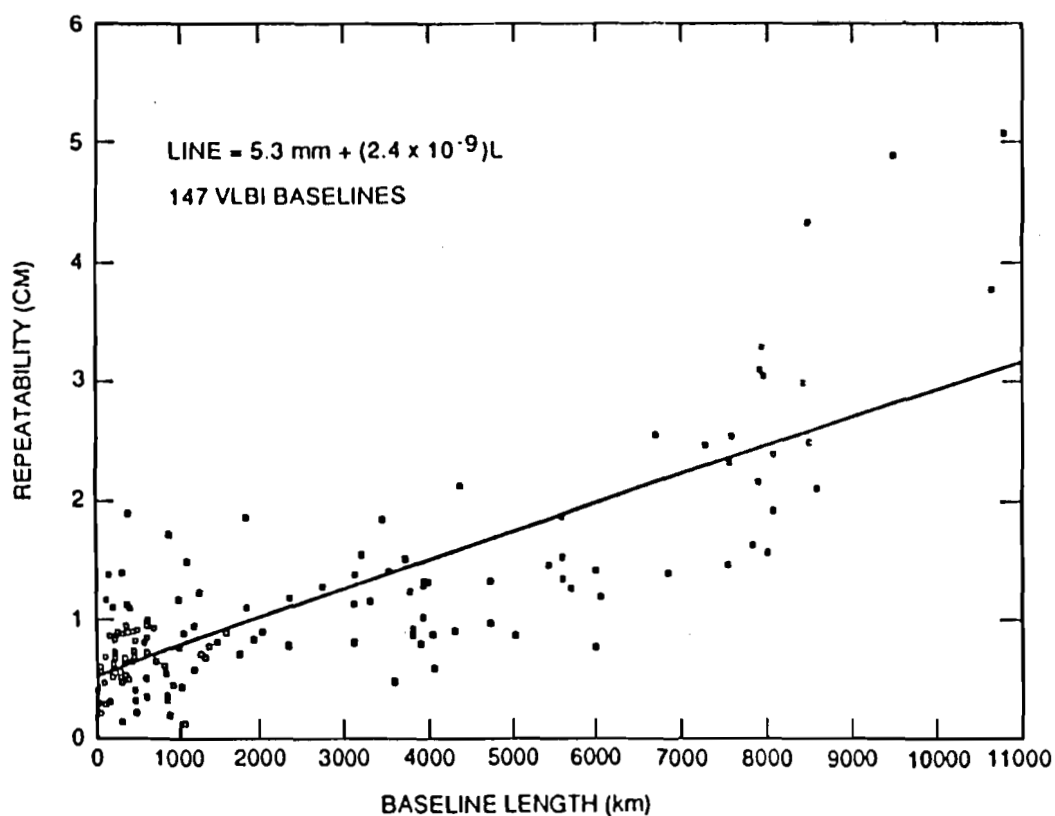


Figure 5: VLBI Baseline length repeatability (taken from Melbourne and Reigber, 1989).

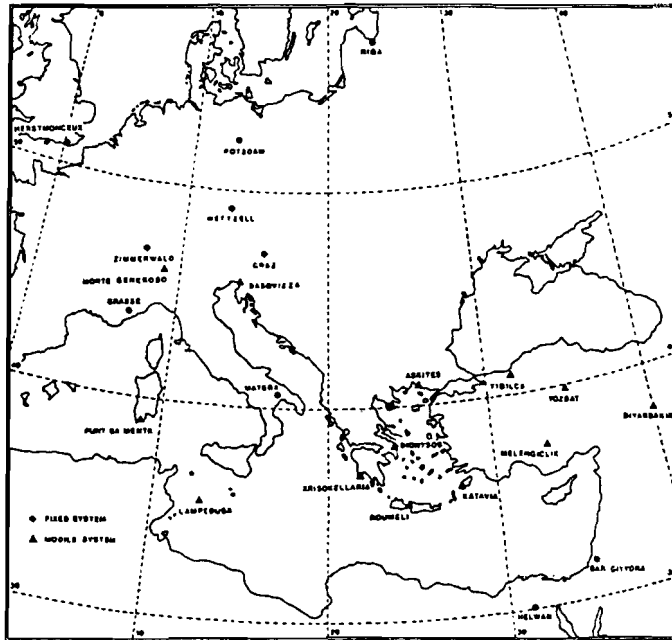


Figure 6: Fixed and mobile SLR stations in Europe and in the Mediterranean (from Wilson 1989).

SLR is a kind of technique which can provide us with precise geocentric positions of stations. The accuracy of station positions has reached 2 cm with the newer ranging systems, and 3-6 cm with older systems (Melbourne and Reigber, 1989, p. 136). Significant progress in SLR can be expected from the Lageos II program, established by the National Aeronautics and Space Administration (NASA) and the space agency of Italy (ASI). It may be expected that the accuracy of SLR networks will be greatly improved, when the Lageos II program and/or other new satellites and stations are put into effect. For more details on the recent progress on SLR, see (Melbourne and Reigber, 1989; Zerbini, 1989).

In addition, there are some worldwide GPS tracking networks. CIGNET tracking sites and DMA monitor tracking sites are plotted in Figure 7. CIGNET stands for the cooperative international GPS network, and DMA for the U.S. defense mapping agency. The former is designed for civilian use and the latter for or mainly for military use. A few centimeters of accuracy can be expected for these networks when combined with VLBI. Other networks include the French



DORIS tracking network whose about fifty points will be distributed around the world.

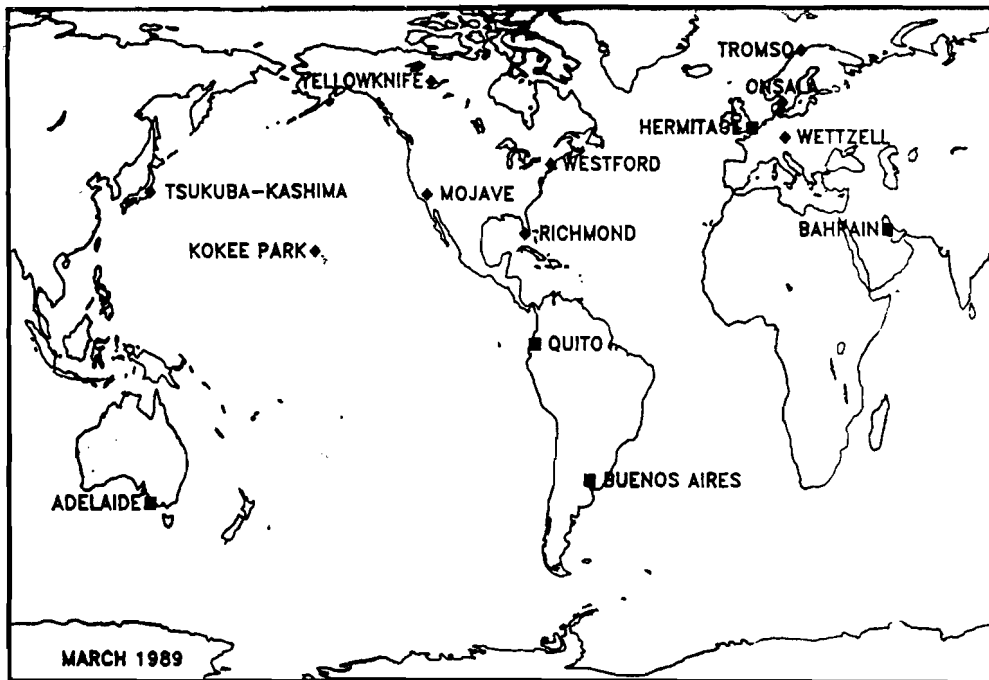


Figure 7: The CIGNET and DMA tracking sites from (Mader et al., 1989).

The combination of SLR, VLBI and GPS tracking networks should improve the distribution of space stations on the one hand, and the accuracy on the other hand. It is reported that more emphasis will be placed on the development of 1 mm system accuracy for SLR and VLBI systems, even at the expense of increased costs. This information is very welcome and should be valuable for the purpose of vertical datum connection.

## 2. Determination of orthometric heights and their accuracy.

The vertical datum connection proposed by Rummel and Teunissen (1988) needs orthometric heights at each station. Orthometric heights are heights between a space station and a fundamental benchmark, both of which located in the same datum zone, obtained by levelling operations and gravity. Levelling measurements are very precise but not reliable. Levelling is time consuming

and work intensive. Weather factors and conditions of levelling lines will affect the speed of measurements and their accuracy. An accuracy formula can be generally written as  $\sigma\sqrt{L}$ , where  $L$  is the distance of a levelling line,  $\sigma$  is a scale factor, depending on the used instruments, measurement conditions and skills of operators. In other words, accuracy of levelling is proportional to the square root of the distance  $L$ . The scale  $\sigma$  is reported in precise levelling differently from person to person. An acceptable value to be used in this simulation is 1 mm/km (Remmer, 1989; personal communication). As we know, levelling measurements are affected not only by random errors but some systematic errors too. The accuracy measure here does not take the systematic errors into consideration.

#### The geoid heights and their accuracy.

There are two typical ways to compute the geoid heights and their accuracy: the integral method and spherical harmonic expansion. The Stokes' formula for computation of geoid heights reads:

$$N = \frac{R}{4\pi\gamma} \int_{\sigma} \text{St}(\psi) \Delta g \, d\sigma \quad (60)$$

where  $\psi$  is the spherical distance between the computed point and integration point,  $\Delta g$  is gravity anomaly, which belongs to a certain datum zone.

Suppose that  $\Delta g$  contains errors such as gravity reduction error and gravity measurement error, denoted by  $\epsilon_{\Delta g}$ , the error of  $N$  is then:

$$\epsilon_N = \frac{R}{4\pi\gamma} \int_{\sigma} \text{St}(\psi) \epsilon_{\Delta g} \, d\sigma \quad (61)$$

from which we can obtain the variance formula of  $N$  (Heiskanen and Moritz, 1967; Strang van Hees, 1986):

$$\Sigma_N = \left(\frac{R}{4\pi\gamma}\right)^2 \int_{\sigma_i} \int_{\sigma_k} \text{St}(\psi_{Pi}) \text{St}(\psi_{Pk}) \Sigma_{\Delta g_i \Delta g_k} \, d\sigma_i \, d\sigma_k \quad (62)$$

where  $\Sigma_N$  is the variance of the geoid in station  $P$ ,  $\Sigma_{\Delta g_i \Delta g_k}$  the variance-covariance of gravity anomalies.

Following Strang van Hees (1986), expression (62) can be transformed into

$$\Sigma_N = \frac{1}{(4\pi\gamma)^2} \int_{\sigma_i} (S_{Pi}^2 E_i - F_{Pi}) d\sigma_i \quad (63)$$

$$E_i = R^2 \int_{\sigma_k} \Sigma \Delta g_i \Delta g_k d\sigma_k \quad (64)$$

$$F_{Pi} = \frac{1}{2} R^2 \int_{\sigma_k} (\text{St}(\psi_{Pi}) - \text{St}(\psi_{Pk}))^2 d\sigma_k \quad (65)$$

If  $F_{Pi}$ , being a small quantity, is neglected we have

$$\Sigma_N = \frac{1}{(4\pi\gamma)^2} \int_{\sigma_i} \text{St}^2(\psi_{Pi}) E_i d\sigma_i \quad (66)$$

If the integration point  $i$  is close to the station  $P$ , the integral will become infinite. We therefore divide (66) into two parts

$$\Sigma_N = \frac{1}{(4\pi\gamma)^2} \int_{\sigma_i - \delta} \text{St}^2(\psi_{Pi}) E_i d\sigma_i + \delta \Sigma_N \quad (67)$$

The influence of the immediate surrounding  $\delta$  can approximately be expressed as (Strang van Hees, 1986)

$$\Delta \Sigma = \left\{ \frac{r_0 \text{ (km)}}{981} \right\}^2 \sigma_{\Delta g}^2 \quad (68)$$

where  $\delta$  is a small circle zone with radius  $r_0$  centered at station  $P$ .

The first term on the right hand side of equation (67) can be computed by either direct numerical integration or frequency domain methods. For more details, readers are referred to (ibid).

The covariance function of geoid heights between stations  $P$  and  $Q$  is written from (61)

$$\begin{aligned}
\Sigma_{N_P N_Q} &= E(\epsilon_P, \epsilon_Q) \\
&= E\left\{ \frac{R}{4\pi\gamma} \int_{\sigma_i} \text{St}(\psi_{P1}) \epsilon_{\Delta g} d\sigma_i, \frac{R}{4\pi\gamma} \int_{\sigma_k} \text{St}(\psi_{Qk}) \epsilon_{\Delta g} d\sigma_k \right\} \\
&= \frac{R^2}{(4\pi\gamma)^2} \int_{\sigma_i} \int_{\sigma_k} \text{St}(\psi_{P1}) \text{St}(\psi_{Qk}) \Sigma_{\Delta g_i \Delta g_k} d\sigma_i d\sigma_k \quad . \quad (69)
\end{aligned}$$

The difference of geoid heights is

$$N_{PQ} = N_Q - N_P \quad (70)$$

therefore

$$\Sigma_{PQ} = \Sigma_{N_P} + \Sigma_{N_Q} - 2\Sigma_{N_P N_Q} \quad (71)$$

Thus, we have

$$\Sigma_{N_P N_Q} = \frac{1}{2} (\Sigma_{N_P} + \Sigma_{N_Q} - \Sigma_{PQ}) \quad (72)$$

The terms of  $\Sigma_{N_P}$  and  $\Sigma_{N_Q}$  are computed according to (67).  $\Sigma_{N_{PQ}}$  is obtained in the following manner

$$\Sigma_{N_{PQ}} = \frac{1}{(4\pi\gamma)^2} \int_{\sigma_i} \{ [\text{St}(\psi_{P1}) - \text{St}(\psi_{Q1})]^2 E_i - F_{PQi} \} d\sigma_i \quad (73)$$

$$F_{PQi} = \frac{1}{2} R^2 \int_{\sigma_k} [\text{St}(\psi_{P1}) - \text{St}(\psi_{Q1}) - \text{St}(\psi_{Pk}) + \text{St}(\psi_{Qk})]^2 \Sigma_{\Delta g_i \Delta g_k} d\sigma_k \quad (74)$$

If  $F_{PQ_i}$  is small, (73) will become (Strang van Hees, 1986)

$$\Sigma_{N_{PQ}} = \frac{1}{(4\pi\gamma)^2} \int_{\sigma_i} [\text{St}(\psi_{P1}) - \text{St}(\psi_{Q1})]^2 E_i d\sigma_i \quad (75)$$

Using equations (67) and (75), we can compute the variance-covariance function of geoid heights by numerical integration techniques.

The alternative method to compute the geoid is to use a spherical harmonic expansion of the gravitational potential:

$$W(r, \theta, \lambda) = \frac{GM}{r} \left[ 1 + \sum_{\ell=2}^{\infty} \left( \frac{a}{r} \right)^{\ell} \sum_{m=0}^{\ell} (C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda) P_{\ell m}(\cos \theta) \right] + \frac{1}{2} \omega^2 r^2 \cos^2 \theta \quad (76)$$

where  $r, \theta, \lambda$  are the polar coordinates of a point at which  $W$  is to be determined.

$a$  is the earth's semi-major axis associated with the potential coefficients,

$C_{\ell m}, S_{\ell m}$  are fully normalized potential coefficients,

$P_{\ell m}$  are fully normalized Legendre functions.

Assume that as a reference potential a rotating, level ellipsoid is chosen with the same  $GM$  and centrifugal potential as the actual earth (Rapp and Cruz, 1986). We define

$$\Delta C_{\ell m} = C_{\ell m} - C_{\ell m}^{\text{ref}} = C_{\ell m} + \begin{cases} \frac{J_{\ell}}{\sqrt{2\ell+1}} & \text{for } \ell = 2, 4, 6, \text{ and } 8 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta S_{\ell m} = S_{\ell m} \quad (77)$$

Then the disturbing potential is

$$T = W - U$$

$$= \frac{GM}{r} \sum_{\ell=2}^{\infty} \left( \frac{a}{r} \right)^{\ell} \sum_{m=0}^{\ell} (\Delta C_{\ell m} \cos m\lambda + \Delta S_{\ell m} \sin m\lambda) P_{\ell m}(\cos \theta) \quad (78)$$

There are several global geopotential models available. The maximum degree ranges from 20 to 360 depending on whether they are solely based on the analysis of satellite orbits, or combinations with satellite altimetry data and/or terrestrial mean gravity anomalies. Rapp (1981) developed a 180 degree gravitational field by combining SEASAT altimeter data, terrestrial  $1^{\circ} \times 1^{\circ}$  gravity anomalies and some prior potential coefficients. He estimated not only the values of the potential coefficients but also their accuracy. The errors of the coefficients reach 100% above degree 120. Rapp and Cruz (1986a,b) computed spherical harmonic expansions even up to degree and order 250 and 360. The accuracy results seem to be not significantly different from the former one, and only the accuracy of solution OSU 86D is computed. A recent earth gravitational model, based on only satellite orbit analysis, is GEM-T1 with its variance-covariance matrix of the estimated coefficients given too (Marsh et al., 1988; Lerch et al., 1988). Because we have a finite number of  $\Delta C_{\ell m}$  and  $\Delta S_{\ell m}$  values (78) should be rewritten as

$$T = \frac{GM}{r} \sum_{\ell=2}^{N_0} \left(\frac{a}{r}\right)^{\ell} \sum_{m=0}^{\ell} (\Delta C_{\ell m} \cos m\lambda + \Delta S_{\ell m} \sin m\lambda) P_{\ell m}(\cos \theta) \\ + \frac{GM}{r} \sum_{\ell=N_0+1}^{\infty} \left(\frac{a}{r}\right)^{\ell} \sum_{m=0}^{\ell} (C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda) P_{\ell m}(\cos \theta). \quad (79)$$

When using a finite degree and order gravitational potential model to compute T, we find for the error

$$\varepsilon_T = \frac{GM}{r} \sum_{\ell=2}^{N_0} \left(\frac{a}{r}\right)^{\ell} \sum_{m=0}^{\ell} (\varepsilon_{C_{\ell m}} \cos m\lambda + \varepsilon_{S_{\ell m}} \sin m\lambda) P_{\ell m}(\cos \theta) \\ + \frac{GM}{r} \sum_{\ell=N_0+1}^{\infty} \left(\frac{a}{r}\right)^{\ell} \sum_{m=0}^{\ell} (C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda) P_{\ell m}(\cos \theta). \quad (80)$$

Suppose that T' is the disturbing potential at point Q. Then the covariance of T and T' is for uncorrelated errors in the potential coefficients

$$\begin{aligned} \Sigma_{TT'} &= \text{cov}(T, T') \\ &= \frac{(GM)^2}{r r'} \sum_{\ell=2}^{N_0} \left( \frac{a^2}{r r'} \right)^\ell \varepsilon_\ell^2 P_\ell^2(\cos \psi) + \frac{(GM)^2}{r r'} \sum_{\ell=N_0+1}^{\infty} \left( \frac{a^2}{r r'} \right)^\ell \sigma_\ell^2 P_\ell^2(\cos \psi) \end{aligned} \quad (81)$$

where

$$\varepsilon_\ell^2 = \sum_{m=0}^{\ell} (\sigma_{C_{\ell m}}^2 + \sigma_{S_{\ell m}}^2) \quad (82)$$

Thereby  $\sigma_{C_{\ell m}}^2$  and  $\sigma_{S_{\ell m}}^2$  are the estimated error variances of the potential coefficients and  $\sigma_\ell$  is the expected root mean square variation of the potential coefficient per degree

$$\sigma_\ell^2 = \sum_{m=0}^{\ell} (C_{\ell m}^2 + S_{\ell m}^2)$$

Equation (82) is used to compute the accuracy of the estimated potential T. But it is not possible to obtain infinite numbers of values of  $\sigma_\ell^2$ . Generally, model values of  $\sigma_\ell^2$  are used. The  $\sigma_\ell^2$  are related to anomaly degree variance  $c_\ell$  model values by:

$$c_\ell = \gamma^2 (\ell-1)^2 \sigma_\ell^2 \quad (83)$$

Tscherning and Rapp (1974) discussed a number of anomaly degree variance models. Moritz suggested a two component global model:

$$c_\ell = \alpha_1 \frac{\ell-1}{\ell+A} s_1^{\ell+2} + \alpha_2 \frac{\ell-1}{(\ell-2)(\ell+B)} s_2^{\ell+2} \quad (84)$$

for which the parameters were estimated by Rapp (1979).

Using equation (81), we can obtain the error covariance function of geoid heights

$$\Sigma_{N_P N_Q} = \frac{1}{\gamma\gamma'} \Sigma_{TT},$$

$$= \frac{(GM)^2}{\gamma\gamma' r r'} \left\{ \sum_{\ell=2}^{N_0} \left( \frac{a^2}{r r'} \right)^\ell \varepsilon_{\ell}^2 P_{\ell}(\cos \psi) + \sum_{\ell=N_0+1}^{\infty} \left( \frac{a^2}{r r'} \right)^\ell \sigma_{\ell}^2 P_{\ell}(\cos \psi) \right\}. \quad (85)$$

The variance value of the geoid is therefore

$$\Sigma_N = \frac{(GM)^2}{\gamma^2 r^2} \left\{ \sum_{\ell=2}^{N_0} \left( \frac{a^2}{r^2} \right)^\ell \varepsilon_{\ell}^2 + \sum_{\ell=N_0+1}^{\infty} \left( \frac{a^2}{r^2} \right)^\ell \sigma_{\ell}^2 \right\}. \quad (86)$$

It should be noted here again that the formulae derived above do not take into consideration the correlation information between the geopotential coefficients. The general computation formula therefore should be

$$\Sigma_{N_P N_Q} = F_P \Sigma_{CS} F_Q^T + \frac{(GM)^2}{\gamma\gamma' r r'} \sum_{\ell=N_0+1}^{\infty} \left( \frac{a^2}{r r'} \right)^\ell \sigma_{\ell}^2 P_{\ell}(\cos \psi) \quad (87)$$

where  $F_P$  and  $F_Q$  is the coefficient vector that relates the  $C_{\ell m}$  and  $S_{\ell m}$  to the geoid heights at points P and Q, respectively.  $\Sigma_{CS}$  is the variance-covariance matrix of  $C_{\ell m}$  and  $S_{\ell m}$ .

### Simulation results.

In this simulation, the world is divided into seven datum zones. In principle, each continent forms its own vertical datum zone. There are two datum zones for the European and Asian continent. Since the North Sea is of special interest, England is thought of as a separate datum zone. There are 63 space stations introduced around the world for the purpose of datum connection. The division of the datum zones and the distribution of the space stations are shown in Figure 8. The number of space stations is different from one datum zone to another. Table 1 lists the distribution of the space stations. It can be seen from Table 1 that the space stations are mainly distributed over North



America and Europe, which amounts to nearly two third of the total stations. The accuracy of space stations is assumed to be 5 cm. Accuracy of orthometric heights is derived from levelling operations. Its standard deviation of unit weight is taken to be 1 mm/km (Remmer, 1989). The benchmarks are also shown in Figure 8.

Table 1: Distribution of Space Stations.

Zone	0	1	2	3	4	5	6
Name	Europe	England	Africa	Asia	Australia	N. America	S. America
Stations	18	3	6	4	2	23	7

There are error covariance sets of three gravitational fields used in the test computations. The first one is that of Rapp-81 coefficient set. The second is based on the publication (Torge et al., 1989). The last one is that of the GEM-T1 field.

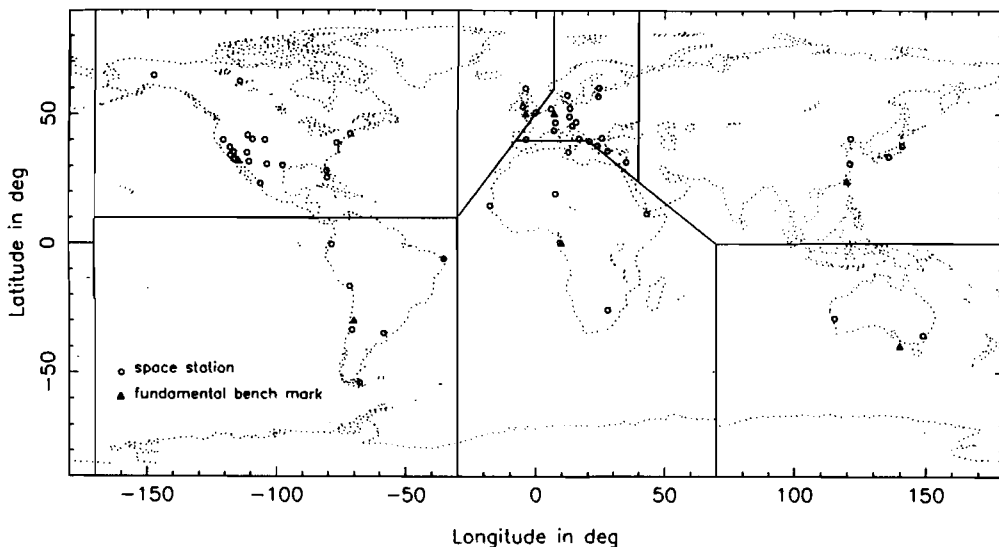


Figure 8: Space stations and datum zones with fundamental benchmarks.

1. Preliminary results using Rapp-81 model.

The accuracy of geoid heights  $N$  is computed using equations (84) and (86), neglecting the correlation terms of geopotential coefficients, where  $\alpha_1 = 3.405 \text{ mgal}^2$ ,  $\alpha_2 = 140.03 \text{ mgal}^2$ ,  $A = 1$ ,  $B = 2$  and  $N_0 = 180$ . The standard deviation of the geoid is about 1.4 meter. Table 2 gives the accuracy estimation of parameters  $\Delta W_0$  and  $C_{Q_1 0}$ . Note that the parameters in fact should be  $\Delta W_0/\gamma$  and  $C_{Q_1 0}/\gamma$ . But for the conciseness of the tables, we still use in the following the notations of  $\Delta W_0$  and  $C_{Q_1 0}$ . This should not be confusing. It can be seen from the table that the accuracy is generally far from being satisfactory. This is due to the poor accuracy of the geoid, because the estimated parameters strongly depend on the geoid in this case (Table 3). On the other hand, because the vertical datum connection should be a problem of determination of relative quantities such as  $C_{Q_1 0}$ , the weak correlation of the geoid in this case may be responsible for the results (omitting the correlations of the geopotential coefficients when computing the variance-covariance matrix of the geoid). It also can be seen from Table 2 that the accuracy estimation is related to some extent to the number of space stations in a datum zone. If the number of space stations increases, the datum connection to datum zone 0 is of better accuracy. For instance, the accuracy of the connection of the North American datum zone to the European one is better than others. Comparing the accuracy of England with that of Australia, the accuracy may also have some relation to the distribution of space stations within one vertical datum zone. We conclude that the accuracy estimation of vertical datum connection will depend on the actual accuracy of the geoid, the correlation length, the number of space stations and their accuracy and distribution, and finally on the distance of datum points from space stations.

Table 2: Accuracy estimation of the parameters (cm).

Zone	0	1	2	3	4	5	6
Param.	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
Accuracy	32.02	74.93	46.46	45.61	65.03	34.87	54.77

Table 3: Effects of errors on the estimated parameters (%).

Param.	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
H	0.08	0.05	0.12	0.10	0.11	0.10	0.14
h	0.10	0.10	0.11	0.11	0.11	0.10	0.11
N	99.82	99.85	99.77	99.79	99.78	99.80	99.75

Table 4 lists the undetected gross error bounds of the observations for  $\alpha = 0.05$ ,  $\beta = 0.80$ . It is obvious that the method in this case has weak reliability. The reason may again be directly related to the weak correlation of the geoid. With the improvement of gravitational fields and inclusion of the correlation among the coefficients, the situation may improve. Further investigations are needed.

When using the Rapp-81 gravitational model, with no correlation information, the detectability of the method is not good (Table 5). Again, the result shows some relation to the number of space stations and their distribution.

Now we will take a look at the external reliability of the method. Table 6 gives the external reliability values. Comparing columns 4 and 7 with others, it can be seen that the smaller the number of space stations in a datum zone, the worse the reliability is. Additional space stations should be helpful in improving the external reliability. An outlier, e.g. caused by a wrong connection, would have the most serious effect on the parameters of its own datum zone. It seems from column 3 of Table 6 that outliers in the observations will have a cumulative effect on the parameter  $\Delta W_0$ . This is due to the positive undetected gross error bounds. They are equivalent to a systematic bias in the observations. On the other hand, the coefficients of parameter  $\Delta W_0$  in all observation equations are -1, which means that this systematic bias will be absorbed in  $\Delta W_0$ .

Table 4: Undetected Gross Error Bounds (cm).

419.4600	406.9836	410.8769	416.7769	405.5856	392.3233	403.7178
415.1195	409.1365	404.4953	423.4323	422.2618	420.4619	417.7337
406.0411	413.7830	412.8762	423.0321	437.1161	432.6139	467.0324
409.7197	521.3350	395.5798	420.6921	416.9847	417.4384	411.3092
409.5715	424.8413	400.6016	405.8873	410.1068	461.6329	405.0199
467.2394	496.2621	460.8895	537.5300	446.9145	400.3593	427.6280
397.3408	403.1771	437.6098	441.7659	451.6982	418.7378	417.4137
455.8443	426.4463	464.4016	416.9977	471.4726	421.4405	423.5813
445.5869	443.6262	423.0482	414.0756	456.4157	457.2131	438.5298

Table 5: Detectability of the method (cm).

Parameters	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
Detectability	62.75	146.86	91.07	89.39	127.46	68.34	107.34

## 2. Simulation of a tailored gravitational model.

Based on (Torge et al., 1989), we make the following assumptions on the relative accuracy of the geoid:

$\sigma(\Delta N) =$	5 cm	$S \leq 100$ km
	13 cm	$100 < S \leq 1000$ km
	20 cm	$1000 < S \leq 1000$ km
	94 cm	$3000 < S \leq 7000$ km
	196 cm	$7000 < S$

It should be pointed out that the assumptions may not apply to all parts of the world. But at least they should hold true in Europe, because error sources of GPS and levelling seem not to exist here, if we consider the comparison results given in (Torge et al., 1989).

Table 6: External reliability values of the method.

OBS.	ZONE	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
1	5	-5.14	-3.04	-2.69	-2.88	-5.60	11.12	-1.94
17	0	-18.24	-36.41	-20.39	-17.32	-15.30	-19.07	-21.32
18	0	-22.67	-21.75	-26.69	-19.32	-24.29	-20.79	-23.91
19	2	-5.17	8.44	35.49	-1.78	19.76	-6.32	9.40
21	3	-20.28	2.65	-7.02	52.20	-2.27	-6.32	-27.37
22	0	-13.87	-15.69	-15.24	-13.21	-13.29	-13.49	-14.83
23	4	8.50	-16.99	18.45	33.59	129.66	-1.67	17.70
25	5	-7.26	-4.71	-2.66	-1.87	-9.48	11.12	-2.77
26	0	-19.28	-24.62	-19.06	-18.71	-21.27	-17.73	-17.75
33	0	-13.98	-19.28	-14.49	-13.25	-11.74	-14.28	-15.58
34	3	-20.18	2.36	-2.58	49.84	-2.11	-9.21	-32.26
36	1	5.93	107.47	9.54	9.26	1.85	11.67	9.89
37	1	6.98	128.25	8.82	10.27	-0.75	14.31	11.75
38	1	10.04	106.20	14.98	12.36	4.54	15.40	15.53
39	4	5.42	-15.60	25.97	43.08	139.30	-9.09	4.46
40	6	4.80	0.15	5.10	4.78	23.91	16.14	52.44
47	6	6.02	17.29	13.97	-17.09	-11.09	20.13	57.82
50	2	-7.56	8.28	43.74	0.80	-11.65	0.13	-0.47
52	3	-20.80	1.58	-0.69	51.20	-0.98	-10.63	-34.76
54	3	-21.27	0.87	-6.06	54.27	-1.18	-8.03	-29.30
57	6	2.01	4.51	0.60	2.62	6.66	16.29	48.31
61	2	-4.99	3.07	50.10	3.80	26.60	-6.56	8.89
SUM		-418.98	42.99	32.03	44.44	66.68	5.65	20.10

Table 7 gives the accuracy values of the parameters. Compared with Table 2, it is clear that the accuracy of the shift parameter  $\Delta W_0$  becomes relatively seen poorer, because the strongly correlated geoid error decreases the importance of the number of space stations. The strong correlation will imply some linear relation among the observations. Therefore, a new observation can be well derived from old ones. In other words, the new observation contributes less information to the parameters. On the other hand, the parameters of vertical datum connection are of good accuracy. This is due to the good relative accuracy of the geoid, though the absolute accuracy of the geoid remains the same. The following is the theoretical explanation of the problem.

Table 7: Accuracy estimation of the parameters (cm).

Zone	0	1	2	3	4	5	6
Param.	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
Accuracy	144.53	3.22	2.34	2.25	3.31	1.58	2.89

Table 8: Effects of errors on the estimated parameters (%).

Param.	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
H	0.005	18.03	30.64	31.94	32.63	27.43	34.80
h	0.003	54.65	46.24	45.37	44.91	48.38	43.46
N	99.992	27.32	23.12	22.69	22.46	24.19	21.74

We take again the previous example with 11 stations and 9 vertical datums. The observation equations have been shown in Figure 2. Rearranging the observation equations by subtracting  $Y_1$  from  $Y_i$  ( $i \neq 1$ ), we have the new observation equations shown in Figure 9. The matrix form is as follow

$$Y_I = -\Delta W_0 / \gamma + A_1 C + \varepsilon_1 \quad (88)$$

$$Y_{II} = A C + \varepsilon_2 \quad (89)$$

where

$$Y_I = h_1 - H_1 - N_1 = Y_1 \quad (90)$$

$$A_1 = (C_{11}, C_{12}, \dots, C_{18}) \quad (91)$$

$$Y_{II} = (Y'_2, Y'_3, \dots, Y'_{11})^T \quad (92)$$

$$Y'_i = Y_i - Y_1 \quad (93)$$

and the matrix A is determined by  $b_{kj}$  and  $c_{1j}$ .

Obviously, the shift parameter  $\Delta W_0$  has disappeared in equation (89). This indicates that the vertical datum connection is a relative problem. We can solve the datum connection by applying a least squares adjustment to equation (89). The result will depend on the relative accuracy of the geoid, and that of geometric heights. Therefore, the result can be expected to improve if the correlations between space stations are taken into account. Inserting the estimates of vertical datum connection into (88), we have

$$\Delta W_0 / \gamma = -Y_1 + A_1 C \quad (94)$$

where

$$C = (A^T \Sigma_{y_{II}}^{-1} A)^{-1} A^T \Sigma_{y_{II}}^{-1} Y_{II} \quad (95)$$

$$\Sigma_{y_{II}} = B \Sigma_y B^T \quad (96)$$

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ -1 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad (97)$$

From equations (90) and (94), one observes that the accuracy of the parameter  $\Delta W_0$  will depend mainly on the absolute accuracy of the geoid (see also column 2 of Table 8). To improve the vertical datum connection further, one should pay attention to the accuracy of space stations and levelling measurements and take correlation information into consideration (see table 8).

It is also seen from Table 7 that it is possible to realize 5 cm accuracy of datum connection, though there may be some difficulty in some parts of the world, due to deviations from the assumptions we have made. But at least, concerning the dense gravity distribution in Europe and North America, and our assumptions based on the comparison among GPS, levelling and gravity data over a long distance, it should be no problem to reach the accuracy of less than 5 cm for vertical datum connection between England and the European mainland, and between Europe and North America.

Good relative accuracy of the geoid improves greatly the internal reliability (see Table 9). This can be explained from equations (45) and (49). With the increase of the accuracy of datum connection, the detectability of the method is improving, which can be seen from Table 10. Table 11 gives the values of external reliability. An outlier has again the most serious effect on the parameters of its own datum zone. Therefore, there must be at least two space stations to avoid the worst situation. An outlier in the reference datum zone affects all parameters, as can be seen from Table 11.



		$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$	$C_{70}$	$C_{80}$
$Y_1$	0	a	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$	$C_{16}$	$C_{17}$	$C_{18}$
$Y'_2$			$C'_{21}$	$C'_{22}$	$C'_{23}$	$C'_{24}$	$C'_{25}$	$C'_{26}$	$C'_{27}$	$C'_{28}$
$Y'_3$			$b'_{31}$	$C'_{32}$	$C'_{33}$	.	.	.	.	.
$Y'_4$			$\vdots$	$b'_{42}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$Y'_5$					$b'_{53}$					
$Y'_6$					$b'_{63}$					
$Y'_7$						$b'_{74}$				
$Y'_8$							$b'_{85}$			
$Y'_9$								$b'_{96}$		
$Y'_{10}$									$b'_{107}$	
$Y'_{11}$										$b'_{118}$

Figure 9: Structure of the coefficient matrix for the new observational equations after the rearrangement, where  $a = -1$ ,  $C'_{kj} = C_{kj} - C_{1j}$ ,  $b'_{ki} = b_{ki} - c_{1i}$ .

Table 9: Undetected Gross Error Bounds(cm).

20.92	20.35	20.59	20.36	19.45	18.80	18.25
20.33	18.74	18.97	21.66	25.57	26.42	25.19
19.00	19.55	18.51	21.67	27.11	27.03	22.67
19.17	28.53	18.20	20.78	22.03	23.10	22.25
21.55	24.67	20.92	19.64	18.73	24.90	19.66
22.63	23.82	21.59	28.06	23.08	20.92	27.88
18.86	18.77	26.35	22.23	31.67	26.75	25.38
27.18	21.01	26.08	20.08	24.72	21.23	25.18
20.86	25.09	31.78	35.15	24.66	21.41	27.28

Table 10: Detectability of the method (cm).

Parameters	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
Detectability	83.28	6.30	4.60	4.41	6.48	3.10	5.66

Table 11: External reliability values of the method.

OBS.	ZONE	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
2	5	-0.18	-0.07	-0.08	-0.12	-0.19	0.43	-0.04
6	5	-0.24	-0.03	0.02	-0.22	-0.51	0.45	-0.05
8	5	-0.22	-0.13	-0.17	0.06	-0.04	0.39	-0.14
15	0	-0.88	-1.04	-1.01	-0.84	-0.86	-0.86	-0.97
20	2	-0.16	0.08	1.51	-0.19	0.36	-0.05	0.81
21	3	-1.22	0.07	-0.51	2.99	-0.22	-0.41	-1.78
23	4	0.39	-0.64	0.86	1.62	6.13	0.15	0.93
33	0	-0.84	-0.87	-0.87	-0.76	-0.67	-0.85	-0.98
34	3	-1.14	0.07	-0.18	2.72	-0.16	-0.54	-1.91
36	1	0.44	6.12	0.71	0.64	0.25	0.82	0.74
37	1	0.38	5.28	0.52	0.50	0.08	0.75	0.68
38	1	0.30	5.33	0.59	0.45	0.09	0.64	0.61
39	4	0.28	-0.85	1.62	2.74	8.69	-0.31	0.26
40	6	0.24	0.14	0.39	0.19	1.30	0.98	3.02
46	6	0.09	0.30	0.18	0.17	0.46	0.97	2.78
50	2	-0.35	0.40	2.10	-0.02	-0.45	-0.17	0.07
52	3	-1.09	0.04	-0.08	2.57	-0.12	-0.55	-1.87
54	3	-1.08	0.01	-0.37	2.64	-0.11	-0.41	-1.58
55	0	-0.84	-1.07	-1.12	-0.71	-0.98	-0.76	-0.90
56	5	-0.07	0.08	-0.04	-0.23	-0.17	0.44	0.28
57	6	0.15	0.38	0.22	0.08	0.51	1.18	3.43
61	2	-0.13	0.36	2.65	0.22	1.34	-0.20	0.72
63	2	-0.09	0.29	1.83	-0.32	0.00	0.14	1.13
SUM		-20.97	2.00	3.51	3.42	6.13	0.99	3.29

3. The simulation results based on the GEM-T1 variance-covariance matrix.

GEM-T1 is a satellite derived earth model, developed by a group of American scientists around Goddard Space Flight Center, NASA. The results are a set of geopotential coefficients up to degree and order 36 with the full error variance-covariance matrix. It is presently thought of as the best satellite derived earth model. The variance-covariance matrix is used to analyze the error estimation of the vertical datum connection. It is indicated in the previous chapters that it is terrestrial gravity anomalies that contribute most to the datum connection problem but not satellite-based earth models. Therefore, this variance-covariance matrix is preferred to be thought of to be an assumed error model as obtained from terrestrial gravity, although it is related to the reality.

For the purpose of comparison, we estimate the accuracy of the vertical datum connection by using three error models. The first is the GEM-T1 variance-covariance model only, which is represented in Figure 10. The second

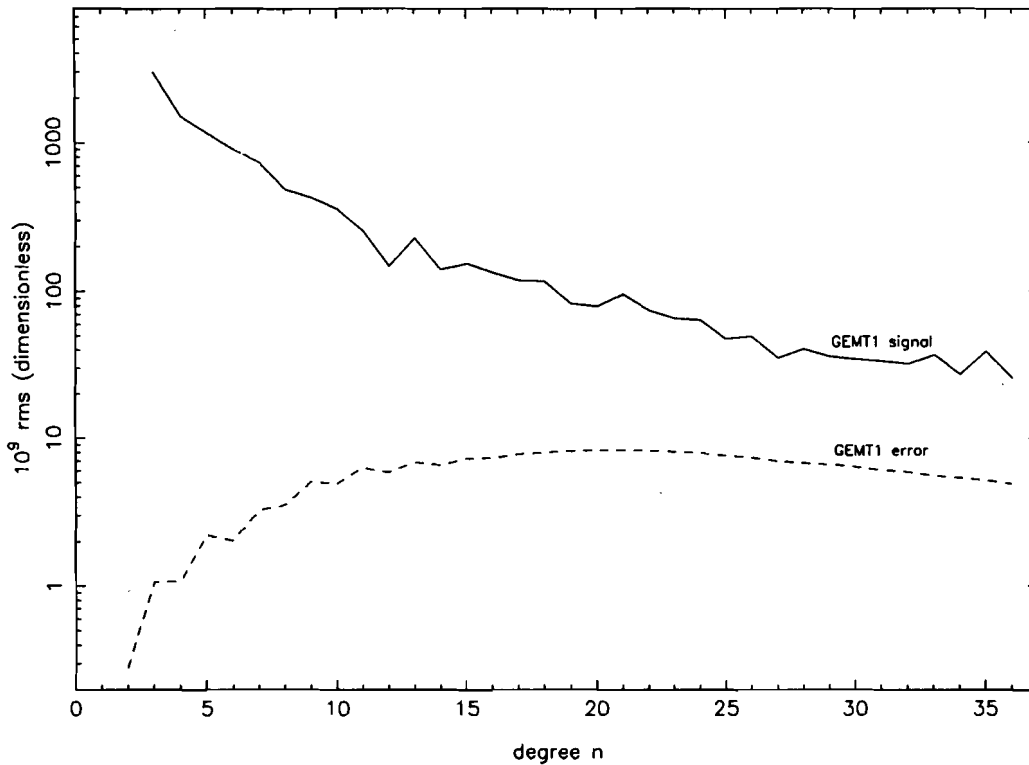


Figure 10: 10<sup>9</sup> degree rms signal and error of GEMT1.

error model is that of GEM-T1 plus the omission error computed from Moritz-Jekeli model starting from degree 37. The last one is obtained with GEM-T1, Rapp's coefficient errors from 37 to 180 without consideration of correlation, and Moritz-Jekeli model from degree 181 again for the omission errors. Generally speaking, the correlation of each error model is relatively weak. Table 12 lists some basic error parameters of these three models for 63 space stations.

Table 12: Error parameters of the three error models for 63 stations (m).

accuracy	GEM-T1	GEM+MJ	GEM+RP+MJ
minimum	1.15	1.98	1.64
mean	1.43	2.15	1.84
maximum	1.87	2.47	2.21

The accuracy values computed by using these three error models are given in Table 13. It can be seen that the parameters of the datum connection are not satisfactory.

Table 13: Accuracy estimation of datum parameters (cm).

zone		1	2	3	4	5	6
parameter	$\Delta W_0$	$C_{10}$	$C_{20}$	$C_{30}$	$C_{40}$	$C_{50}$	$C_{60}$
GEM-T1	21.32	58.84	45.38	38.97	54.89	26.46	55.50
GEM+MJ	41.34	111.26	71.67	70.40	93.61	52.20	82.22
GEM+RP+MJ	33.42	88.55	59.48	55.48	77.15	40.23	70.96

Among them the results with GEM-T1 error model are the best and those of GEM+MJ error model are the worst. All are largely influenced by the absolute accuracy of the geoidal heights. The comparison of the results of these three error models makes us conclude, that the improvement of global geopotential models will increase the accuracy of the datum connection, because in this case the omission error will be largely decreased, which is confirmed by Table 12. The results are strongly related to the number of stations and factors mentioned before, similar to example 1.

A last word should be added. The results of this example do not necessarily represent the attainable accuracy of datum connection, because the applied error models have to be interpreted as assumed error models of height anomalies purely from terrestrial gravity anomalies. This is an important deviation from reality.

**An alternative datum connection model using terrestrial gravity anomalies and satellite geopotential coefficients.**

The method presented by Rummel and Teunissen (1988) should be perfect, if only the parameters of vertical datum connection are of interest. But it seems difficult to implement the satellite derived geopotential coefficients into the model, which would be very useful to accurately determine the parameter  $\Delta W_0$  for realization of an absolute vertical datum definition. The main idea of the method is to solve for the vertical datum connection using space station coordinates, orthometric heights and suitable terrestrial gravity anomalies. Because the number of the parameters for vertical datum connection is not large in general, a large number of space stations including their densification networks over the world should improve not only the vertical datum connection, but also the gravity field itself. On the other hand, nearly all gravity field models using terrestrial gravity anomalies are affected by datum problems. Therefore, we shall recommend an alternative datum connection model as follows.

The disturbing potential again can be written as (Rapp and Cruz, 1986):

$$T(r, \phi, \lambda) = \frac{GM}{r} \sum_{\ell=2}^{\infty} \left(\frac{a}{r}\right)^{\ell} \sum_{m=0}^{\ell} \sum_{\alpha=0}^{\ell} C_{\ell m}^{\alpha} Y_{\ell m}^{\alpha}(\phi, \lambda) \quad (98)$$

where  $r$ ,  $\phi$ ,  $\lambda$  are geocentric coordinates

$$\bar{C}_{\ell m}^{\alpha} = \begin{cases} \bar{C}_{\ell m} & , \alpha = 0 \\ \bar{S}_{\ell m} & , \alpha = 1 \end{cases}$$

$$\bar{Y}_{\ell m}^{\alpha} = \begin{cases} \bar{P}_{\ell m} (\sin \phi) \cos m\lambda & , \alpha = 0 \\ \bar{P}_{\ell m} (\sin \phi) \sin m\lambda & , \alpha = 1 \end{cases} .$$

Therefore we can easily write the model for simultaneous determination of the vertical datum connection, the parameter  $\Delta W_0$  of absolute datum definition and the potential coefficients using terrestrial gravity anomalies and satellite derived geopotential coefficients

$$\Delta g^{(i)} = \frac{2}{r} \Delta W_0 - \frac{2}{r} C_{Q_1 0} - \left( \frac{2}{r} + \frac{\partial}{\partial r} \right) T \quad , \text{ for datum zone } i \quad (99a)$$

where  $\Delta g^{(i)}$  is the corrected gravity anomaly in datum zone  $i$ . The observation stations give

$$h - H^{(i)} = - \frac{\Delta W_0}{r} + \frac{C_{Q_1 0}}{r} + \frac{1}{\gamma} T \quad , \text{ for datum zone } i \quad . \quad (99b)$$

And the observation equations of satellite derived potential coefficients simply are

$$X_s = X \quad . \quad (99c)$$

It can be seen from (99) that the increase of the quality of satellite derived potential models will decrease the effect of datum problem on gravity potential, which further implicitly means the accurate determination of datum parameters. They are related to the weight ratio between geopotential coefficients obtained from terrestrial gravity anomalies and satellite observations, respectively.

## Conclusions.

Vertical datum connection is to determine some relative parameters  $C_{10}$ . It requires orthometric heights obtained by levelling, a set of relative geocentric coordinates, and suitable gravity anomalies. The accuracy of the vertical datum connection is mainly related to the relative accuracy of geometric heights and of the geoid. The higher the relative accuracy of geometric heights and of the geoid, the higher the accuracy of datum connection. In other words, the absolute accuracy of geometric or geoid heights does not affect the parameters  $C_{10}$ . Only when the shift parameter  $\Delta W_0$  is of interest, both absolute geometric and geoid heights at one station are necessary. The accuracy of  $\Delta W_0$  is certainly determined by the absolute accuracy of  $h$  and  $N$ . The accuracy of the datum connection also depends on the number of stations and their distribution. The simulation shows that the accuracy of vertical datum connection can be expected to reach 5 cm or even better between England and Europe, and between Europe and North America, considering the dense gravity data and good relative accuracy of the geoid which have been proved by some tests using GPS, levelling and gravity data.

The reliability measures are also related to the relative accuracy of the geoid, and some of the other factors mentioned above. The larger the number of space stations, the better is the reliability. The simulation indicates that an outlier, e.g. a wrong station connection, will have the most serious effect on the parameters in its own datum zone. Therefore, there should be at least two space stations, suitably distributed in each datum zone to avoid the worst situation, that an outlier is directly transferred to the estimated parameter. The simulation also indicates that the outliers in the observations would have an accumulative effect on the parameter  $\Delta W_0$ . This can be explained by positive undetected gross errors, because they are thought of as a systematic bias in the observations and the coefficients of parameter  $\Delta W_0$  in all observation equations are -1.

The detectability depends on the accuracy of the method. Therefore it will possess the same properties as the accuracy.

An alternative datum connection model is the simultaneous determination of the vertical datum connection, the parameter  $\Delta W_0$  of absolute datum definition



and the potential coefficients using terrestrial gravity anomalies and satellite derived potential coefficients.

The purpose of this work was to evaluate a technique, the method of vertical datum connection proposed in (Rummel & Teunissen, 1988), before the background of the currently available observational data. One limiting factor is the sparseness of precise satellite tracking or VLBI stations. It can be expected that this deficiency will disappear in the near future. More severe is the still insufficient quality of and global coverage with gravity. A gradiometric space mission, space-borne GPS, or the high-precision, almost continuous radio tracking of low flying space crafts by DORIS or PRARE shall very likely improve the situation profoundly before the end of this century. In addition, satellite derived gravity models are uniform, not affected by off-sets between various vertical datums.

The technique of datum connection should be related to identifiable, physical terrain points. As the technique is essentially one of relative connection, it is probably preferable to avoid the term *absolute* geoid computation in connection with the determination of  $\Delta W_0$ . This becomes clear if one realizes that the determination of  $\Delta W_0$  requires the measurement of at least one precise distance between two points. Even if one believes in absolute distances one has to realize that the scale of this distance determines that of the geoid. In addition, as the earth is pulsating distances between points become functions of time and consequently the "absolute" geoid too. Already Bruns (1878) pointed out the vagueness of the geoid concept with a number of still valid arguments. Baarda (1979) formulated his theory in such a way these complications are avoided by definition.

Finally, one should realize that a uniform worldwide vertical datum at a level of better than, say 5 cm, is not only a matter of data coverage and precision, but one of a suitable computation model. At this level all dynamic phenomena affecting the solid earth enter. This makes datum determination a continuous process - a worldwide vertical datum monitoring. The effect of the pulsating earth should less be seen as an additional inconvenience or complication, but more as an additional challenge. It could place geodesy right into the center of the study of the earth as a system with solid earth, hydrosphere, cryosphere and atmosphere as components. The establishment and

maintenance of a precise global datum, spanning all continents and islands, and connecting geometric and gravimetric methods would on the one hand provide a "static" framework for monitoring dynamic processes, in particular global ocean circulation, on the other hand it would be the key for a proper understanding of sea level rise, its causes and its relation to vertical crustal motion on a global scale.

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